Vanishing of one dimensional L^2-cohomologies of loop groups

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Abstract

Let G be a simply connected compact Lie group. Let $L_e(G)$ be the based loop group with the base point e which is the identity element. Let ν_e be the pinned Brownian motion measure on $L_e(G)$ and let $\alpha \in L^2(\wedge^1 T^* L_e(G), \nu_e) \cap \mathbb{D}^{\infty,p}(\wedge^1 T^* L_e(G), \nu_e)$ $(1 < p < 2)$ be a closed 1-form on $L_e(G)$. Using the results in rough path analysis, we prove that there exists an $f \in L^1_{loc}(L_e(G), \nu_e)$ such that $df = \alpha$. Moreover we prove that $\dim \ker \Box = 0$ for the Hodge-Kodaira type operator acting on 1-forms on $L_e(G)$.

1 Introduction

Let (M, g) be a compact Riemannian manifold. Let d be the exterior differential operator on M. Let d^* be the adjoint operator of d in the L^2 space of differential forms with respect to the Riemannian volume. Let $\Box = dd^* + d^*d$. Celebrated Hodge-Kodaira theorem asserts that $\dim \ker \Box|_p = b_p$. Here $\Box|_p$ denotes the Hodge-Kodaira operator restricted to the space of p-forms and b_p is the (real coefficient) Betti number of M. This theorem does not hold any more in non-compact Riemannian manifold. On the other hand, in infinite dimension, there exist natural measures, such as (pinned) Brownian motion measures, on spaces of paths over a Riemannian manifold. Several researchers have been trying to establish a differential geometry and analysis including Hodge-Kodaira type theorem based on Brownian motion measures. Since the path space $P_x(M) = C([0, 1] \to M \mid \gamma(0) = x)$ has trivial topology, one natural guess is that there are no harmonic forms on $P_x(M)$ except 0-dimension. When M is a euclidean space and $x = 0$, the path space with the Brownian motion measure is the Wiener space and Shigekawa [28] proved that cohomologies of all dimensions vanish on the Wiener space. When M is a Riemannian manifold, the Bismut tangent space is used to define vector fields on $P_x(M)$. However, if the curvature of M does not vanish, then the Lie bracket of the vector fields do not belong to the Bismut tangent space. This shows a difficulty to study exterior differential operators on $P_x(M)$. We refer the reader to [11, 22] for this problem. Now let us consider the pinned case. Let $L_x(M) = C([0, 1] \to M \mid \gamma(0) = \gamma(1) = x)$. We have difficulties for the definition of exterior differential operator similarly to $P_x(M)$. Instead of working on $P_x(M)$, some researchers studied the properties of differential calculus over submanifolds in the Wiener space [6, 18, 1, 26]. These submanifolds are in some sense isomorphic to $L_x(M)$. The tangent space of the submanifold is defined naturally and the Lie brackets of the vector fields on the submanifold also belong to the tangent space. That is, the exterior differential operator is well-defined. The topology of $L_x(M)$ is non-trivial and we may expect that the dimension of harmonic forms coincide with the Betti number of $L_x(M)$. Kusuoka announced this result in the framework of submanifolds in Wiener spaces in his ICM plenary lecture in 1990 [19]. See [20, 21] also. We explain his results in Section 2 breifly. In the present paper, we study the Hodge-Kodaira type theorem for 1-forms on the based loop group $L_e(G)$, where G is a compact Lie group. When G is simply connected, $\pi_2(G) = 0$ and so $\pi_1(L_e(G)) = 0$ and the first Betti number is 0. Therefore we may conjecture that the kernel of "the Hodge-Kodaira type operator" acting on 1-forms is 0. Indeed, this is one of the main results of this paper. The proof of this fact and the approach to the problem is different from Kusuoka's ones. We explain the rough idea of our proof. Using a map from Wiener space to $L_e(G)$, we change the problem to a problem on an "open subset" $\mathcal{D}_\varepsilon$ of the Wiener space. The map is given by the solution of a stochastic differential equation on the compact Lie group and flows which are generated by certain vector fields on the Wiener space. The "open subset" $\mathcal{D}_\varepsilon$ is homotopy equivalent to $L_e(G)$ in some sense. However the "open" is defined in the sense of rough path analysis. This topology is finer topology than usual uniform convergence topology of the Wiener space. The most important next step is to establish a Poincaré's type vanishing lemma on a ball like set $U_r(\varphi)$ in the sense of rough path analysis. That is, we prove that a closed 1 form on $U_r(\varphi)$ is exact. Note that $\mathcal{D}_\varepsilon$ has a covering by a countable sets of the ball like sets. In the third step, using the topological property of $\pi_1(L_e(G)) = 0$, we prove that the closed 1-form on $\mathcal{D}_\varepsilon$ is exact putting together the locally established Poincaré's type lemma on $U_r(\varphi)$. Finally, using the vanishing theorem on 1-dimensional cohomologies of $\mathcal{D}_\varepsilon$ and hypoellipticity results of Bochner type Laplacian and the essential self-adjointness of the Ornstein-Uhlenbeck operator [2], we prove that the dimension of the harmonic 1 forms is 0 on $L_e(G)$.

The paper is organized as follows. In Section 2, we state main results in this paper and make some remarks. In Section 3, first, we recall the necessary results in rough path analysis and introduce the subset $U_{r,\varphi}$, $U_r(\varphi)$ in (3.47) and prove a Poincaré type vanishing lemma for 1-forms on the subsets in Lemma 3.13 and Theorem 3.14. This kind of vanishing lemma was studied by Kusuoka [21]. Also Shigekawa [31] studied Hodge-Kodaira type operator with absolute boundary condition on convex domains. However $U_r(\varphi)$ is not an H-convex domain and the Poincaré type vanishing lemma is non-trivial. To prove Lemma 3.13 and Theorem 3.14, we prove the Poincaré inequalities on finite dimensional approximation of $U_{r,\varphi}$ in Claim 2 in the proof of Lemma 3.13. The argument is similar to that of [5]. The point is that the Poincaré constant are independent of the dimensions. After proving these results, we introduce the notion of H-connectedness and H-simply connectedness. We prove the H-connectivity of the connected union of $U_r(\varphi)$ in Lemma 3.17. Also we state basic properties of the solution of a stochastic differential equation on a compact Lie group. The domain $\mathcal{D}_\varepsilon$ is defined by the solution and we prove the H-connectedness and the H-simply connectedness of $\mathcal{D}_\varepsilon$ when G is simply connected. This and Stokes theorem (Lemma 3.22) are used to prove the existence of a function F such that $dF = \beta$ for a closed 1-form β on $\mathcal{D}_\varepsilon$ in Section 5. In Section 4, we introduce a submanifold S in the Wiener space using the solution of the stochastic differential equation which is introduced in Section 2. S with the induced probability measure on it is isomorpic to $L_e(G)$ with the pinned Brownian motion measure. See Proposition 4.1. By this Proposition, we identify $L_e(G)$ and S. Also we define an ordinary differential equation on the Wiener space and using the solution map, we define a retract map from $\mathcal{D}_\varepsilon$ to S. This kind of retract map are used in [7, 14, 1]. By the retract map, we obtain a closed form on $\mathcal{D}_\varepsilon$ from a closed form on S and $L_e(G)$ and we can apply the results in Section 3. In Section 5, we prove our main results.

2 Statement of results and remarks

Let W^d be the set of continuous paths on $\mathbb{R}^d$ defined on $[0,1]$ starting at 0. We denote by μ the Wiener measure on W^d whose Cameron-Martin subspace is $H = H^1([0,1] \to \mathbb{R}^d \mid h_0 = 0)$. We recall the definition of Sobolev spaces ([17]) over the Wiener space (W^d, H, μ). Let $\mathfrak{F}C_b^\infty(W^d, E)$ be the set of all smooth cylindrical functions with values in a separable Hilbert space E. When $E = \mathbb{R}$, we may omit to denote E. $\mathbb{D}^{k,p}(W^d, E)$ stands for the set of L^p functions with respect to μ on W^d with values in E which are k-times H-differentiable and all their derivatives are also in $L^p(\mu)$. Let G be a compact Lie group and $\langle \cdot, \cdot \rangle$ be a bi-invariant Riemannian metric on G. Let $P_e(G)$ be the set of continuous paths which are defined on the time interval $[0,1]$ and the starting point is e. Let $L_e(G)$ be the subset of $P_e(G)$ which consists of paths whose end point is also e. Let ν, ν_e be the Brownian motion measure on $P_e(G)$ and the pinned Brownian motion measure on $L_e(G)$ respectively. These measures are defined by the diffusion semigroup $e^{t\Delta/2}$, where Δ is the Laplace-Bertlami operator which is defined by the bi-invariant Riemannian metric. Let $T_e(G) = \mathfrak{g}$ be the Lie algebra of G. We identify it as the set of right invariant vector fields. The Riemannian metric $\langle \cdot, \cdot \rangle$ defines a inner product on $\mathfrak{g}$. We fix an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_d\}$ which enables us to identify $\mathfrak{g}$ and $\mathbb{R}^d$, where $d = \dim G$. Therefore we identify H and a set of H^1-paths over $\mathfrak{g}$ staring at 0 in this way. Also we denote $H_0 = \{h \in H \mid h_1 = 0\}$. We recall the definition of H-derivative on $P_e(G)$ and $L_e(G)$. For a smooth cylindrical function $F(\gamma)$ on $P_e(G)$(or $L_e(G)$), we define the H-derivative of F by a measurable map $G = G(\gamma)$ (actually smooth map in this case) from $P_e(G)$(or $L_e(G)$) to H(or H_0) which satisfies that

$$(G(\gamma), h) = \lim_{\varepsilon \to 0} \frac{F(e^{\varepsilon h}\gamma) - F(\gamma)}{\varepsilon}$$

for all $h \in H$ (or $h \in H_0$), where $(\cdot, \cdot)$ is the inner product of H(or H_0). We denote $G(\gamma)$ by $(\partial F)(\gamma)$. This derivative corresponds to the derivative which is defined by a right-invariant vector field on $L_e(G)$. By the right translation, we can define a right invariant Riemannian metric on $L_e(G)$ and the Levi-Civita covariant derivative ∇. See [2] and [4]. The covariant derivative ∇ is a map on the smooth cylindrical tensor fields such that $\nabla T \in \mathfrak{F}C_b^\infty(\otimes^{p+1} T^* L_e(G))$ for $T \in \mathfrak{F}C_b^\infty(\otimes^p T^* L_e(G))$ $(p = 0, 1, 2, \ldots)$. The Sobolev spaces $\mathbb{D}^{k,q}(\otimes^p T^* L_e(G))$ $(k \in \mathbb{N}, q \geq 1)$ is the completion of $\mathfrak{F}C_b^\infty(\otimes^p T^* L_e(G))$ by the norm $\| \ \|_{k,q}$ such that

$$\|T\|_{k,q} = \left(\sum_{k=0}^{q} \|\nabla^k T\|_{L^q(\nu_e)}^q \right)^{1/q}.$$

