SOLUTION OF THE POMPEIU PROBLEM (I)

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Dedicated to Professor Louis Nirenberg with admiration

Abstract. A nonempty bounded open set $\Omega \subset \mathbb{R}^2$ is said to have the *Pompeiu property* if and only if the only continuous function f on $\mathbb{R}^2$ for which the integral of f over $\sigma(\Omega)$ is zero for all rigid motions σ of $\mathbb{R}^2$ is $f \equiv 0$. In this paper, a longstanding open problem, the Pompeiu problem (or equivalently, the Schiffer conjecture), is completely solved. More precisely, we prove that among bounded open sets of $\mathbb{R}^2$, each of which has a connected Lipschitz boundary, only the disks fail to have the Pompeiu property. In addition, we also give an affirmative answer to a longstanding Morera's problem.

1. Introduction

Let Ω be a nonempty bounded open subset of $\mathbb{R}^2$, and let $\mathcal{M}$ denote the set of rigid motions of $\mathbb{R}^2$ onto itself (each $\sigma \in \mathcal{M}$ can be thought of as a rotation followed by a translation). We say that Ω has the *Pompeiu property* if and only if the only continuous function f on $\mathbb{R}^2$ for which

$$\int_{\sigma(\Omega)} f(x,y)dx\,dy = 0 \quad \text{for every } \sigma \in \mathcal{M} \tag{1.1}$$

is the function $f \equiv 0$. Here $\sigma(\Omega)$ denotes the image of Ω under the rigid motion σ (see [P_1], [P_2], [B] or [W_1]).

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The *Pompeiu problem*, which has puzzled many mathematicians during the past 78 years, asks: which sets Ω have the Pompeiu property?

The Pompeiu problem takes its name from the Rumanian mathematician Dimitrie Pompeiu, who was the first to consider equation (1.1). In his papers on the subject, Pompeiu asserted [P_1, P_3] that a disk of any radius $\rho_0 > 0$ possesses Pompeiu property and even published an erroneous proof [P_2]. (The error occurs on page 286, formula (5)). The error was perpetuated by Nicolesco [Ni_1, Ni_2], who searched to establish generalizations of Pompeiu's result. Chakalov [Ch] seems to have been the first to point out that disks do not have the Pompeiu property (refer to [Z_3]). In fact, the function $f(x, y) = \sin ax$, for a suitable choice $a > 0$ satisfying $J_1(a\rho_0) = 0$ (where $J_1(z)$ denotes the Bessel function of the first kind of order one), provides a counter-example (see [Ch]). Nevertheless, Pompeiu did prove in [P_2, P_3] that the square has the Pompeiu property under the additional assumption that f tends to a limit at infinity. This superfluous restriction was later removed by C. Christov, who also showed that a triangle or a parallelogram has the Pompeiu property (see [C_1, C_2, C_3]).

In a celebrated paper [BST], Brown, Schreiber and Taylor [BST] proved that a bounded set $\Omega \subset \mathbb{R}^2$ has the Pompeiu property if and only if for any $\alpha \in \mathbb{C} \setminus \{0\}$ the complexified Fourier transform of the characteristic function of Ω,

$$\hat{\chi}_\Omega(\zeta) = \int_\Omega e^{-i(x\zeta_1 + y\zeta_2)} dx\, dy, \qquad \zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2,$$

does not vanish identically on the set $\{\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2 \big| \zeta_1^2 + \zeta_2^2 = \alpha\}$. Applying the above characterization and asymptotic estimates of the growth of certain Fourier-Laplace transforms, Brown, Schreiber and Taylor [BST] then showed that any polygonal region, proper ellipse, or convex set with at least a true corner has the Pompeiu property. Berenstein [B] subsequently observed that if domain $\Omega \subset \mathbb{R}^2$ is also simply connected then "for every $\alpha \in \mathbb{C} \setminus \{0\}$" in the statement of [BST] can be replaced by "for every $\alpha > 0$".

It is very tempting to conjecture (see, for example, [W_1, p.185], [Z_1], [B] and [K, p.168]) the following:

Among bounded open sets of $\mathbb{R}^2$, each of which has a connected smooth boundary, only the disks fail to have the Pompeiu property.

In 1976, by using Brown-Schreiber-Taylor's result [BST], Williams [W_1] proved a remarkable connection between the Pompeiu problem and a symmetry problem in partial differential equations, known as *Schiffer's conjecture*:

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary of class C^2. Does the existence of a nontrivial solution u of the over-determined Neumann eigenvalue problem

$$\begin{cases} \triangle u + \alpha u = 0 & \text{in } \Omega, \quad \alpha > 0, \\ u\big|_{\partial\Omega} = c = constant, & \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0 \end{cases} \tag{1.2}$$

imply that Ω *is a disk, and that* u *is symmetric about the center of the disk? Here* ν *denotes the unit interior normal to* $\partial\Omega$.

It was proved in [W_1] (also see [B]) that for a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary of class C^2, the failure of the Pompeiu property is equivalent to the existence of a nontrivial solution of (1.2).

Inspired by the results of Kinderlehrer-Nirenberg [KN] and Caffarelli [C] on the regularity of free-boundaries, in 1981 Williams [W_2] proved that if a bounded domain $\Omega \subset \mathbb{R}^2$ has a connected Lipschitz boundary $\partial\Omega$, and if Ω fails to have the Pompeiu property, then $\partial\Omega$ is real analytic. In [CKS], the real analyticity of the boundary of Ω is proved under a substantially weaker condition than Lipschitz boundary by Caffarelli, Karp and Shahgholian. Concerning convex set with a real analytic boundary, a result of Brown and Kahane [BK] states that if the minimum diameter of the plane domain is less than or equal to half the maximum diameter, then the domain has the Pompeiu property.

Let us remark that when Ω is a disk of radius ρ_0 with the center at the origin in $\mathbb{R}^2$, there exist infinitely many couples (α_j, u_j) that solve (1.2). That is, if we choose $\{\alpha_j\}$ such that $\sqrt{\alpha_j}\,\rho_0$ are zeros of the Bessel function of the first kind of order one, then $u_j(x,y) = C_j J_0(\sqrt{\alpha_j}|r|),\ r < \rho_0$, where $r = \sqrt{x^2+y^2}$. Berenstein [B] proved that in $\mathbb{R}^2$ the disk can be characterized as the only simply-connected domain with $C^{2,\gamma}$ $(0<\gamma<1)$ boundary for which there exist infinitely many (α_j, u_j) that solve (1.2). His result has been extended by Berenstein and Yang [BY_2] to any number of dimensions.