We introduce a submanifold which is isomorphic to $L_e(G)$ through the solution of the stochastic differential equation in the sense of Stratonovich on G:

$$\begin{aligned} dX(t,a,w) &= (L_{X(t,e,w)})_* \circ dw_t, \\ X(0,a,w) &= a \in G. \end{aligned} \tag{2.1}$$

Here $L_a b = ab$ for $a, b \in G$ and w_t is the d-dimensional standard Brownian motion on $\mathbb{R}^d \cong \mathfrak{g}$ whose starting point is 0. That is, $w = (w_t) \in W^d$. We fix an ∞-quasi-continuous version of $X(1,e,w)$. Let

$$S = \left\{ w \in W^d \mid X(1,e,w) = e \right\}.$$

$\delta_e(X(1,e,w))$ is a positive generalized Wiener function [32] and it defines a probability measure

$$d\mu_e(w) = p(1,e,e)^{-1}\delta_e(X(1,e,w))d\mu(w)$$

on S. Note that μ_e has no mass on any Borel measurable subset A with $C_q^s(A) = 0$, where C_q^s denotes the (q,s)-capacity of A and q (the parameter for integrability) is any number which is greater than 1 and s (the parameter for differentiability) is a sufficiently large positive number which depends on the dimension of G. We refer the reader to [32] and [17] for these notions and results. The map $X : (S, \mu_e) \to (L_e(G), \nu_e)$ is isomorphism in the sense of Proposition 4.1. In particular, $X_*\mu_e = \nu_e$. Also the covariant derivative ∇_S and Sobolev spaces $\mathbb{D}^{k,q}(\otimes^p T^*S)$ are defined by using the connection as the submanifold in the Wiener space W^d. We denote by $\| \ \|_{k,q}$ the Sobolev norm. See [2]. We denote the subspace of p-forms on S by $\mathbb{D}^{k,q}(\wedge^p T^*S)$. The exterior differential operator d_S on S can be defined in the same way as finite dimensions. The first main theorem of this paper is formulated on the submanifold S instead of $L_e(G)$.

Theorem 2.1. *Let G be a simply connected compact Lie group. There exists a sequence of non-negative bounded ∞-quasi-continuous functions $\rho_n \in \mathbb{D}^\infty(W^d)$ $(n \in \mathbb{N})$ for which the following statements hold.*
(1) *For any $r > 1$, $k \in \mathbb{N}$, $\lim_{n\to\infty} C_r^k\left(\{w \in W^d \mid \rho_n(w) = 1\}^c\right) = 0$ and $\lim_{n\to\infty} \|\rho_n - 1\|_{r,k} = 0$.*
(2) *Let $1 < p < 2$. Let $\theta \in \mathbb{D}^{1,2}(\wedge^1 T^*S, d\mu_e) \cap \mathbb{D}^{\infty,p}(\wedge^1 T^*S, d\mu_e)$ and assume that $d_S\theta = 0$ $\mu_e - a.s.$ on S. Let $1 < q < p$ and k be a sufficiently large positive integer. Then there exist f and f_n which satisfy* (i)-(v) *below.*

(i) *f is a μ_e-almost everywhere defined measurable function on S and f_n are (q,k)-quasi-continuous function on W^d and $f_n \in \mathbb{D}^{k,q}(W^d)$.*

(ii) *For any n, $f_n(w) = f(w)$ μ_e-almost everywhere on $\{\rho_n(w) \neq 0\} \cap S$. Assume that k is sufficiently large. Then (q,k)-quasi-continuous modification of $d_S f_n$ is equal to θ for μ_e-almost all element of $\{\rho_n(w) \neq 0\} \cap S$.*

(iii) *Let $\eta \in \mathbb{D}^\infty(W^d)$ be an ∞-quasi-continuous function. Then it holds that $f\rho_n\eta \in L^1(S, \mu_e)$.*

(iv) *For any n and ∞-quasi-continuous map $\eta \in \mathbb{D}^\infty(W^d, H^*)$,*

$$\int_S f(w)\rho_n(w)\left(-(d_S\rho_n(w), \eta(w)) + \rho_n(w)\widetilde{d_S^*\eta}(w)\right) d\mu_e(w)$$
$$= \int_S \left(\tilde{\theta}(w)\rho_n(w) + f(w)d_S\rho_n(w), \rho_n(w)\eta(w)\right) d\mu_e(w),$$

where $\widetilde{d_S^*\eta}$ is an ∞-quasi-continuous modification of $d_S^*\eta$ and so on.

(v) *Let $K > 0$ and ψ_K be a smooth function on $\mathbb{R}$ such that $\psi_K(u) = u$ $(|u| \le K)$, $\psi_K(u) = -K - 1$ $(u \le -K - 1)$, $\psi_K(u) = K + 1$ $(u \ge K + 1)$ and set $f^K = \psi_K(f)$. Then $f^K \in \mathbb{D}^{1,2}(S, \mu_e)$ and $d_S f^K = \psi_K'(f)\theta$ holds.*

Note that ρ_n can be chosen independent of θ. Actually ρ_n can be given more explicitly using the iterated integrals in the sense of rough path. The second main result is concerned with the Hodge-Kodaira operator acting on 1-forms.

Definition 2.2. *Let d be the exterior differential operator acting on* 1*-forms on $L_e(G)$. Let d^* be the adjoint operator of d. We consider the closable form on $L^2(\wedge^1 T^*S)$.*

$$\mathcal{E}(\alpha,\alpha) = (d\alpha, d\alpha)_{L^2(\wedge^2 T^* L_e(G))} + (d^*\alpha, d^*\alpha)_{L^2(L_e(G))},$$

which is defined on $\mathfrak{F}C_b^\infty(\wedge^1 T^* L_e(G))$. The Hodge-Kodaira operator $\square$ acting on 1*-forms is the non-negative generator of the closed form of the closure of this form.*

It is proved in [30] that $\left(dd^* + d^*d, \mathfrak{F}C_b^\infty(\wedge^1 T^* L_e(G))\right)$ is essentially self-adjoint. The following is the second main theorem which follows from Theorem 2.1.

Theorem 2.3. *Let G be a simply connected compact Lie group. Then $\ker \square = \{0\}$. Also it holds that*

$$L^2(\wedge^1 T^* L_e(G)) = \overline{\{df \mid f \in \mathfrak{F}C_b^\infty(L_e(G))\}} \oplus \overline{\{d^*\alpha \mid \alpha \in \mathfrak{F}C_b^\infty(\wedge^2 T^* L_e(G))\}}.$$

Finally, we make further remarks.

(1) Let (M,g) be a complete Riemannian manifold. Let φ be a smooth function and assume that $d\mu(x) = e^{-\varphi(x)}dv(x)$ is a probability measure on M, where $dv(x)$ denotes the Riemannian volume element. If $H_1(M,\mathbb{Z}) = 0$, it is not difficult to see that

$$L^2(\wedge^1 T^* M, d\mu) = \overline{\{df \mid f \in C_0^\infty(M)\}} \oplus \overline{\{d^*\alpha \mid \alpha \in C_0^\infty(\wedge^2 T^*(M))\}}.$$

Our results is a generalization of this result in infinite dimensional spaces. Note that for $p \geq 2$ even if $H_p(M,\mathbb{Z}) = 0$, the similar kind of result is non-trivial.

(2) As noted in the introduction, there are some difficulties to define a de Rham complex of differential forms in the Sobolev space category on the general path spaces $P_x(M)$, $L_x(M)$ However, we can define them on submanifolds in Wiener spaces. See [18, 19, 1, 6]. The proof in this paper can be applied to prove the vanishing of the 1-dimensional L^2 cohomology of the submanifold which is isomorphic to $L_x(M)$ in the case where $\pi_2(M) = 0$ which is equivalent to $\pi_1(L_x(M)) = 0$. Generally, $H_1(L_x(M),\mathbb{Z}) = 0$ does not imply $\pi_1(L_x(M)) = 0$ although the converse is always true. However, the proof in this paper may be extended to the case where $H_1(L_x(M),\mathbb{Z}) = 0$ since we use such kind of topological property of the loop space when we apply the Stokes theorem which is valid for singular chains.

(3) We mention the works of Kusuoka in the introduction. We explain Kusuoka's results. Kusuoka defined a local Sobolev spaces $\mathcal{D}^{\infty,q}_{loc}(U, d\mu)$ where U is a subset of W^d and q is the index of the integrability. Based on this Sobolev spaces and several results on the capacity which he introduced, Kusuoka announced the following theorems in [19]. Let M be a compact Riemannian manifold which is isometrically embedded in $\mathbb{R}^d$. Let $P(x) : \mathbb{R}^d \to T_xM$ be the projection operator and

$$\begin{aligned} dX(t,x,w) &= P(X(t,x,w)) \circ dw_t, \\ X(0,x,w) &= x \in M. \end{aligned}$$

There exists a probability measure $d\mu_x = p(1,x,x)^{-1}\delta_x(X(1,x,w))d\mu$ on the submanifold:

$$S = \{w \in W^d \mid X(1,x,w) = x\} \subset W^d.$$

Theorem 2.4. *There exists an isomorphism:*

$$\left\{\alpha \in \mathcal{D}^{\infty;q}_{loc}(\wedge^p T^*S) \mid d_S\alpha = 0\right\}/\left\{d_S\beta \mid \mathcal{D}^{\infty;q}_{loc}(\wedge^{p-1}T^*S)\right\} \quad \simeq \quad H^p(\mathcal{M}_x, \mathbb{R}),$$

where

$$\begin{aligned}\mathcal{M}_x \quad = \quad & \Big\{h \in H \mid \xi(1,x,h) = x, \text{ where } \xi(t,x,h) \text{ is the solution to} \\ & \dot{\xi}(t,x,h) = P(\xi(t,x,h))\dot{h}(t), \xi(0,x,h) = x,\ t \ge 0\Big\}\end{aligned}$$

and $H^p(\mathcal{M}_x, \mathbb{R})$ is the de Rham cohomology of $\mathcal{M}_x$.