In (1.2), if $\alpha = \lambda_2$ the second eigenvalue of the Dirichlet problem in Ω, then the Schiffer conjecture holds because it is a consequence of the isoperimetric inequality of Payne and Weinberg [Pa] that Ω is a disk. In a recent paper [L_1], the author proved that the Schiffer conjecture holds if and only if the third order interior normal derivative of the corresponding Neumann eigenfunction u is constant on the boundary of Ω.

The Pompeiu problem (or equivalently, the Schiffer conjecture) has been included in famous Yau's set of open problems (see [Y, problem section IV, problem 80]). This problem is not only related to the over-determined Neumann eigenvalue problems, but also to problems in harmonic analysis, mathematical physics, mechanics, plasma physics [Te], nuclear reactors [No] and topography (see [SK], [SSW]). Many papers have contributed partial solutions to this problem (see [BST], [W_1], [W_2], [B], [BY_1], [BY_2], [BS], [BK], [Z_2], [Av], [GS_1], [GS_2], [GS_3], [D], [E_1], [E_2], [E_3], [J], [CKS], [L_1] etc). For a magnificent exposition for the history of the problem, and for information about various aspects of the Pompeiu problem, we also refer the reader to Zalcman [Z_1, Z_3, Z_4].

It will be of interest to only give a medical motivation for the problem. Imagine a thin slice, say through the subdermal tissue of the body, parallel to the surface of the skin. Suppose we have a "medical machine" whose X-ray lens can measure the mass of subdermal tissue in the plane of this slice. Assume that the normal subdermal tissue has a two-dimensional density τ_0 (for example, the density of the tumour is bigger than τ_0 and that of the pustule is less than τ_0). Now, our "medical machine" will provide the two-dimensional total mass of the subdermal tissue of this slice, which is just under this piece of the lens. We would be free to move this lens on the skin and the question arises as to whether we can reconstruct the "true" two-dimensional (density) picture of the subdermal tissue in the plane ? In particular, is it possible that two different pieces of subdermal tissue produce the same set of mass for some shape of the lens ? The expected answer is that the "true" two-dimensional picture could be reconstructed if and only if the shape of the lens is not a disk.

It is also interesting to compare the Schiffer conjecture with a result of James Serrin [S] that if $\Omega \subset \mathbb{R}^2$ is a bounded open connected set with smooth boundary $\partial\Omega$ on which there exists a function u satisfying $\triangle u = -1$ on Ω, with $u = 0$ and $\frac{\partial u}{\partial \nu}$=constant on $\partial\Omega$, then Ω must be a disk.

Studying the Pompeiu problem leads to another problem of a similar nature, which we shall refer to as the *Morera problem* because of its relationship to the classical theorem of Morera (see [BST] and [Z_2, Z_3]). Let $\Omega \subset \mathbb{C}$ be a Jordan domain with piecewise smooth boundary Γ. Γ is said to have the *Morera property* if each continuous complex-valued function in the complex plane which satisfies

$$\int_{\sigma(\Gamma)} f(z)dz = 0 \tag{1.3}$$

for every rigid motion σ of the plane is entire (analytic on all of $\mathbb{C}$), where $z = x+iy$. The Morera problem asks: which Jordan curves Γ have the Morera property ? The Morera problem has been open for a long time (see [Z_2, Z_3] and [BST]).

In this paper, we prove the Schiffer conjecture, or equivalently, we prove that among bounded open sets of $\mathbb{R}^2$, each of which has a connected Lipschitz boundary, only the disks fail to have the Pompeiu property. Therefore the Pompeiu problem is completely solved. In addition, we also give an affirmative answer to the longstanding Morera problem.

The main ideas of proof of the Schiffer conjecture are as follows: Let D be the biggest disk that is contained in Ω (Maybe there are more than one, but we only choose one of them). If $D = \Omega$, then the desired conclusion has been obtained. Suppose by contradiction that $D \neq \Omega$. Then, by a rigid motion of the rectangular coordinate frame (a rotation followed by a translation), we may choose a new rectangular coordinate frame such that the origin is at the center of the disk D, and the part boundary $(\partial\Omega)\cap\{\theta\big|0\le\theta<\theta_0\}$ can be written as a strictly increasing function $\theta=\theta(r)$ in some interval $[r_0,r_1)$ with $\theta(r_0)=0$, where r_0 is the radius of D, θ is the polar angle from the positive x-axis, and r is the length of line segment joining the origin and a point $(x,y)\in\partial\Omega$. The analytic of u on $\bar\Omega$ (see, proof of Theorem 3.1) implies that in $\bar\Omega\cap\{(r,\theta)\in\mathbb{R}^2\big|0\le\theta<\theta_0\}$, u can be expanded in a series of type

$$u(x,y)=A_0\,J_0(\sqrt{\alpha}\,r)+\sum_{m=1}^{\infty}A_mJ_m(\sqrt{\alpha}\,r)\cos m(\theta-\tau_m),$$

where $r=\sqrt{x^2+y^2}$, and A_m $(m=0,1,\cdots)$ are constants. The boundary conditions of (1.2) lead to a power series in power of r on $[r_0,r_1)$, which vanishes identically in the interval. By discussing coefficients of this series, we finally obtain that $u(x,y)$ has the form

$$u(x,y)=A_0J_0\big(\sqrt{\alpha\big(x^2+y^2\big)}\,\big),\quad\text{for any }(x,y)\in\Omega.$$

This yields a contradiction because this kind of u can't be a constant on the part of $(\partial\Omega)\cap\{(r,\theta)\in\mathbb{R}^2\big|0\le\theta<\theta_0\}$, from which the desired result is proved.

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2. The Helmholtz equation

Let $g(\xi)$ be a real-valued function defined in an open set Ω in $\mathbb{R}^n$ $(n\ge1)$. For $\zeta\in\Omega$ we call g *real analytic at* ζ if there exist $a_\beta\in\mathbb{R}^1$ and a neighborhood U of ζ (all depending on ζ) such that

$$g(\xi)=\sum_{\beta}a_\beta(\xi-\zeta)^\beta$$

for all ξ in U. We say g is *real analytic in* Ω, if g is real analytic at each $\zeta\in\Omega$.

A subset S of $\mathbb{R}^2$ is an 1-dimensional Lipschitz (respectively, real analytic) surface if and only if S is nonempty and if for every point (x,y) in S, there is a Lipschitz (respectively, real analytic) diffeomorphism of the open unit disk $B(0;1)$ in $\mathbb{R}^2$ onto an open neighborhood J of (x,y) such that $B(0;1)\cap\{z=(z_1,z_2)\in\mathbb{R}^2\big|z_2=0\}$ maps onto $J\cap S$.

Lemma 2.1 (Unique continuation of real analytic function, see, for example, [Jo, p.65]). *Let Ω be a connected open set in $\mathbb{R}^n$, and let g be real analytic in Ω. Let $\zeta \in \Omega$. Then g is determined uniquely in Ω if we know the $D^\beta g(\zeta)$ for all $\beta \in \mathbb{Z}^n$. In particular g is determined uniquely in Ω by its values in any non-empty open subset of Ω.*

Lemma 2.2 (The real analyticity of the solutions of real analytic elliptic equations, see [MS], [MN; Theorem A], [Mu], [Mo$_1$], [Mo$_2$] or [Mo$_3$, §6.6]). *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ be a bounded open set with real analytic boundary. Let A be a strongly elliptic linear operator (of order $2m$) with real analytic coefficients on $\bar{\Omega}$. If f and g_j are real analytic on $\bar{\Omega}$ and $\partial\Omega$, respectively, and if u is a solution of*

$$\begin{cases} Au = f & \text{in } \Omega, \\ \frac{\partial^j u}{\partial \nu^j} = g_j & \text{on } \partial\Omega, \end{cases}$$

where ν is the unit inward normal to $\partial\Omega$, then u is real analytic up to boundary (i.e., u can be extended analytically across the boundary $\partial\Omega$).

Lemma 2.3 (Holmgren's uniqueness theorem, see [Ra; Theorem 2 of p.42] or [T, p.433]). *Let $P(x, D)$ be a linear elliptic partial differential operator of order m, with real analytic coefficients on a connected open set $U \subset \mathbb{R}^n$, and let Γ be a piece of real analytic hypersurface in U. Suppose that a real analytic function u defined in U satisfies $Pu = 0$ and for all $|\beta| \le m-1$, $\partial^\beta u = 0$ on Γ, then $u \equiv 0$ in U.*

Lemma 2.4 (see [B, p.130] or [L$_1$, Lemma 2.5]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and let u be the solution of (1.2), then $u\big|_{\partial\Omega} \equiv c \ne 0$.*

For each integer $m \ge 0$, let $\mathcal{P}_m(\mathbb{R}^n)$ denote the set of homogeneous polynomials of degree m in n variables, i.e., the set of functions u of the form

$$u(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha \quad \text{for } x \in \mathbb{R}^n,$$

with coefficients $a_\alpha \in \mathbb{C}$. A *solid spherical harmonic of degree* m is an element of the subspace

$$\mathcal{H}_m(\mathbb{R}^n) = \{u \in \mathcal{P}_m(\mathbb{R}^n)\big|\triangle u = 0 \text{ on } \mathbb{R}^n\}.$$

Put $\mathcal{H}_m(S^{n-1}) = \{\psi\big|\psi = u\big|_{\mathbb{S}^{n-1}} \text{ for some } u \in \mathcal{H}_m(\mathbb{R}^n)\}$. An element of $\mathcal{H}_m(\mathbb{S}^{n-1})$ is called a *surface spherical harmonic of degree* m. It is well-known (for example, [Mc, Corollary 8.3]) that the restriction map $u \to u\big|_{\mathbb{S}^{n-1}}$ is one-to-one, and hence is an isomorphism from $\mathcal{H}_m(\mathbb{R}^n)$ onto $\mathcal{H}_m(\mathbb{S}^{n-1})$. In particular, $\mathcal{H}_0(\mathbb{S}^{n-1})$ consists of just the constant functions on $\mathbb{S}^{n-1}$. One can easily check that

$$\dim \mathcal{H}_m(\mathbb{S}^1) := N(2,m) = \begin{cases} 1 & \text{if } m = 0, \\ 2 & \text{if } m \geq 1, \end{cases}$$

$$\dim \mathcal{H}_m(\mathbb{S}^{n-1}) := N(n,m) = \frac{2m+n-2}{n-2}\binom{m+n-3}{n-3}, \quad n \geq 3,\ m \geq 0.$$

It will be convenient to introduce the *Beltrami operator*, $\triangle_{\mathbb{S}^{n-1}}$, a differential operator on the unit sphere defined by

$$\triangle_{\mathbb{S}^{n-1}}\psi = (\triangle\tilde{\psi})\big|_{\mathbb{S}^{n-1}} \quad \text{where } \tilde{\psi}(\xi) = \psi(\omega) \quad \text{for } \xi \neq 0.$$

That is, $\tilde{\psi}$ is the extension of ψ to a homogeneous function of degree 0. It is well known (see, for example, [M]) that if $\psi \in \mathcal{H}_m(\mathbb{S}^{n-1})$, then $-\triangle_{\mathbb{S}^{n-1}}\psi = m(m+n-2)\psi$.

We consider the Helmholtz equation

$$\triangle u + \beta^2 u = 0, \tag{2.1}$$

by seeking solutions of the form

$$u(\xi) = f(\beta r)\psi(\omega) \quad \text{where } \xi = r\omega \quad \text{and } r = |\xi|. \tag{2.2}$$

One can easily verify that if u has the form (2.2), and if $z = \beta r$, then

$$\triangle u(\xi) = \beta^2 \left(f''(z)\psi(\omega) + \frac{n-1}{z} f'(z)\psi(\omega) + \frac{1}{z^2} f(z)\triangle_{\mathbb{S}^{n-1}}\psi(\omega) \right).$$

Thus, when $\psi \in \mathcal{H}_m(\mathbb{S}^{n-1})$, the function (2.2) is a solution of the Helmholtz equation (2.1) if and only if f is a solution of

$$f''(z) + \frac{n-1}{z} f'(z) + \left(1 - \frac{m(m+n-2)}{z^2}\right) f(z) = 0. \tag{2.3}$$

This ordinary differential equation can be transformed, by putting

$$g(z) = z^{(n/2)-1} f(z),$$

into Bessel equation of order μ,

$$g''(z) + \frac{1}{z}g'(z) + \left(1 - \frac{\mu^2}{z^2}\right)g(z) = 0, \tag{2.4}$$

with $\mu = m + \frac{n}{2} - 1$.