Let $H^1 \cap L_x(M)$ be the subset of H^1-paths of $L_x(M)$. Noting that $H^1 \cap L_x(M)$ and $\mathcal{M}_x$ is C^∞-homotopy equivalent, the conclusion of Theorem 2.4 is natural.

Let $\Box = d_S^* d_S + d_s d_S^*$ and $\Box|_p$ be the restriction on p-forms. They are defined as the Friedrichs extension of them on some cores.

Theorem 2.5. *There exists a map $j_p : \ker \Box|_p \to H^p(\mathcal{M}_x, \mathbb{R})$ such that*
(1) *j_p is surjective for $p = 0, 1, 2, \ldots$.*
(2) *j_p is injective for $p = 0, 1$.*

Therefore our results give another proof to some special cases of his results. Our proof is based on the vanishing theorem (Theorem 3.14) which is new. Note that our proof is an extension of the proof of the weak Poincaré inequalities on $L_e(G)$ in [5]. We can prove the weak Poincaré inequality on $L_x(M)$ when M is simply connected. Using the retract map in the proof, we may prove a vanishing theorem on a "contractible domain" of $L_x(M)$. Moreover, combining the usage of the Čech cohomology, we may prove the isomorphism between $H_1(H^1 \cap L_x(M), \mathbb{R})$ and $\ker \Box|_1$ based on our proof. But we do not pursuit this dierction in this paper.

3 Preliminary

The solutions of Itô's stochastic differential equations are measurable functions on W^d, but, they are not continuous in the uniform convergence topology of W^d in general. The reason of the discontinuity is clarified by the rough path analysis [24, 25]. In rough path analysis, we need to consider objects which consists of the path and the iterated integrals. To explain the iterated integrals, we take two continuous paths $x = x_t = (x_t^1, \ldots, x_t^d)$, $y = y_t = (y_t^1, \ldots, y_t^d)$ $(0 \le t \le 1)$ on $\mathbb{R}^d$. Suppose that x or y is a bounded variation path. Then we can define for $0 \le s \le t \le 1$

$$\begin{aligned}C(x,y)_{s,t} &= \int_s^t (x_u - x_s) \otimes dy_u \\ &= \sum_{1 \le i,j \le d} \left(\int_s^t (x_u^i - x_s^i) dy_u^j\right) e_i \otimes e_j \in \mathbb{R}^d \otimes \mathbb{R}^d \qquad (3.1)\end{aligned}$$

as a Stieltjes integral. Here $e_i = {}^t(0, \ldots, \overset{i}{1}, \ldots, 0)$. We introduce a function spaces for these iterated integrals. Let $\Delta = \{(s,t) \in \mathbb{R}^2 \mid 0 \le s \le t \le 1\}$. Let V be a normed linear space. For a continuous mapping $\phi : \Delta \to V$, define

$$\|\phi\|_{m,\theta} \quad = \quad \left\{\int_0^1 \int_0^t \frac{|\phi(s,t)|^m}{(t-s)^{2+m\theta}} ds dt\right\}^{1/m},$$

where, $m \geq 4$ is an even number and $0 < \theta < 1$. We denote the all continuous mapping ϕ from Δ to V with $\|\phi\|_{m,\theta} < \infty$ by $W_{m,\theta}(\Delta \to V)$. This spaces are separable Banach spaces. For $w \in W^d$, define $\bar{w}_{s,t} = w_t - w_s$ $((s,t) \in \Delta)$. We denote by $W_{m,\theta}(\mathbb{R}^d)$ all $w \in W^d$ with $\|\bar{w}\|_{m,\theta} < \infty$. We denote $\|w\|_{m,\theta}$ instead of $\|\bar{w}\|_{m,\theta}$. Note that the Hölder norm $\|w\|_{H,\theta} = \sup_{0<s,t<1} \frac{|w_t - w_s|}{|t-s|^\theta}$ is weaker than the norm of $\| \ \|_{m,\theta}$. Wiener measure μ satisfies that $\mu(W_{m,\theta/2}) = 1$ for all $0 < \theta < 1$. If x and y are Lipschitz continuous paths, then $C(x,y) \in W_{m,\theta}(\Delta \to \mathbb{R}^d \otimes \mathbb{R}^d)$ for all (m,θ) with $m(1-\theta) > 2$. See Lemma 3.5. Let $w = w_t = (w_t^1, \ldots, w_t^d) \in W^d$ and $w(N)_t$ be the dyadic polygonal approximation of w. That is $w(N)_t = w_t$ for $t = \frac{k}{2^N}$ $(k = 0, 1, \ldots, 2^N)$ and $t \to w(N)_t$ $(\frac{k}{2^N} \leq t \leq \frac{k+1}{2^N}, 0 \leq k \leq 2^N)$ are linear function. Also $w(N)^i$ denotes the dyadic polygonal approximation of w^i and define $w(N)^{\perp,i} = w^i - w(N)^i$, $w(N)^\perp = w - w(N)$. We need a probabilistic argument to define the integral $C(w^i, w^j)_{s,t}$, $C(w,w)_{s,t}$, etc in contrast with $C(w(N), w)$, $C(w(N)^i, w^j)$ etc. That is, they are Stratonovich integrals. Actually we need to fix a version of them below.

Theorem 3.1. *Let Ω be the subset of W^d which consists of w satisfying the following* (i)-(iii).

(i) $\lim_{N\to\infty} w(N)$ *converges in $W_{m,\theta/2}(\mathbb{R}^d)$ for all (m,θ) with $m(1-\theta) > 2$.*

(ii) $\lim_{N\to\infty} C(w(N), w(N))$ *converges in $W_{m,\theta}(\Delta \to \mathbb{R}^d \otimes \mathbb{R}^d)$ for all (m,θ) with $m(1-\theta) > 2$.*

(iii) $\lim_{N\to\infty} C(w(N)^\perp, w(N))$ *and* $\lim_{N\to\infty} C(w(N), w(N)^\perp)$ *converge to* 0 *in $W_{m,\theta}(\Delta \to \mathbb{R}^d \otimes \mathbb{R}^d)$ for all (m,θ) with $m(1-\theta) > 2$.*

Then Ω^c is a slim set and it holds that $H \subset \Omega$ and $\Omega + H \subset \Omega$.

In rough path analysis, it is proved in many papers that the Wiener measure of the total set of paths which satisify (i), (ii) above is 1. We need the property (iii) for our applications. The property (iii) is essential in [5] also. The fact that Ω^c is a slim set is proved in [15]. We give the proof of Theorem 3.1 for the sake of completeness, together with that of Theorem 3.2.

We use the following notation. For $w \in \Omega$, we define

$$C(w,w)_{s,t} = \lim_{N\to\infty} C(w(N), w(N))_{s,t} \tag{3.2}$$

$$C(w^i, w^j)_{s,t} = \lim_{N\to\infty} C(w(N)^i, w(N)^j)_{s,t} \tag{3.3}$$

where $1 \leq i, j \leq d$. Then it holds that for any $w = (w^i) \in \Omega$ and $0 \leq s \leq t \leq 1$,

$$C(w^i, w^j)_{s,t} = (w_t^i - w_s^i)(w_t^j - w_s^j) - C(w^j, w^i)_{s,t} \tag{3.4}$$

and $\|C(w(N)^{\perp,i}, w(N)^{\perp,j})\|_{m,\theta}$ converges to 0 for all $1 \leq i,j \leq d$ and (m,θ) with $m(1-\theta) > 2$.

Theorem 3.2. *Let us fix an even number m and a positive number θ with $m(1-\theta) > 2$. Let $\mathfrak{T}$ be the weakest topology such that $w(\in W^d) \to w(k/2^N)$ are continuous mappings for all k, N. The mappings $w(\in \Omega) \to C(w^i, w^j) \in W_{m,\theta}(\Delta \to \mathbb{R})$ and $w(\in \Omega) \to w \in W_{m,\theta/2}$ are ∞-quasi-continuous maps for all i,j with respect to the topology $\mathfrak{T}$.*

To prove these theorems, we recall the following lemmas.

Lemma 3.3. *Let $u \in \mathbb{D}^{s,q}(W^d)$ and $\tilde{u}$ be the (q,s)-quasi-continuous version of u. Then*

$$C_q^s\left(\{w \in W^d \mid |\tilde{u}(w)| > R\}\right) \leq R^{-1} M_{s,q} \|u\|_{s,q}.$$

We refer the proof of Lemma 3.3 to [26].

Lemma 3.4. *Let* $x, y \in W_{m,\theta/2}(\mathbb{R})$ *and set* $(\bar{x}\cdot\bar{y})_{s,t} = (x_t - x_s)(y_t - y_s)$ $(0 \le s \le t \le 1)$. *Then*

$$\|\bar{x}\cdot\bar{y}\|_{m,\theta} \le C_{m,\theta}\|x\|_{m,\theta/2}\|y\|_{m,\theta/2}.$$

Proof. Using the inequality $\|y\|_{\theta/2} \le C_{m,\theta}\|y\|_{m,\theta/2}$, we have

$$\begin{aligned}\|\bar{x}\cdot\bar{y}\|_{m,\theta}^m &= \int_0^1\int_0^t \frac{|(x_t-x_s)(y_t-y_s)|^m}{(t-s)^{2+m\theta}}dsdt \\ &\le \int_0^1\int_0^t \frac{|(x_t-x_s)|^m(C_{m,\theta}\|y\|_{m,\theta/2})^m}{(t-s)^{2+m\theta/2}}dsdt = C_{m,\theta}^m\|x\|_{m,\theta/2}^m\|y\|_{m,\theta/2}^m.\end{aligned}$$

□

Proof of Theorem 3.1 *and Theorem* 3.2 Let $z(N) = w(N) - w(N-1)$ $(N = 1, 2, \ldots)$, where $w(0) = 0$. Then $\{z(N); N = 1, 2, \ldots\}$ are piecewise linear function space valued independent random variables. Using explicit form of $z(N)$, we have

$$E[|w(N)_t - w(N)_s|^2] \le d|t-s| \tag{3.5}$$

$$E[|z(N)_t - z(N)_s|^2] \le C_d \min\left(|t-s|, 2^{-N}\right) \tag{3.6}$$

$$E[|w(N)_t^{\perp} - w(N)_s^{\perp}|^2] \le C_d' \min\left(|t-s|, 2^{-N}\right). \tag{3.7}$$

We estimate L^2-norm of $\|z(N)^i\|_{m,\theta/2}^m$.

$$\begin{aligned}&\left\|\|z(N)^i\|_{m,\theta/2}^m\right\|_{L^2(\mu)} \\ &= \left\{\int_{W^d} d\mu \iint_{(s,t)\in\Delta,(s',t')\in\Delta} \frac{(z(N)_t^i - z(N)_s^i)^m(z(N)_{t'}^i - z(N)_{s'}^i)^m}{|t-s|^{2+m\theta/2}|t'-s'|^{2+m\theta/2}}dsdt\, ds'dt'\right\}^{1/2} \\ &\le \iint_{(s,t)\in\Delta} \frac{E[(z(N)_t^i - z(N)_s^i)^{2m}]^{1/2}}{|t-s|^{2+m\theta/2}}dsdt \\ &= C_m \iint_{(s,t)\in\Delta} \frac{E[(z(N)_t^i - z(N)_s^i)^2]^{m/2}}{|t-s|^{2+m\theta/2}}dsdt \\ &= C_m' \iint_{(s,t)\in\Delta} \frac{\min(|t-s|, 2^{-N})^{m/2}}{|t-s|^{2+m\theta/2}}dsdt \\ &\le C \iint_{(s,t)\in\Delta} |t-s|^{\frac{m}{2}(1-\varepsilon-\theta)-2}2^{-\varepsilon mN/2}dsdt.\end{aligned}$$

Thus if $m(1-\theta) > 2$, choosing an appropriate $\varepsilon > 0$, there exists a positive number $C_{m,\theta,\varepsilon}$

$$\left\|\|z(N)^i\|_{m,\theta/2}^m\right\|_{L^2(\mu)} \le C_{m,\theta,\varepsilon}2^{-\varepsilon mN/2}. \tag{3.8}$$

By the calculation similar to the above, if $m(1-\theta) > 2$,

$$\left\|\|w(N)\|_{m,\theta/2}^m\right\|_{L^2(\mu)} \le C_{m,\theta} \tag{3.9}$$

$$\left\|\|w(N)^{\perp,i}\|_{m,\theta/2}^m\right\|_{L^2(\mu)} \le C_{m,\theta,\varepsilon}2^{-\varepsilon mN/2}. \tag{3.10}$$

Hence by Lemma 3.4,

$$\left\| \left\| \overline{w(N)^i} \cdot \overline{z(N+1)^j} \right\|_{m,\theta}^m \right\|_{L^2(\mu)} \le C_{m,\theta,\varepsilon} 2^{-\varepsilon m(N+1)/2},$$

where $\left(\overline{w(N)^i} \cdot \overline{z(N+1)^j}\right)_{s,t} = \left(w(N)^i_t - w(N)^i_s\right)\left(z(N+1)^j_t - z(N+1)^j_s\right)$. Similarly,

$$\left\| \left\| \overline{w(N)^{\perp,i}} \cdot \overline{w(N)^j} \right\|_{m,\theta} \right\|_{L^2(\mu)} \le C_{m,\theta,\varepsilon} 2^{-\varepsilon m(N+1)/2},$$

We estimate $C(z(N+1)^i, w(N)^j)_{s,t}$. Below we assume $i \ne j$. By the independence of $w(N)^{\perp,i}$ and $w(N)^j$,

$$\begin{aligned} E\left[C(z(N+1)^i, w(N)^j)^m_{s,t}\right] &= C_m E\left[\left(\int_s^t (z(N+1)^i_u - z(N+1)^i_s)^2 du\right)^{m/2}\right] \\ &\le C_m E\left[\int_s^t \left(z(N+1)^i_u - z(N+1)^i_s\right)^m du\right]\left(\int_s^t 1 du\right)^{(m-2)/2} \\ &\le C \min\left(|t-s|^m, 2^{-(N+1)m/2}\right). \end{aligned}$$

Using this,

$$\begin{aligned} \left\| \| C(z(N+1)^i, w(N)^j) \|^m_{m,\theta} \right\|_{L^2(\mu)} &\le \iint_{(s,t)\in\Delta} \frac{E\left[C(z(N+1)^i, w(N)^j)^{2m}_{s,t}\right]^{1/2}}{|t-s|^{2+m\theta}} dsdt \\ &\le 2^{-(N+1)m\varepsilon/2} \iint_{(s,t)\in\Delta} |t-s|^{m(1-\varepsilon-\theta)-2} dsdt. \end{aligned}$$

Hence if $m(1-\theta) > 1$, then we have

$$\left\| \| C(z(N+1)^i, w(N)^j) \|^m_{m,\theta} \right\|_{L^2(\mu)} \le C_{m,\theta,\varepsilon} 2^{-(N+1)m\varepsilon/2}.$$

Similarly if $m(1-\theta) > 1$,

$$\begin{aligned} \left\| \| C(w(N)^{\perp,i}, w(N)^j) \|^m_{m,\theta} \right\|_{L^2(\mu)} &\le C_{m,\theta,\varepsilon} 2^{-Nm\varepsilon/2}, \\ \left\| \| C(z(N)^i, z(N)^j) \|^m_{m,\theta} \right\|_{L^2(\mu)} &\le C_{m,\theta,\varepsilon} 2^{-Nm\varepsilon/2}. \end{aligned}$$

Let

$$\begin{aligned} A_{N,i} &= \left\{ w \;\middle|\; \|z(N+1)^i\|_{m,\theta/2} > N^{-2} \right\}, \\ B_{N,i,j} &= \left\{ w \;\middle|\; \|C(w(N+1)^i, w(N+1)^j) - C(w(N)^i, w(N)^j)\|_{m,\theta} > N^{-2} \right\}, \\ C_{N,i,j} &= \left\{ w \;\middle|\; \|\overline{w(N)^{\perp,i}} \cdot \overline{w(N)^j}\|_{m,\theta} > N^{-2} \right\} \\ D_{N,i,j} &= \left\{ w \;\middle|\; \|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta} > N^{-2} \right\}. \end{aligned}$$

Note that $\|z(N+1)^i\|_{m,\theta/2}^m$, $\|C(w(N+1)^i, w(N+1)^j) - C(w(N)^i, w(N)^j)\|_{m,\theta}^m$, $\|\overline{w(N)^{\perp,i}} \cdot \overline{w(N)^j}\|_{m,\theta}^m$, $\|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta}^m$, are Wiener chaos at most the degree $2m$. Hence by the hypercontractivity of the Ornstein-Uhlenbeck semi-group, their L^2-norms and the (q,s)-Sobolev norms are equivalent for any $q \geq 2, s > 0$. By Lemma 3.3, (3.8) and (3.10) and

$$\max\left(C_q^s(A_{N,i}), C_q^s(C_{N,i,j}), C_q^s(D_{N,i,j})\right) \leq M_{s,q,m,\theta,\varepsilon} N^{2m} 2^{-\varepsilon m N/2}. \tag{3.11}$$

Since

$$\begin{aligned} &C(w(N+1)^i, w(N+1)^j) - C(w(N)^i, w(N)^j) \\ &= \left(w(N)_t^i - w(N)_s^i\right)\left(z(N+1)_t^j - z(N+1)_s^j\right) - C(z(N+1)^j, w(N)^i)_{s,t} \\ &+ C(z(N+1)^i, w(N)^j)_{s,t} + C(z(N+1)^i, z(N+1)^j)_{s,t}, \end{aligned}$$

using the subadditivity of the capacity, we have

$$C_q^s(B_{N,i,j}) \leq M_{s,q,m,\theta,\varepsilon} N^{2m} 2^{-\varepsilon m N/2}. \tag{3.12}$$

Here we note that $A_{N,i}, B_{N,i,j}, C_{N,i,j}, D_{N,i,j}$ depend on (m,θ) satisfying $m(1-\theta) > 2$. Let

$$E = \cup_{i,j,m,\theta\in\mathbb{Q}} \left\{ (\limsup_{N\to\infty} A_{N,i}) \cup (\limsup_{N\to\infty} B_{N,i,j}) \cup (\limsup_{N\to\infty} C_{N,i,j}) \cup (\limsup_{N\to\infty} D_{N,i,j}) \right\}.$$

By (3.11) and (3.12), E is a slim set. Moreover $w(N)$, $C(w(N), w(N))$ converges uniformly with respect to $\| \ \|_{m,\theta/2}$ and $\| \ \|_{m,\theta}$ and $C(w(N)^\perp, w(N))$, $C(w(N), w(N)^\perp)$ converges to 0 with respect to $\| \ \|_{m,\theta}$ on $\cap_{i,j,m,\theta\in\mathbb{Q}} \left\{\cap_{N=K}^\infty (A_{N,i}^c \cap B_{N,i,j}^c \cap C_{N,i,j}^c \cap D_{N,i,j}^c)\right\}$ for all m, θ with $m(1-\theta) > 2$. Therefore It suffices to set $\Omega = E^c$. □

We note that $C(w^i, \eta^j)$ is meaningless even if $w = (w^i), \eta = (\eta^j) \in \Omega$ generally. Also let $\Omega_N = \{w(N) \mid w \in \Omega\}$ and $\Omega_N^\perp = \{w - w(N) \mid w \in \Omega\}$. Note that Ω_N is the same as the set of all piecewise linear continuous path w such that $t \to w_t$ $(\frac{k}{2^N} \leq t \leq \frac{k+1}{2^N}, 0 \leq k \leq 2^N - 1)$ is a linear function and this space is isomorphic to $\mathbb{R}^{2^N d}$. Let μ_N and $\mu_N^\perp$ be the law of $w(N)$ and $w(N)^\perp$. Clearly, they are independent. We use the following bound.