Let J_μ denote the usual Bessel function of the first kind of order μ, which has the series representation

$$J_\mu(z) = \sum_{q=0}^{\infty} \frac{(-1)^q (z/2)^{\mu+2q}}{q!\Gamma(\mu+q+1)} \quad \text{for } |\arg z| < \pi. \tag{2.5}$$

The Bessel function of the second second kind, Y_μ, is defined by

$$Y_\mu(z) = \frac{J_\mu(z)\cos \pi\mu - J_{-\mu}(z)}{\sin \pi\mu} \quad \text{for } |\arg z| < \pi, \quad \text{if } \mu \notin \mathbb{Z},$$

and by

$$Y_m(z) = \lim_{\mu \to m} Y_\mu(z) \quad \text{for } |\arg z| < \pi, \quad \text{if } m \in \mathbb{Z}.$$

The functions J_μ and Y_μ form a basis for the solution space of Bessel's equation (2.4), so the functions

$$j_m(z) = j_m(n,z) = \sqrt{\frac{\pi}{2}} \frac{J_{m+(n/2)-1}(z)}{z^{(n/2)-1}}$$

and

$$y_m(z) = y_m(n,z) = \sqrt{\frac{\pi}{2}} \frac{Y_{m+(n/2)-1}(z)}{z^{(n/2)-1}}$$

form a basis for the solution space of the original differential equation (2.3). Consequently, let u have the form (2.2) with $\psi \in \mathcal{H}_m(\mathbb{S}^{n-1})$, if $f(z) = j_m(n,z)$, then $\triangle u + \beta^2 u = 0$ on $\mathbb{R}^n$; if $f(z) = y_m(n,z)$, then $\triangle u + \beta^2 u = 0$ on $\mathbb{R}^n \setminus \{0\}$.

Lemma 2.5 (see [Mc, p.336]). *Let θ be the polar angle in the usual parametric representation of the unit circle $\mathbb{S}^1$,*

$$\omega = (\cos\theta, \sin\theta).$$

If $m \geq 1$, then $N(2,m) = 2$ and the functions

$$\psi_{m1}(\omega) = \frac{1}{\sqrt{\pi}} \cos m\theta \quad \text{and} \quad \psi_{m2}(\omega) = \frac{1}{\sqrt{\pi}} \sin m\theta$$

form an orthonormal basis for $\mathcal{H}_m(\mathbb{S}^1)$.

Lemma 2.6. *Let $G \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $u \in C^2(\bar{G})$ be a solution of the Helmholtz equation*

$$\triangle u + \beta^2 u = 0 \quad in\ G, \quad \beta > 0. \tag{2.6}$$

For each fixed point $(x_0, y_0) \in G$, let $B((x_0, y_0); \rho_0)$ be the biggest disk that is contained in G. Then, u has the following representation in $B((x_0, y_0); \rho_0)$:

$$u(x,y) = u(r,\theta) = A_0\, J_0(\beta r) + \sum_{m=1}^{\infty} A_m J_m(\beta r) \cos m(\theta - \tau_m), \tag{2.7}$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, and and A_m ($m = 0, 1, 2, \cdots$) are constants.

Proof. By a translation of the coordinate frame we may assume that $(x_0, y_0) = (0, 0) \in G$. From Lemma 2.5, we know that the functions

$$\frac{1}{\sqrt{\pi}} \cos m\theta \quad \text{and} \quad \frac{1}{\sqrt{\pi}} \sin m\theta$$

form an orthonormal basis for $\mathcal{H}_m(\mathbb{S}^1)$. Since $Y_m(\beta\, r)$ possesses singularity at $r = 0$, and since u belongs to $C^2\big(B((0,0);\rho_0)\big)$, it follows that $\frac{1}{\sqrt{2\pi}} J_0(\beta r)$, $\frac{1}{\sqrt{\pi}} J_1(\beta r) \cos\theta$, $\frac{1}{\sqrt{\pi}} J_1(\beta r) \sin\theta$, $\frac{1}{\sqrt{\pi}} J_2(\beta r) \cos 2\theta$, $\frac{1}{\sqrt{\pi}} J_2(\beta r) \sin 2\theta$, $\cdots$, $\frac{1}{\sqrt{\pi}} J_m(\beta r) \cos m\theta$, $\frac{1}{\sqrt{\pi}} J_m(\beta r) \sin m\theta$, $\cdots$ form an orthogonal basis in $L^2(B((x_0, y_0); \rho_0))$ for the solution space of the Helmholtz equation $\triangle u + \beta^2 u = 0$ in $B((0,0);\rho_0)$. In fact, we have

$$\begin{aligned}
&\int_{B((0,0);\rho_0)} \big(\frac{1}{\sqrt{\pi}} J_m(\beta r) \cos m\theta\big)\big(\frac{1}{\sqrt{\pi}} J_k(\beta r)\ \cos k\theta\big) dx\, dy \\
&= \int_0^{\rho_0} \frac{1}{\pi} J_m(\beta r) J_k(\beta r) r\, dr \int_0^{2\pi} (\cos m\theta)(\cos k\theta) d\theta = 0, \\
&= \begin{cases} \int_0^{\rho_0} \big(J_m(\beta r)\big)^2 r\, dr & \text{when m=k} \\ \quad 0 & \text{when } \ m \neq k; \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\int_{B((0,0);\rho_0)} \big(\frac{1}{\sqrt{\pi}} J_m(\beta r) \sin m\theta\big)\big(\frac{1}{\sqrt{\pi}} J_k(\beta r)\ \sin k\theta\big) dx\, dy \\
&= \int_0^{\rho_0} \frac{1}{\pi} J_m(\beta r) J_k(\beta r) r\, dr \int_0^{2\pi} (\sin m\theta)(\sin k\theta) d\theta \\
&= \begin{cases} \int_0^{\rho_0} \big(J_m(\beta r)\big)^2 r\, dr & \text{when m=k} \\ \quad 0 & \text{when } \ m \neq k; \end{cases}
\end{aligned}$$

$$\int_{B((0,0);\rho_0)} \big(\frac{1}{\sqrt{\pi}} J_m(\beta r)\cos m\theta\big)\big(\frac{1}{\sqrt{\pi}} J_k(\beta r)\sin k\theta\big) dx\, dy$$
$$= \int_0^{\rho_0} \frac{1}{\pi}\big(J_m(\beta r)\big)\big(J_k(\beta r)\big) r\, dr \int_0^{2\pi} (\cos m\theta)(\sin k\theta) d\theta = 0.$$

Note that 1, $\cos\theta$, $\sin\theta$, $\cos 2\theta$, $\sin 2\theta$, $\cdots$, $\cos m\theta$, $\sin m\theta$, $\cdots$ is a complete orthogonal system on $[0, 2\pi]$. This implies that our orthogonal system is also complete in $L^2(B((0,0);\rho_0))$, therefore, it is an orthogonal basis. Thus, u has the following expression in $B((0,0);\rho_0)$:

$$u(x,y) = u(r,\theta) = A_0\, J_0(\beta r) + \sum_{m=1}^{\infty} J_m(\beta r)\big(a_m \cos m\theta + b_m \sin m\theta\big), \tag{2.8}$$

where

$$A_0 = \Big(\int_{B((0,0);\rho_0)} u(r,\theta) J_0(\beta r) r\, dr\, d\theta\Big) \Big/ \Big(2\pi \int_0^{\rho_0} r\big(J_0(\beta r)\big)^2 dr\Big),$$
$$a_m = \Big(\int_{B((0,0);\rho_0)} u(r,\theta) J_m(\beta r)(\cos m\theta) r\, dr\, d\theta\Big) \Big/ \Big(\pi \int_0^{\rho_0} r\big(J_m(\beta r)\big)^2 dr\Big),$$
$$b_m = \Big(\int_{B((0,0);\rho_0)} u(r,\theta) J_m(\beta r)(\sin m\theta) r\, dr\, d\theta\Big) \Big/ \Big(\pi \int_0^{\rho_0} r\big(J_m(\beta r)\big)^2 dr\Big).$$

Note that the right-hand side of (2.8) not only converges in $L^2(B((0,0);\rho_0))$ but also converges pointwise in $B((0,0);\rho_0)$. Put $A_m = \sqrt{a_m^2 + b_m^2}$. When $A_m \ne 0$, we can find a number $\tau_m \in [0, 2\pi)$ such that

$$\cos\tau_m = \frac{a_m}{A_m}, \qquad \sin\tau_m = \frac{b_m}{A_m}.$$

Therefore

$$u(x,y) = A_0\, J_0(\beta r) + \sum_{m=1}^{\infty} A_m J_m(\beta r)\big((\cos\tau_m)\cos m\theta + (\sin\tau_m)\sin m\theta\big)$$
$$= A_0\, J_0(\beta r) + \sum_{m=1}^{\infty} A_m J_m(\beta r)\cos m(\theta - \tau_m).$$

□

Remark 2.7. A similar asymptotic expansion as in Lemma 2.6 can been seen in [Pa, p.527] and [Av, p.1032].

3. Proof of the Schiffer conjecture

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with connected Lipschitz boundary. Assume that there exists an $\alpha > 0$ and a function $u \neq 0$ satisfying (1.2). Then Ω is a disk. Moreover, u has the form*

$$u(x,y) = A_0 J_0\big(\sqrt{\alpha\big((x-x_0)^2 + (y-y_0)^2\big)}\big), \quad \textit{for all } (x,y) \in \Omega, \tag{3.1}$$

where (x_0, y_0) is the center of the disk Ω.

Proof. Since u is a solution of the over-determined Neumann problem (1.2) and Ω is a bounded Lipschitz domain with connected boundary, it follows from [W$_2$] that the boundary $\partial\Omega$ of the bounded domain Ω is actually real analytic.

Let D be the biggest disk that is contained in Ω (Maybe there are more than one, but we only choose one of them). Let the radius of D be r_0 and the center at (x_0, y_0). If $D = \Omega$, then the desired conclusion has been obtained. Suppose by contradiction that $D \neq \Omega$. Then, by a rigid motion of the rectangular coordinate frame (a rotation followed by a translation), we may choose a new rectangular coordinate frame such that (x_0, y_0) is at the origin, and the part boundary $(\partial\Omega) \cap \{\theta \big| 0 \le \theta < \theta_0\}$ can be written as a strictly increasing function

$$r = r(\theta), \quad 0 \le \theta < \theta_0$$

with $r_0 = r(0)$, where θ is the polar angle (counterclockwise) from the positive x-axis, and r is the length of line segment joining the origin and a point $(x,y) \in \partial\Omega$ (If needed, we can alternatively consider the case that $r = r(\theta)$, $\theta_0 < \theta < 0$, is strictly decreasing on $(\theta_0, 0]$ with $\theta_0 < 0$). Thus the inverse $\theta = \theta(r)$ is also strictly increasing in some interval $[r_0, r_1)$, where r_1 is a positive real number satisfying $r_1 > r_0$. Note that for any $(x,y) \in \mathbb{R}^2$,

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

The real analyticity of $\partial\Omega$ implies that $\theta(r)$ is also real analytic in $[r_0, r_1)$; hence $\theta(r)$ has the Taylor series expansion:

$$\theta(r) = \sum_{k=0}^{\infty} \frac{\theta^{(k)}(r_0)}{k!} r^k, \quad r \in [r_0, r_1). \tag{3.2}$$

Since $\partial\Omega$ is real analytic, it follows from the real analyticity of the solutions for real analytic elliptic equations (Lemma 2.2) that the solution u of (1.2) is real analytic upto the boundary, i.e., the solution u is real analytic on $\bar{\Omega}$ and can be analytically extended across the boundary $\partial\Omega$. Thus there exists an open domain $G \supset \bar{\Omega}$ such that u can be analytically extended to G. We denote it by $\tilde{u}$. From Lemma 2.1, $\tilde{u} = u$ on $\bar{\Omega}$. Therefore, there is a disk D_ϵ of radius $\epsilon + \sqrt{|D|/\pi}$ with the center at the origin satisfying $D_\epsilon \subset G$. We may choose θ_0 small enough such that $\bar{\Omega} \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$ is contained in D_ϵ. It follows from the proof of Lemma 2.6 that in D_ϵ, the extended function $\tilde{u}$ can be expressed as

$$\tilde{u}(x,y) = \tilde{u}(r,\theta) = A_0\, J_0(\sqrt{\alpha}\, r) + \sum_{m=1}^{\infty} A_m J_m(\sqrt{\alpha}\, r) \cos m(\theta - \tau_m), \tag{3.3}$$

where $r = \sqrt{x^2+y^2}$; $\;A_m = \sqrt{a_m^2 + b_m^2}$; $\;A_m \cos\tau_m = a_m$, $\;A_m \sin\tau_m = b_m\;$ and

$$\begin{aligned}
A_0 &= \left(\int_{D_\epsilon} \tilde{u}(r,\theta) J_0(\beta r) r\, dr\, d\theta\right) \Big/ \left(2\pi \int_0^{\sqrt{|D_\epsilon|/\pi}} r\big(J_0(\beta r)\big)^2 dr\right),\\
a_m &= \left(\int_{D_\epsilon} \tilde{u}(r,\theta) J_m(\beta r)(\cos m\theta) r\, dr\, d\theta\right) \Big/ \left(\pi \int_0^{\sqrt{|D_\epsilon|/\pi}} r\big(J_m(\beta r)\big)^2 dr\right),\\
b_m &= \left(\int_{D_\epsilon} \tilde{u}(r,\theta) J_m(\beta r)(\sin m\theta) r\, dr\, d\theta\right) \Big/ \left(\pi \int_0^{\sqrt{|D_\epsilon|/\pi}} r\big(J_m(\beta r)\big)^2 dr\right).
\end{aligned}$$