Lemma 3.5. *Let $w \in W_{m,\theta/2}(\mathbb{R})$ and $\varphi \in H$. Suppose that $m(1-\theta) > 2$. Then*

$$\begin{aligned} \|\varphi\|_{m,\theta/2} &\leq \|\varphi\|_H, &(3.13)\\ \|C(w,\varphi)\|_{m,\theta} &\leq \|w\|_{m,\theta/2}\|\varphi\|_H, &(3.14)\\ \|C(\varphi,w)\|_{m,\theta} &\leq 2\|w\|_{m,\theta/2}\|\varphi\|_H, &(3.15)\\ \|D\|C(w,\varphi)\|_{m,\theta}^m\|_H &\leq 2m\|C(w,\varphi)\|_{m,\theta}^{m-1}\|w\|_{m,\theta/2}\|\varphi\|_{m,\theta/2} &(3.16)\\ \|D\|C(\varphi,w)\|_{m,\theta}^m\|_H &\leq \|C(\varphi,w)\|_{m,\theta}^{m-1}\|w\|_{m,\theta/2}\|\varphi\|_{m,\theta/2}. &(3.17) \end{aligned}$$

Proof. (3.13) follows from

$$\frac{|\varphi_t - \varphi_s|^m}{(t-s)^{m\theta/2}} = \frac{|(\dot\varphi, 1_{[s,t]})_{L^2}|^m}{(t-s)^{m\theta/2}} \leq \|\varphi\|_H^m. \tag{3.18}$$

We prove (3.14). Using the Hölder inequality, we have

$$\begin{aligned}
&\frac{\left|\int_s^t (w(u)-w(s))\dot{\varphi}(u)du\right|^m}{(t-s)^{2+m\theta}} \\
&\quad \le \frac{1}{(t-s)^{m\theta/2}}\left(\int_s^t \frac{|w(u)-w(s)|}{|u-s|^{(2+m\theta/2)/m}}|\dot{\varphi}(u)|du\right)^m \\
&\quad \le \int_s^t \frac{|w(u)-w(s)|^m}{|u-s|^{2+m\theta/2}}du \frac{1}{(t-s)^{m\theta/2}}\left(\int_s^t |\dot{\varphi}(u)|^{m/(m-1)}du\right)^{m-1},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(t-s)^{m\theta/2}}\left(\int_s^t |\dot{\varphi}(u)|^{m/(m-1)}du\right)^{m-1} &\le \frac{1}{(t-s)^{m\theta/2}}\left(\int_s^t |\dot{\varphi}(u)|^2 du\right)^{m/2}(t-s)^{\frac{m-2}{2}} \\
&\le (t-s)^{\frac{(m-2)-m\theta}{2}}\|\varphi\|_H^m \\
&\le \|\varphi\|_H^m.
\end{aligned}$$

Hence

$$\begin{aligned}
\|C(w,\varphi)\|_{m,\theta}^m &\le \int_0^1\int_0^t\left(\int_s^t \frac{|w(u)-w(s)|^m}{|u-s|^{2+m\theta/2}}du\right)dsdt\|\varphi\|_H^m \\
&= \int_0^1\int_0^t\left(\int_0^u \frac{|\varphi(u)-\varphi(s)|^m}{|u-s|^{2+m\theta/2}}ds\right)dudt\|\varphi\|_H^m \\
&= \|w\|_{m,\theta/2}^m\|\varphi\|_H^m.
\end{aligned}$$

We prove (3.15). Noting that for 1-dimensional paths x, y,

$$C(x,y)_{s,t} = (x_t - x_s)(y_t - y_s) - C(y,x)_{s,t}, \tag{3.19}$$

we have

$$\begin{aligned}
\|C(\varphi,w)\|_{m,\theta} &\le \|C(w,\varphi)\|_{m,\theta} + \left(\int_0^1\int_0^t \frac{|(w_t-w_s)(\varphi_t-\varphi_s)|^m}{(t-s)^{2+m\theta}}dsdt\right)^{1/m} \\
&\le \|C(w,\varphi)\|_{m,\theta} + \|w\|_{m,\theta/2}\|h\|_H,
\end{aligned}$$

where we have used (3.18). This and (3.14) prove (3.15). We consider (3.16). Let $h \in H$. We have

$$(D_w)_h\left(\int_s^t (w_u - w_s)d\varphi_u\right) = (\varphi_t-\varphi_s)(h_t-h_s) - \int_s^t (\varphi_u-\varphi_s)\dot{h}_u du.$$

Therefore

$$\begin{aligned}
&(D_w)_h\left(\|C(w,\varphi)\|_{m,\theta}^m\right) \\
&\quad = m\int_0^1\int_0^t \frac{\left((\varphi_t-\varphi_s)(h_t-h_s) - C(\varphi,h)_{s,t}\right)C(w,\varphi)_{s,t}^{m-1}}{(t-s)^{2+m\theta}}dsdt.
\end{aligned}$$

Using the Hölder inequality, (3.14) and (3.18), we get

$$\begin{aligned}
(D_w)_h(\|C(w,\varphi)\|_{m,\theta}^m) &\le m\left(\|\varphi\|_{m,\theta/2}\|h\|_H + \|C(\varphi,h)\|_{m,\theta}\right)\|C(w,\varphi)\|_{m,\theta}^{m-1} \\
&\le 2m\|\varphi\|_{m,\theta/2}\|h\|_H\|C(w,\varphi)\|_{m,\theta}^{m-1}
\end{aligned}$$

which proves (3.16). As for (3.17), noting that

$$(D_w)_h\left(\int_s^t(\varphi_u-\varphi_s)dw_u\right)=\int_s^t(\varphi_u-\varphi_s)\dot{h}_u du,$$

we can prove (3.17) similarly to (3.16). ☐

Lemma 3.6. *Suppose that $m(1-\theta)>2$. Let $(x,y)=(w(N)^i,w(N)^j),(w^i,w^j)$ for $i\neq j$ or $(x,y)=(w(N)^i,w(N)^{\perp,j}),(w(N)^{\perp,i},w(N)^j)$ for any i,j. Then the following estimates hold for almost all w.*

$$\|D\|x\|_{m,\theta/2}^m\|_H \leq m\|x\|_{m,\theta/2}^{m-1} \tag{3.20}$$

$$\|D\|C(x,y)\|_{m,\theta}^m\|_H \leq m\left(\|x\|_{m,\theta/2}^2+4\|y\|_{m,\theta/2}^2\right)^{1/2}\|C(x,y)\|_{m,\theta}^{m-1}. \tag{3.21}$$

Proof. We consider the case where $x=w(N)^i$ in (3.20). The proof of other cases are similar to it. We have

$$\begin{aligned}|D_h\|x\|_{m,\theta/2}^m| &= \left|m\int_0^1\int_0^t\frac{(h(N)_t^i-h(N)_s^i)(w(N)_t^i-w(N)_s^i)^{m-1}}{(t-s)^{2+m\theta/2}}dsdt\right|\\ &= m\|h(N)^i\|_{m,\theta/2}\|w(N)^i\|_{m,\theta/2}^{m-1}\\ &\leq m\|h\|_H\|w(N)^i\|_{m,\theta/2}^{m-1}\end{aligned}$$

which implies (3.20). We prove (3.21). Let $(x,y)=(w(N)^i,w(N)^j)$ $(i\neq j)$. Then

$$\begin{aligned}&|D_h\|C(x,y)\|_{m,\theta}^m|\\ &=m\left|\int_0^1\int_0^t\frac{\left(C(h(N)^i,w(N)^j)_{s,t}+C(w(N)^i,h(N)^j)_{s,t}\right)\left(C(x,y)_{s,t}^{m-1}\right)}{(t-s)^{2+m\theta}}dsdt\right|\\ &\leq m\left(\|C(h(N)^i,w(N)^j)\|_{m,\theta}+\|C(w(N)^i,h(N)^j)\|_{m,\theta}\right)\|C(x,y)\|_{m,\theta}^{m-1}\\ &\leq m\left(\|w(N)^i\|_{m,\theta/2}\|h(N)^j\|_H+2\|w(N)^j\|_{m,\theta/2}\|h(N)^i\|_H\right)\|C(x,y)\|_{m,\theta}^{m-1}.\end{aligned}$$

This implies (3.21). We can check the other cases in similar ways. ☐

Lemma 3.7. *Let $x\in W_{m,\theta/2}(\mathbb{R})$ and w be a 1-dimensional Brownian motion. Let $C(x,w)_{s,t}=\int_s^t(x_u-x_s)dw_u$, $C(w,x)_{s,t}=\int_s^t(w_u-w_s)dx_u$ be the Wiener integrals. We fix a continuous versions of them. Then we have*

$$E\left[\|C(x,w)\|_{m,\theta}^m+\|C(w,x)\|_{m,\theta}^m\right] \leq C_m\|x\|_{m,\theta/2}^m, \tag{3.22}$$

where C_m is a constant which depends on m only. Also we have

$$\lim_{N\to\infty}E\left[\|C(x,w(N))-C(x,w)\|_{m,\theta}^m+\|C(w(N),x)-C(w,x)\|_{m,\theta}^m\right]=0. \tag{3.23}$$

Proof. We have

$$\begin{aligned}
E\left[\int_0^1\int_0^t \frac{\left(\int_s^t (x_u-x_s)dw_u\right)^m}{|t-s|^{2+m\theta'}}dsdt\right] &= C_m\int_0^1\int_0^t \frac{\left(\int_s^t (x_u-x_s)^2du\right)^{m/2}}{(t-s)^{2+m\theta'}}dsdt\\
&\le C_m\int_0^1\int_0^t \frac{(t-s)^{\frac{m}{2}-1}\int_s^t (x_u-x_s)^m du}{(t-s)^{2+m\theta'}}dsdt\\
&\le C_m\int_0^1\int_0^t \frac{\int_s^t (x_u-x_s)^m du}{(t-s)^{2+m\theta/2}}dsdt\\
&\le C_m\|x\|_{m,\theta/2}^m.
\end{aligned}$$

Noting that $C(w,x)_{s,t}=(w_t-w_s)(x_t-x_s)-C(x,w)_{s,t}$ and

$$E\left[\int_0^1\int_0^t \frac{(w_t-w_s)^m}{(t-s)^{2+m\theta/2}}dsdt\right]<\infty$$

we complete the proof of (3.22). We prove (3.23). We have

$$\left\|\|C(x,w(N)^{\perp})\|_{m,\theta}^m\right\|_{L^2(\mu)} \le \iint_{\{(s,t)\in\Delta\}}\frac{E\left[C(x,w(N)^{\perp})_{s,t}^{2m}\right]^{1/2}}{(t-s)^{2+m\theta}}dsdt. \tag{3.24}$$

Note that

$$E\left[C(x,w(N)^{\perp})_{s,t}^{2m}\right]^{1/2} \le C_m\psi_N(s,t)$$

where $\psi_N(s,t)=E\left[C(x,w(N))_{s,t}^2\right]^{m/2}$. Also

$$\psi_N(s,t)\le E\left[C(x,w)_{s,t}^2\right]^{m/2}=:\psi(s,t).$$

This follows from that $w=w(N)+w(N)^{\perp}$ and $w(N)$ and $w(N)^{\perp}$ are independent. It holds that $\lim_{N\to\infty}\psi_N(s,t)=0$ for all (s,t) and $\iint_\Delta \frac{\psi(s,t)}{(t-s)^{2+m\theta}}dsdt<\infty$. Hence the Lebegsue dominated convergence theorem implies that the quantity on the right-hand side of (3.24) converges to 0. For the term containing $C(w(N)^{\perp},x)$, it suffices to note the relation $C(w(N)^{\perp},x)=C(x,w(N)^{\perp})-\bar{x}\cdot w(\bar{N})^{\perp}$ and (3.7). □

The reader may find the following statement in Remark 3.2 in [5]. We apply this lemma to Dirichlet forms on open subsets in euclidean spaces. For the sake of completeness, we give the proof.

Lemma 3.8. *Let (X,μ) and (Y,ν) be probability spaces. Let $dm=d\mu\otimes d\nu$. Assume that we are given Dirichlet forms $(\mathcal{E}_X,\mathrm{D}(\mathcal{E}_X))$, $(\mathcal{E}_Y,\mathrm{D}(\mathcal{E}_Y))$ on $L^2(X,\mu)$ and $L^2(Y,\nu)$. Moreover we assume that $\mathcal{E}_X$, $\mathcal{E}_Y$ has the square field operators Γ_X and Γ_Y respectively. Let U be a measurable subset of $X\times Y$ with $m(U)>0$. Let $U_x=\{y\in Y\mid (x,y)\in U\}$ and $U^y=\{x\in X\mid (x,y)\in U\}$. Let $A=\{x\in X\mid \nu(U_x)>0\}$ and $B=\{y\in Y\mid \mu(U^y)>0\}$. Let $d\nu_x=d\nu|_{U_x}/\nu(U_x)$ for $x\in A$ and $d\mu_y=d\mu|_{U^y}/\mu(U^y)$ for $y\in B$. We assume that*
(1) There exists $\tilde{A}\subset A$ such that $\mu(A\setminus\tilde{A})=0$ and $\delta=\inf_{x,x'\in\tilde{A}}\nu\left(U_x\cap U_{x'}\right)>0$. Moreover there exists a positive number C_2 such that for any $x\in\tilde{A}$ and $g\in\mathrm{D}(\mathcal{E}_Y)$,

$$\mathrm{Var}_{\nu_x}(g)\le C_2\int_{U_x}\Gamma_Y g(y)d\nu_x(y). \tag{3.25}$$

Here $\mathrm{Var}_{\nu_x}(g)$ *denotes the variance of* g *with respect to the probability measure* ν_x*.*
(2) *There exists* $\tilde{B} \subset B$ *such that* $\nu(B \setminus \tilde{B}) = 0$ *and there exists a positive number* C_1 *such that for any* $y \in \tilde{B}$ *and* $h \in \mathrm{D}(\mathcal{E}_X)$

$$\mathrm{Var}_{\mu_y}(h) \leq C_1 \int_{U^y} \Gamma_X h(x) d\mu_y(x). \tag{3.26}$$

Let us denote $z = (x, y) \in X \times Y$ *and* $dm_U = dm|_U/m(U)$*. Then we have for* $f = f(z) = f(x, y)$*,*

$$\mathrm{Var}_{m_U}(f) \leq \frac{3}{\delta} \int_U \left(\frac{C_1}{m(U)} \Gamma_X f(x, y) + C_2 \Gamma_Y f(x, y) \right) dm_U(z). \tag{3.27}$$

Proof. Let $x, x' \in \tilde{A}$, $y \in U_x, y' \in U_{x'}$, $z \in U_x \cap U_{x'}$. Noting that

$$\begin{aligned}&\big(f(x,y) - f(x',y')\big)^2 \\ &\quad \leq 3\Big\{(f(x,y) - f(x,z))^2 + (f(x,z) - f(x',z))^2 + (f(x',z) - f(x',y'))^2\Big\},\end{aligned} \tag{3.28}$$

and $\nu\left(U_x \cap U_{x'}\right) > \delta$, we have

$$\begin{aligned}\big(f(x,y) - f(x',y')\big)^2 &\leq \frac{3}{\delta} \int_{U_x \cap U_{x'}} (f(x,y) - f(x,z))^2 d\nu(z) \\ &\quad + \frac{3}{\delta} \int_{U_x \cap U_{x'}} \big(f(x,z) - f(x',z)\big)^2 d\nu(z) \\ &\quad + \frac{3}{\delta} \int_{U_x \cap U_{x'}} \big(f(x',z) - f(x',y')\big)^2 d\nu(z) \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{3.29}$$

We estimate I_i.

$$\begin{aligned}&\int_{x,x' \in \tilde{A}, y \in U_x, y' \in U_{x'}} I_1 d\mu(x) d\mu(x') d\nu(y) d\nu(y') \\ &\quad \leq \frac{3}{\delta} \int_{x \in \tilde{A}, y, z \in U_x} (f(x,y) - f(x,z))^2 d\nu(y) d\nu(z) d\mu(x) m(U) \\ &\quad \leq \frac{3C_2 m(U)}{\delta} \int_{x \in \tilde{A}, y \in U_x} 2\nu(U_x) \Gamma_Y f(x,y) d\nu(y) d\mu(x).\end{aligned} \tag{3.30}$$

$$\begin{aligned}&\int_{x,x' \in \tilde{A}, y \in U_x, y' \in U_{x'}} I_2 d\mu(x) d\mu(x') d\nu(y) d\nu(y') \\ &\quad = \frac{3}{\delta} \int_{x,x' \in \tilde{A}} \left(\nu(U_x) \nu(U_{x'}) \int_{z \in U_x \cap U_{x'}} \big(f(x,z) - f(x',z)\big)^2 d\nu(z) \right) d\mu(x) d\mu(x') \\ &\quad \leq \frac{3}{\delta} \int_{x,x' \in \tilde{A} \cap U^z, z \in Y} \Big\{ \big(f(x,z) - f(x',z)\big)^2 d\mu(x) d\mu(x') \Big\} d\nu(z) \\ &\quad = \frac{3}{\delta} \int_{x,x' \in U^z, z \in \tilde{B}} \Big\{ \big(f(x,z) - f(x',z)\big)^2 d\mu(x) d\mu(x') \Big\} d\nu(z) \\ &\quad \leq \frac{3}{\delta} \int_{\tilde{B}} d\mu(z) 2C_1 \mu(U^z) \int_{U^z} \Gamma_X f(x,z) d\mu(x).\end{aligned} \tag{3.31}$$

As to I_3, we have the same estimate for I_1:

$$\int_{x,x'\in\tilde{A},y\in U_x,y'\in U_{x'}} I_1 d\mu(x)d\mu(x')d\nu(y)d\nu(y')$$
$$\leq \frac{3C_2 m(U)}{\delta}\int_{x\in\tilde{A},y\in U_x} 2\nu(U_x)\Gamma_Y f(x,y)d\nu(y)d\mu(x). \tag{3.32}$$

Since

$$\int_{x,x'\in\tilde{A},y\in U_x,y'\in U_{x'}} (f(x,y)-f(x',y'))^2 d\mu(x)d\mu(x')d\nu(y)d\nu(y')$$
$$= 2m(U)^2\int_U\left(f(z)-\int_U f(z)dm_U(z)\right)^2 dm_U(z), \tag{3.33}$$

the above estimates complete the proof. □

Let U be a bounded open subset of $\mathbb{R}^{n+m}$. We denote $z=(x,y)\in\mathbb{R}^n$, where $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$. Let A be the image of the projection of U with respect to the first variable x. Clearly, A is also an open subset. For $x\in A$, set $U_x=\{y\in\mathbb{R}^m \mid (x,y)\in U\}$. U_x is also an open subset. Using the notation above, we prepare the following. The proof of this result is easy and we omit it.