Obviously, $\tilde{u}$ still satisfies the Helmholtz equation $\triangle\tilde{u} + \alpha\tilde{u} = 0$ in D_ϵ because each term of the right-hand side in (3.3) satisfies the same equation. Note that the right-hand side of (3.3) also converges to $\tilde{u}$ pointwise in D_ϵ. In particular, the representation (3.3) remains valid in $\bar{\Omega} \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$ when $\tilde{u}$ is replaced by u. Differentiating (3.3) with respect to θ, we get

$$\frac{\partial u}{\partial \theta} = -\sum_{m=1}^{\infty} m A_m J_m(\sqrt{\alpha}\, r) \sin m(\theta - \tau_m) \quad \text{in } \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}. \tag{3.4}$$

On the other hand, we have

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}, \quad \text{in } \Omega.$$

From the boundary conditions of (1.2) we get (see [B] or [L$_1$]) that $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \equiv 0$ on $\partial\Omega$, which implies $\frac{\partial u}{\partial \theta} \equiv 0$ on $\partial\Omega$. In particular, on $(\partial\Omega) \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$,

$$\sum_{m=1}^{\infty} m A_m J_m(\sqrt{\alpha}\, r) \sin m(\theta - \tau_m) \equiv 0,$$

i.e.,

$$\sum_{m=1}^{\infty} mA_m J_m(\sqrt{\alpha}\, r) \sin m(\theta(r) - \tau_m) \equiv 0, \quad \text{for all } r \in [r_0, r_1). \tag{3.5}$$

Since the curve $\theta(r)$, $r_0 \leq r < r_1$ is real analytic, and since the function u is real analytic on $\bar{\Omega}$, we immediately get that the left-hand side of (3.5) is a real analytic function of one variable r in the interval $[r_0, r_1)$. Clearly, for all $r_0 \leq r < r_1$,

$$\begin{aligned}
0 \equiv & \sum_{m=1}^{\infty} mA_m J_m(\sqrt{\alpha}\, r) \sin m\big(\theta(r) - \tau_m\big) \\
= & \sum_{m=1}^{\infty} mA_m J_m(\sqrt{\alpha}\, r)(\cos m\tau_m)(\sin m\theta(r)) - (\sin m\tau_m)(\cos m\theta(r)) \\
= & \sum_{m=1}^{\infty} mA_m J_m(\sqrt{\alpha}\, r)\Bigg[(\cos m\tau_m)\bigg(m\theta(r) - \frac{m^3}{3!}\big(\theta(r)\big)^3 + \frac{m^5}{5!}(\theta(r))^5 - \cdots \\
& + (-1)^k \frac{m^{2k+1}}{(2k+1)!}\big(\theta(r)\big)^{2k+1} + \cdots\bigg) \\
& - (\sin m\tau_m)\bigg(1 - \frac{m^2}{2!}\big(\theta(r)\big)^2 + \frac{m^4}{4!}(\theta(r))^4 - \cdots \\
& + (-1)^k \frac{m^{2k}}{2k!}\big(\theta(r)\big)^{2k} + \cdots\bigg)\Bigg] \\
= & A_1 J_1(\sqrt{\alpha}\, r)\Bigg\{(\cos \tau_1)\Bigg[\bigg(\theta'(r_0)r + \frac{\theta''(r_0)}{2!}r^2 + \frac{\theta'''(r_0)}{3!}r^3 + \cdots\bigg) \\
& - \frac{1}{3!}\bigg(\theta'(r_0)r + \frac{\theta''(r_0)}{2!}r^2 + \frac{\theta'''(r_0)}{3!}r^3 + \cdots\bigg)^3 \\
& + \frac{1}{5!}\bigg(\theta'(r_0)r + \frac{\theta''(r_0)}{2!}r^2 + \frac{\theta'''(r_0)}{3!}r^3 + \cdots\bigg)^5 - \cdots\Bigg] \\
& + (-\sin \tau_1)\Bigg[1 - \frac{1}{2!}\bigg(\theta'(r_0)r + \frac{\theta''(r_0)}{2!}r^2 + \frac{\theta'''(r_0)}{3!}r^3 + \cdots\bigg)^2 \\
& + \frac{1}{4!}\bigg(\theta'(r_0)r + \frac{\theta''(r_0)}{2!}r^2 + \frac{\theta'''(r_0)}{3!}r^3 + \cdots\bigg)^4 - \cdots\Bigg]\Bigg\} \\
& + 2A_2 J_2(\sqrt{\alpha}\, r)\Bigg\{(\cos 2\tau_2)\Bigg[2\bigg(\theta'(r_0)r + \frac{\theta''(r_0)}{2!}r^2 + \frac{\theta'''(r_0)}{3!}r^3 + \cdots\bigg) \\
& - \frac{2^3}{3!}\bigg(\theta'(r_0)r + \frac{\theta''(r_0)}{2!}r^2 + \frac{\theta'''(r_0)}{3!}r^3 + \cdots\bigg)^3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2^5}{5!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^5-\cdots\bigg]\\
&+(-\sin 2\tau_2)\bigg[1-\frac{2^2}{2!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^2\\
&+\frac{2^4}{4!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^4-\cdots\bigg]\bigg\}\\
&+3A_3J_3(\sqrt{\alpha}\,r)\bigg\{(\cos 3\tau_3)\bigg[3\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)\\
&-\frac{3^3}{3!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^3\\
&+\frac{3^5}{5!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^5-\cdots\bigg]\\
&+(-\sin 3\tau_3)\bigg[1-\frac{3^2}{2!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^2\\
&+\frac{3^4}{4!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^4-\cdots\bigg]\bigg\}\\
&+4A_4J_4(\sqrt{\alpha}\,r)\bigg\{(\cos 4\tau_4)\bigg[4\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)\\
&-\frac{4^3}{3!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^3\\
&+\frac{4^5}{5!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^5-\cdots\bigg]\\
&+(-\sin 4\tau_4)\bigg[1-\frac{4^2}{2!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^2\\
&+\frac{4^4}{4!}\bigg(\theta'(r_0)r+\frac{\theta''(r_0)}{2!}r^2+\frac{\theta'''(r_0)}{3!}r^3+\cdots\bigg)^4-\cdots\bigg]\bigg\}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
=&\Big[-\frac{\sqrt{\alpha}}{2}(\sin\tau_1)A_1\Big]r+\bigg[\frac{\sqrt{\alpha}}{2}(\cos\tau_1)\theta'(r_0)A_1-\frac{\alpha}{4}(\sin 2\tau_2)A_2\bigg]r^2\\
&+\bigg[\bigg(\frac{\sqrt{\alpha}}{4}(\cos\tau_1)\theta''(r_0)+\frac{\alpha^{3/2}}{2^4}(\sin\tau_1)+\frac{\sqrt{\alpha}}{4}(\sin\tau_1)(\theta'(r_0))^2\bigg)A_1\\
&+\frac{\alpha}{2}(\cos 2\tau_2)\theta'(r_0)A_2-\frac{\alpha^{3/2}}{2^4}(\sin 3\tau_3)A_3\bigg]r^3\\
&+\bigg[(\cos\tau_1)\bigg(-\frac{\alpha^{3/2}}{2^4}\theta'(r_0)+\frac{\sqrt{\alpha}}{12}\theta'''(r_0)-\frac{\sqrt{\alpha}}{12}(\theta'(r_0))^3\bigg)A_1
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sqrt{\alpha}}{4}(\sin\tau_1)\theta'(r_0)\,\theta''(r_0)A_1+\left(\frac{\alpha}{4}(\cos 2\tau_2)\theta''(r_0)+\frac{\alpha^2}{48}\sin 2\tau_2\right.\\
&\left.+\frac{\alpha}{2}(\sin 2\tau_2)(\theta'(r_0))^2\right)A_2+\frac{3\alpha^{3/2}}{2^4}(\cos 3\tau_3)\theta'(r_0)A_3-\frac{\alpha^2}{96}(\sin 4\tau_4)A_4\Bigg]r^4\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{3.6}
$$

We have used the Taylor series expansion of the function $\theta(r)$ and representation (2.5). Since the sum function (being a real analytic function) of power series of right-hand side in (3.6) vanishes identically in $[r_0, r_1)$, it follows that all coefficients of the power series must be zero. Thus, we get the following system of infinitely many equations:

$$
\begin{cases}
-\frac{\sqrt{\alpha}}{2}(\sin\tau_1)A_1=0;\\
\frac{\sqrt{\alpha}}{2}(\cos\tau_1)\theta'(r_0)\,A_1-\frac{\alpha}{4}(\sin 2\tau_2)A_2=0;\\
\left[\frac{\sqrt{\alpha}}{4}(\cos\tau_1)\theta''(r_0)+\frac{\alpha^{3/2}}{2^4}\sin\tau_1+(\frac{\sqrt{\alpha}}{4}\sin\tau_1)(\theta'(r_0))^2\right]A_1\\
\qquad\qquad+\frac{\alpha}{2}(\cos 2\tau_2)\theta'(r_0)\,A_2-\frac{\alpha^{3/2}}{2^4}(\sin 3\tau_3)A_3=0;\\
\left[(\cos\tau_1)\left(-\frac{\alpha^{3/2}}{2^4}\theta'(r_0)+\frac{\sqrt{\alpha}}{12}\theta'''(r_0)-\frac{\sqrt{\alpha}}{12}(\theta'(r_0))^3\right)\right.\\
\qquad\qquad\left.+\frac{\sqrt{\alpha}}{4}(\sin\tau_1)\theta'(r_0)\,\theta''(r_0)\right]A_1\\
\qquad\qquad+\left(\frac{\alpha}{4}(\cos 2\tau_2)\theta''(r_0)+\frac{\alpha^2}{48}(\sin 2\tau_2)+\frac{\alpha}{2}(\sin 2\tau_2)(\theta'(r_0))^2\right)A_2\\
\qquad\qquad+\frac{3\alpha^{3/2}}{2^4}(\cos 3\tau_3)\theta'(r_0)\,A_3-\frac{\alpha^2}{96}(\sin 4\tau_4)A_4=0;\\
\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
\tag{3.7}
$$

Obviously, (3.7) is a system of homogeneous linear equations for A_1, A_2, A_3, $\cdots$. We shall discuss (3.7) by distinguishing two cases:

(i) If $\sin m\tau_m \ne 0$ for all $m = 1, 2, 3, \cdots$, then the system (3.7) has a unique solution $A_m = 0$ for all $m \ge 1$. In this case, we find by (3.3) that the solution u has form

$$u(x,y)=u(r,\theta)=A_0J_0(\sqrt{\alpha}r)\quad\text{for all }(r,\theta)\in\bar{\Omega}\cap\{(r,\theta)\in\mathbb{R}^2\big|0\le\theta<\theta_0\}. \tag{3.8}$$

(ii) If there exists an $m_0 \ge 1$ such that $\sin m_0\tau_{m_0} = 0$, then (3.7) has at least two solutions. One of which is $A_m = 0$ for all $m = 1, 2, 3, \cdots$, and others are non-zero solutions. From the zero solution $A_m = 0$ for all $m = 1, 2, 3, \cdots$, we get

$$u(x,y)=u(r,\theta)=A_0J_0(\sqrt{\alpha}r)\quad\text{for all }(r,\theta)\in\bar{\Omega}\cap\{(r,\theta)\in\mathbb{R}^2\big|0\le\theta<\theta_0\}. \tag{3.9}$$

From any non-zero solution of (3.7) (saying $A_1^*, A_2^*, A_3^*, \cdots$), we find by (3.3) that u can be written as

(3.10)
$$u(x,y) = u(r,\theta) = A_0 J_0(\sqrt{\alpha}\, r) + \sum_{m=1}^{\infty} A_m^* J_m(\sqrt{\alpha}\, r)\cos m(\theta - \tau_m)$$
$$\text{for all } (r,\theta) \in \bar{\Omega} \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}.$$

By subtracting two sides of (3.9) from that of (3.10), we obtain

$$\sum_{m=1}^{\infty} A_m^* J_m(\sqrt{\alpha}\, r)\cos m(\theta - \tau_m) \equiv 0 \tag{3.11}$$
$$\text{for all } (r,\theta) \in \bar{\Omega} \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\},$$

which implies u still has form (3.9) for all $(r,\theta) \in \bar{\Omega} \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$. The above argument shows that in any case, u has the form (3.8) (or (3.9)) in $\bar{\Omega} \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$.

Since $u\big|_{\partial\Omega} \equiv c = constant$ and $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$, we get that $A_0 J_0(\sqrt{\alpha}\, r)$ is a constant on $\partial\Omega \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$, and the interior normal derivatives of $A_0 J_0(\sqrt{\alpha}\, r)$ vanish identically on $\partial\Omega \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$. On the other hand, it is clear that the Bessel function $A_0 J_0(\sqrt{\alpha}\, r)$ can't be a constant on $(\partial\Omega) \cap \{(r,\theta) \in \mathbb{R}^2 \big| 0 \le \theta < \theta_0\}$ because any two different points on the curve $\{(r,\theta) \in \mathbb{R}^2 \big| r = r(\theta),\, 0 \le \theta < \theta_0\}$ (the curve lies on $\partial\Omega$), have different distances to the origin. This is a contradiction, which implies that Ω must be the disk D.