Lemma 3.9. *Suppose that U_x is a convex set and contains 0. Let α be a C^∞ 1-form on U with $\sup_{z\in U}|\alpha(z)|<\infty$. We denote*

$$\alpha(z)=\sum_{i=1}^n \beta_i(x,y)dx^i+\sum_{j=1}^m \gamma_j(x,y)dy^j. \tag{3.34}$$

Let $\pi: U\to A$ be the projection and $s: A\to U$ be the map such that $s(x)=(x,0)\in U$ for $x\in A$. Let

$$(K\alpha)(z)=\int_0^1\sum_{j=1}^m \gamma_j(x,ty)y^j dt. \tag{3.35}$$

If $d\alpha=0$ on U, then it holds that $s^*\alpha$ is a closed form on A and

$$\alpha=\pi^*s^*\alpha+dK\alpha. \tag{3.36}$$

Lemma 3.10. *Take θ' such that $0<\theta<\theta'<1$. Let $z^1,\ldots,z^l\in W_{m,\theta/2}(\mathbb{R})$ and define*

$$U_N(z^1,\ldots,z^l;\varepsilon)$$
$$=\left\{w\in W_{m,\theta'/2}(\mathbb{R})\;\middle|\;\max_{1\leq i\leq l}\{\|w(N)\|_{m,\theta'/2},\|C(w(N),z^i)\|_{m,\theta},\|C(z^i,w(N))\|_{m,\theta}\}<\varepsilon\right\},$$

where ε is a positive number. Then for fixed l, $r>0$ and $\varepsilon>0$, we have

$$C(l,\varepsilon,r,m,\theta,\theta')=\inf\left\{\mu\left(U_N(z^1,\ldots,z^l;\varepsilon)\right)\;\middle|\;\max_{1\leq i\leq l}\|z^i\|_{m,\theta'/2}\leq r, N\in\mathbb{N}\right\}>0. \tag{3.37}$$

Proof. First we prove that for any N,

$$\varepsilon_N = \inf\left\{\mu\left(U_N(z^1,\ldots,z^l;\varepsilon)\right) \;\middle|\; \max_{1\le i\le l}\|z^i\|_{m,\theta'/2}\le r\right\} > 0. \tag{3.38}$$

Note that for any $z^1,\ldots,z^l \in W_{m,\theta/2}(\mathbb{R})$,

$$\mu\left(U_N(z^1,\ldots,z^l;\varepsilon)\right) > 0. \tag{3.39}$$

If (3.38) does not hold, then we can find a sequence $\{z^{i,n}\}$ such that $\sup_{i,n}\|z^{i,n}\|_{m,\theta'/2}\le r$ $\lim_{n\to\infty}\mu(U_N(z^{1,n},\ldots,z^{l,n};\varepsilon)) = 0$. Since the embedding $W_{m,\theta'/2}(\mathbb{R})\subset W_{m,\theta/2}(\mathbb{R})$ is compact, there exists a subsequence $\{z^{i,n(k)}\}$ and $\{y^i\}\subset W_{m,\theta/2}(\mathbb{R})$ such that $\lim_{k\to\infty}\|z^{i,n(k)}-y^i\|_{m,\theta/2} = 0$. By Lemma 3.7 and $E[\|C(x,w(N))\|_{m,\theta}^m]\le E[\|C(x,w)\|_{m,\theta}^m]$ and so on,

$$\lim_{k\to\infty} E\left[\|C(w(N),z^{i,n(k)}) - C(w(N),y^i)\|_{m,\theta} + \|C(z^{i,n(k)},w(N)) - C(y^i,w(N))\|_{m,\theta}\right] = 0.$$

This implies that $\mu\left(U_N(y^1,\ldots,y^l;\varepsilon/2)\right) = 0$. This is a contradiction. Next we prove that $\liminf_{N\to\infty}\varepsilon_N > 0$. We denote

$$U(z^1,\ldots,z^l;\varepsilon) = \left\{w\in W_{m,\theta'/2}(\mathbb{R}) \;\middle|\; \max_{1\le i\le l}\left\{\|w\|_{m,\theta'/2}, \|C(w,z^i)\|_{m,\theta}, \|C(z^i,w)\|_{m,\theta}\right\} < \varepsilon\right\}. \tag{3.40}$$

Then $(w, C(w,z^i), C(z^i,w))$ defines a Gaussian measure with mean 0 on the separable Banach space

$$W_{m,\theta'/2}(\mathbb{R}) \times \prod_{i=1}^{2l} W_{m,\theta}(\Delta\to\mathbb{R}).$$

Therefore the probability of the any ball centered at 0 with a positive radius have positive measure. Thus we obtain for any $\varepsilon > 0$ and $\{z^i\}_{i=1}^l \subset W_{m,\theta/2}(\mathbb{R})$,

$$\mu(U(z^1,\ldots,z^l;\varepsilon)) > 0. \tag{3.41}$$

Now suppose that there exist $\{z^{i,N}\}\subset W_{m,\theta'/2}(\mathbb{R})$ such that their norms are less than r and

$$\lim_{N\to\infty}\mu\left(U_N(z^{1,N},\ldots,z^{l,N};\varepsilon)\right) = 0.$$

We may assume that there exists $y^i\in W_{m,\theta/2}(\mathbb{R})$ such that $\lim_{N\to\infty}\|z^{i,N}-y^i\|_{m,\theta/2} = 0$. We have

$$C(w(N),z^{i,N}) \;=\; C(w(N),z^{i,N}-y^i) + C(w,y^i) - C(w(N)^{\perp},y^i). \tag{3.42}$$

By (3.41), the probability of any small ball of the law of $C(w,y^i)$ is positive. Also the $\|\;\|_{m,\theta/2}$ norms of $C(w(N),z^{i,N}-y^i)$ and $C(w(N)^{\perp},y^i)$ converge to 0 in probability by Lemma 3.7. This is a contradiction and we have proved that $\inf_N\varepsilon_N > 0$. □

Lemma 3.11. *Let w be the 1-dimensional Brownian motion. Let $\eta \in W_{m,\theta'/2}(\mathbb{R})$, $z^1,\ldots,z^{2l} \in W_{m,\theta'/2}(\mathbb{R})$ and $\xi^1,\ldots\xi^{2l} \in W_{m,\theta}(\Delta \to \mathbb{R})$. Let r be a positive number and $0<\delta<1$. Suppose that $\|\eta\|_{m,\theta'/2} < \delta r$ and $\max_{1\le i\le 2l}\|\xi^i\|_{m,\theta} < \delta r$. Let us consider the bounded open subset of Ω_N*

$$
\begin{aligned}
&U_N(\{z^i\}_{i=1}^{2l},\{\xi^i\}_{i=1}^{2l},\eta)\\
&\quad=\Big\{w(N)\ \Big|\ \|w(N)+\eta\|_{m,\theta'/2}<r,\ \max_{1\le i\le l}\|C(w(N),z^i)+\xi^i\|_{m,\theta}<r,\\
&\qquad \max_{1\le i\le l}\|C(z^{i+l},w(N))+\xi^{i+l}\|<r\Big\}. \qquad (3.43)
\end{aligned}
$$

(1) *It holds that for any $C>0$*

$$
\inf\Big\{\mu(U_N(\{z^i\}_{i=1}^{2l},\{\xi^i\}_{i=1}^{2l},\eta))\ \Big|\ \max_{1\le i\le 2l}\|z^i\|_{m,\theta'/2}\le C\Big\}>0. \qquad (3.44)
$$

(2) *Let $W^1(U_N(\{z^i\},\{\xi^i\},\eta),\mu_N)$ be the Sobolev spaces which consists of L^2-functions with respect to μ_N on $U_N(\{z^i\},\{\xi^i\},\eta)$ whose weak derivatives are in $L^2(\mu_N)$. This set coincides with $W^1(U_N(\{z^i\},\{\xi^i\},\eta))$ which is usual Sobolev spaces whose derivatives are in L^2 with respect to the Lebesgue measure. Moreover there exists a bounded linear operator* (*extension operator*) *$T : W^1(U_N(\{z^i\},\{\xi^i\},\eta),\mu_N) \to W^1(\Omega_N,\mu_N)$ such that $Tf|_{U_N(\{z^i\},\{\xi^i\},\eta)} = f$.*
(3) *It holds that for any $f \in W^1(U_N(\{z^i\},\{\xi^i\},\eta),\mu_N)$,*

$$
\mathrm{Var}_{\mu_{N,U_N(\{z^i\},\{\xi^i\},\eta)}}(f) \ \le\ \int_{U_N(\{z^i\},\{\xi^i\},\eta)} |Df(w(N))|_H^2 d\mu_{N,U_N(\{z^i\},\{\xi^i\},\eta)}(w(N)). \quad (3.45)
$$

where $\mu_{N,U_N(\{z^i\},\{\xi^i\},\eta)}$ is the normalized probability measure of μ_N on $U_N(\{z^i\},\{\xi^i\},\eta)$.

Proof. (1) follows from Lemma 3.10. (2) follows from that $U_N(\{z^i\},\{\xi^i\},\eta)$ is a bounded convex domain of Ω_N. See [12] for more general results. (3) also follows from the convexity of the domain. □

Let $\varphi=\varphi_t=(\varphi_t^1,\ldots,\varphi_t^d)$ $(0\le t\le 1)$ be an element of H. From now on, we fix m,θ,θ' such that $m(1-\theta')>2$ and $2/3<\theta<\theta'<1$. Under this assumption, if $\|w\|_{m,\theta/2}<\infty$ and $\|C(w,w)\|_{m,\theta}<\infty$, then $2/\theta$-variation norm of w and $1/\theta$-variation norm of $C(w,w)$ is finite. Let us define

$$
\begin{aligned}
&U_{r,\varphi}\\
&\quad=\Big\{w\in\Omega\ \Big|\ \max_{1\le i\le d}\|w^i\|_{m,\theta'/2}<r,\ \max_{1\le j<k\le d}\|C(w^j,w^k)\|_{m,\theta}<r,\ \max_{1\le i\le j\le d}\|C(\varphi^i,w^j)\|_{m,\theta}<r,\\
&\qquad \sup_{1\le i\le j\le d}\|C(w^i,\varphi^j)\|_{m,\theta}<r\Big\}, \qquad (3.46)
\end{aligned}
$$

and

$$
\begin{aligned}
&U_r(\varphi)\\
&\quad=\Big\{w\in\Omega\ \Big|\ \max_{1\le i\le d}\|w^i-\varphi^i\|_{m,\theta'/2}<r,\ \max_{1\le j<k\le d}\|C(w^j-\varphi^j,w^k-\varphi^k)\|_{m,\theta}<r,\\
&\qquad \max_{1\le i\le j\le d}\|C(\varphi^i,w^j-\varphi^j)\|_{m,\theta}<r,\ \max_{1\le i\le j\le d}\|C(w^i-\varphi^i,\varphi^j)\|_{m,\theta}<r\Big\}. \qquad (3.47)
\end{aligned}
$$

These sets are neighborhoods of 0 and φ respectively and play the similar kind of role to the balls in normed linear spaces. Note that we have the following relation:

$$U_r(\varphi) = \{w + \varphi \mid w \in U_{r,\varphi}\}\,. \tag{3.48}$$

The strict positivity of the measures of these subsets for any $r > 0$ and $\varphi \in H$ can be proved by the argument similar to [5].