When Ω is a disk, say $\Omega = B((x_0, y_0); \sqrt{|\Omega|/\pi}\,)$, by Holmgren's uniqueness theorem (Lemma 2.3) we get

(3.12)
$$u(x,y) = A_0 J_0\Big(\sqrt{\alpha\big((x-x_0)^2 + (y-y_0)^2\big)}\,\Big), \quad \forall (x,y) \in B((x_0,y_0); \sqrt{|\Omega|/\pi}\,).$$

Obviously, u is symmetric about the center of the disk $B\big((x_0,y_0); \sqrt{|\Omega|/\pi}\,\big)$ by (3.12). The theorem is proved. □

By Theorem 3.1, [W$_1$] and [B], we immediately have the following

Theorem 3.2. *If Ω is a bounded domain in $\mathbb{R}^2$ with connected Lipschitz boundary, then Ω has the Pompeiu property if and only it is not a disk.*

Remark 3.3. Let Ω be a bounded domain in $\mathbb{R}^2$ with connected Lipschitz boundary, and let α and u satisfy (1.2). Putting $w = \frac{1}{\alpha c}(u - c)$, we get

$$\begin{cases} \triangle w + \alpha w = -1 & \text{in } \Omega, \\ w = \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

As in [L$_1$], we know that

$$w(x,y) = -\int_\Omega F(x,y;\xi,\zeta)d\xi\,d\zeta, \qquad (x,y)\in\bar{\Omega},$$

where

$$F(x,y;\xi,\zeta)) = \frac{1}{4}Y_0\left(\sqrt{\alpha\big((\xi-x)^2+(\zeta-y)^2\big)}\right)$$

is a fundamental solution for the Helmholtz operator $\triangle+\alpha$ on $\mathbb{R}^2$, and $Y_0(z)$ is the Bessel function of the second kind of order 0. Thus

$$u(x,y) = c\left(1-\frac{\alpha}{4}\int_\Omega Y_0\left(\sqrt{\alpha\big((\xi-x)^2+(\zeta-y)^2\big)}\right)d\xi\,d\zeta\right), \quad (x,y)\in\bar{\Omega},$$

where $c\equiv u\big|_{\partial\Omega}$. By Theorem 3.1, we know that Ω is a disk of radius $\sqrt{|\Omega|/\pi}$ with the center (x_0,y_0), and that $u(x,y) = A_0J_0\big(\sqrt{\alpha\big((x-x_0)^2+(y-y_0)^2\big)}\,\big)$, where $A_0 = \frac{c}{J_0(\sqrt{\alpha|\Omega|/\pi}\,)}$. Let us translated the center of this disk Ω to the origin, we get an identity

$$\begin{aligned} &J_0\big(\sqrt{\alpha|\Omega|/\pi}\,\big)\bigg(1-\frac{\alpha}{4}\int_{B\left(0;\sqrt{|\Omega|/\pi}\right)} Y_0\bigg(\sqrt{\alpha\big((\xi-x)^2+(\zeta-y)^2\big)}\,\bigg)d\xi\,d\zeta\bigg) \\ &\quad = J_0(\sqrt{\alpha(x^2+y^2)}\,), \qquad \text{for all } (x,y)\in\bar{B}(0;\sqrt{|\Omega|/\pi}). \end{aligned} \tag{4.14}$$

This is a new formula, which gives the relationship between the Bessel functions $J_0(z)$ and $Y_0(z)$.

Remark 3.4 It is clear that by some simple modifications, the method developed here works also in $\mathbb{R}^n$, $n\geq 3$, (see [L$_2$]).

4. The Morera Problem

Let f be a continuous complex-valued function in the complex plane $\mathbb{C}$. The well-known Cauchy-Morera theorem says that f is an entire function if and only if

$$\int_\Gamma f(z)dz = 0$$

holds for all piecewise smooth closed Jordan curve Γ in the complex plane.

Recall that a closed Jordan curve Γ is said to have the *Morera property* if each continuous complex-valued function in the complex plane which satisfies

$$\int_{\sigma(\Gamma)} f(z)dz = 0 \tag{4.1}$$

for every rigid motion σ of the plane is entire (analytic on all of $\mathbb{C}$), where $z = x+iy$.

Combining Theorem 3.2, [Z$_2$] and [BST], we have the following:

Theorem 4.1. *Let Γ be a Lipschitz continuous, closed Jordan curve in the complex plane. Then Γ has the Morera property if and only if Γ is not a circle.*

Proof. Let Ω be the plane domain which is enclosed by Γ. When Γ is piecewise smooth, L. Zalcman [Z$_2$] proved that the Pompeiu property of Ω implies the Morera property of the boundary Γ. In fact, if $f \in C^1(\mathbb{R}^2)$ satisfies (4.1), it follows from Green's theorem that

$$\int_{\sigma(\Omega)} \frac{\partial f}{\partial \bar{z}} dxdy = \frac{1}{2i}\int_{\sigma(\Gamma)} f(z)dz = 0, \tag{4.2}$$

so that $\frac{\partial f}{\partial \bar{z}} \equiv 0$, and f is analytic on $\mathbb{C}$. A standard smoothing shows that it is actually sufficient to assume that f is continuous (also see [Z$_3$]). Now, for the Lipschitz continuous closed Jordan curve Γ, we can choose a sequence $\{\Gamma_k\}$ of piecewise smooth closed Jordan curves such that Γ_k approaches Γ as $k \to \infty$. Thus the first equality in (4.2) holds for every Γ_k and the corresponding domain enclosed by Γ_k, so does (4.2) for the Γ and the corresponding Ω enclosed by Γ. Therefore, $\frac{\partial f}{\partial \bar{z}} \equiv 0$ in $\mathbb{C}$ by the Pompeiu property of Ω, which shows that f is analytic on $\mathbb{C}$.

Conversely, Brown, Schreiber and Taylor [BST] showed that the Morera property for Γ implies the Pompeiu property for Ω. Hence, Ω has the Pompeiu property if and only if Γ has the Morera property. Combining this fact and our theorem 3.2, we know that Γ has the Morera property if and only if Γ is not a circle. $\square$

The above theorem enables us to give a simpler criterion for the entire function in the complex plane:

Theorem 4.2. *Let f be a continuous complex-valued function in $\mathbb{C}$, and let Γ be a Lipschiz continuous, closed Jordan curve which is not a circle in the complex plane. If f satisfies (4.1) for all rigid motions σ of the plane, then f is an entire function.*

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