For $w, \eta \in \Omega$, let

$$d_\Omega(w,\eta) = \max\left\{\max_{i,j}\|C(w^i,w^j) - C(\eta^i,\eta^j)\|_{m,\theta}, \max_i \|w^i - \eta^i\|_{m,\theta'/2}\right\}. \tag{3.49}$$

$$V_r(\eta) = \{w \in \Omega \mid d_\Omega(w,\eta) < r\}\,. \tag{3.50}$$

Lemma 3.12. *Let $r > 0$.*
(1) *Let $\varphi_1, \varphi_2 \in H$. Let $0 < \delta < 1$. If*

$$\max_i \|\varphi_1^i - \varphi_2^i\|_H \le \frac{\delta r}{3r + C_{m,\theta,\theta'} + 2\max_i \|\varphi_1^i\|_{m,\theta/2}} \tag{3.51}$$

then it holds that $U_r(\varphi_1) \subset U_{(1+\delta)r}(\varphi_2)$.

If the stronger assumption

$$\max_i \|\varphi_1^i - \varphi_2^i\|_H \le \frac{\delta r}{6r + C_{m,\theta,\theta'} + 2\max_i \left(\|\varphi_1^i\|_{m,\theta/2}, \|\varphi_2^i\|_{m,\theta/2}\right)} \tag{3.52}$$

holds then we have

$$U_r(\varphi_1) \subset U_{(1+\delta)r}(\varphi_2) \subset U_{(1+\delta)^2 r}(\varphi_1).$$

(2) *Let $0 < r < 1$ and $\varphi \in H$. It holds that*

$$U_r(\varphi) \subset V_{\{C_{m,\theta}(1+\|\varphi\|_{m,\theta/2})+1\}r}(\varphi). \tag{3.53}$$

Proof. (1) Let $\varepsilon = \max_i \|\varphi_1^i - \varphi_2^i\|_H$. We assume $\varepsilon < 1$. Let $w \in U_r(\varphi_1)$. Then we have

$$\|w^i - \varphi_2^i\|_{m,\theta'/2} \le \|w^i - \varphi_1^i\|_{m,\theta'/2} + \|\varphi_1^i - \varphi_2^i\|_{m,\theta'/2} < r + C_{m,\theta'}\varepsilon, \tag{3.54}$$

$$\begin{aligned}
&\|C(w^j - \varphi_2^j, w^k - \varphi_2^k)\|_{m,\theta} \\
&\quad= \Big\| C(w^j - \varphi_1^j, w^k - \varphi_1^k) + C(\varphi_1^j - \varphi_2^j, w^k - \varphi_1^k) + C(w^j - \varphi_1^j, \varphi_1^k - \varphi_2^k) \\
&\qquad\quad + C(\varphi_1^j - \varphi_2^j, \varphi_1^k - \varphi_2^k)\Big\|_{m,\theta} \\
&\quad< r + 3\varepsilon r + C_{m,\theta}\varepsilon^2,
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
\|C(\varphi_2^i, w^j - \varphi_2^j)\|_{m,\theta} &= \Big\| C(\varphi_1^i, w^j - \varphi_1^j) + C(\varphi_1^i, \varphi_1^j - \varphi_2^j) + C(\varphi_2^i - \varphi_1^i, w^j - \varphi_1^j) \\
&\qquad + C(\varphi_2^i - \varphi_1^i, \varphi_1^j - \varphi_2^j)\Big\|_{m,\theta} \\
&< r + \|\varphi_i^1\|_{m,\theta/2}\|\varphi_1^j - \varphi_2^j\|_H + 2\|\varphi_2^i - \varphi_1^i\|_H \|w^j - \varphi_1^j\|_{m,\theta/2} \\
&\qquad + C_{m,\theta}\|\varphi_1^i - \varphi_2^i\|_H \|\varphi_1^j - \varphi_2^j\|_H \\
&< r + 2\varepsilon r + C_{m,\theta}\varepsilon^2 + \varepsilon\|\varphi_1^i\|_{m,\theta/2}.
\end{aligned} \tag{3.56}$$

Similarly,

$$\|C(w^i - \varphi_2^i, \varphi_2^j)\|_{m,\theta} \quad < \quad r + \varepsilon r + 2\varepsilon \|\varphi_1^i\|_{m,\theta/2} + C_{m,\theta}\varepsilon^2. \tag{3.57}$$

Therefore if

$$\varepsilon \left(3r + C_{m,\theta'} + C_{m,\theta} + 2 \max_i \|\varphi_1^i\|_{m,\theta/2} \right) \le \delta r,$$

then $w \in U_{r(1+\delta)}(\varphi_2)$. This completes the proof.
(2) Assume $w \in U_r(\varphi)$. Let $i < j$. Since $C(w^i, w^j) - C(\varphi^i, \varphi^j) = C(w^i - \varphi^i, w^j - \varphi^j) + C(\varphi^i, w^j - \varphi^j) + C(w^i - \varphi^i, \varphi^j)$, we have

$$\|C(w^i, w^j) - C(\varphi^i, \varphi^j)\|_{m,\theta} < 3r.$$

Let us consider the case where $i = j$. Since

$$\begin{aligned} & C(w^i, w^i)_{s,t} - C(\varphi^i, \varphi^i)_{s,t} \\ & \quad = \frac{1}{2} \left\{ (w^i - \varphi^i)_t - (w^i - \varphi^i)_s \right\}^2 + C(\varphi^i, w^i - \varphi^i)_{s,t} + C(w^i - \varphi^i, \varphi^i)_{s,t}, \end{aligned} \tag{3.58}$$

using $\|\phi\|_{2m,\theta/2} \le C_{m,\theta} \|\phi\|_{m,\theta/2}$,

$$\begin{aligned} & \|C(w^i, w^i) - C(\varphi^i, \varphi^i)\|_{m,\theta} \\ & \quad \le C_{m,\theta} \|w^i - \varphi^i\|_{m,\theta/2}^2 + \|C(\varphi^i, w^i - \varphi^i)\|_{m,\theta} + \|C(w^i - \varphi^i, \varphi^i)\|_{m,\theta} \\ & \quad \le (C_{m,\theta} r + 2) r. \end{aligned} \tag{3.59}$$

Let $i > j$. Using (3.4), we have

$$\begin{aligned} & C(w^i, w^j)_{s,t} - C(\varphi^i, \varphi^j)_{s,t} \\ & \quad = C(\varphi^j, \varphi^i)_{s,t} - C(w^j, w^i)_{s,t} + \left\{ (w^i - \varphi^i)_t - (w^i - \varphi^i)_s \right\} \left\{ (w^j - \varphi^j)_t - (w^j - \varphi^j)_s \right\} \\ & \quad + (\varphi_t^i - \varphi_s^i) \left\{ (w^j - \varphi^j)_t - (w^j - \varphi^j)_s \right\} + \left\{ (w^i - \varphi^i)_t - (w^i - \varphi^i)_s \right\} (\varphi_t^j - \varphi_s^j). \end{aligned} \tag{3.60}$$

Hence

$$\|C(w^i, w^j) - C(\varphi^i, \varphi^j)\|_{m,\theta} \quad \le \quad 3r + C_{m,\theta} r^2 + 2r \max_i \|\varphi^i\|_{m,\theta/2} \tag{3.61}$$

□

Lemma 3.13. *Let $\beta \in \mathbb{D}^{\infty,q}(W^d, H^*) \cap L^2(W^d, H^*)$, where $q > 1$. Suppose that $d\beta = 0$ on $U_{r,\varphi}$. Then for any $r' < r$, there exists $g \in \mathbb{D}^{\infty,q}(W^d, \mathbb{R}) \cap \mathbb{D}^{1,2}(W^d, \mathbb{R})$ such that $dg = \beta$ on $U_{r',\varphi}$.*

Theorem 3.14. *Let $\beta \in \mathbb{D}^{\infty,q}(W^d, H^*) \cap L^2(W^d, H^*)$, where $q > 1$. We assume that the first derivative of φ is a bounded variation function. Suppose that $d\beta = 0$ on $U_r(\varphi)$. Then for any $r' < r$, there exists $g \in \mathbb{D}^{\infty,q}(W^d, \mathbb{R}) \cap \mathbb{D}^{1,2}(W^d, \mathbb{R})$ such that $dg = \beta$ on $U_{r'}(\varphi)$.*

Proof of Theorem 3.14. Let $T_\varphi w = w + \varphi$. Then $U_r(\varphi) = \{T_\varphi w \mid w \in U_{r,\varphi}\}$. For a measurable function u on W^d, define $T_\varphi^* u(w) = u(w + \varphi)$. Let χ_R be a smooth function on $\mathbb{R}$ such that $\chi_R(x) = 1$ for $|x| \le R$ and $\chi_R(x) = 0$ and $|x| \ge R + 1$. Let $\hat{\chi}_R(w) = \chi(\|w\|_{m,\theta/2}^m)$. Note that $D^l \hat{\chi}_R(w)$ is a bounded function for all l. For any $q > 1, k \in \mathbb{N} \cup \{0\}$, there exist positive constants C_1, C_2 $(C_1 < C_2)$ such that for any $u \in \mathbb{D}^{k,q}(W^d)$

$$C_1 \|u\|_{k,q} \le \|(T_\varphi^* u)\hat{\chi}_R\|_{k,q} \le C_2 \|u\|_{k,q}.$$

This can be checked by using the Cameron-Martin formula and the fact that the stochastic integral $\int_0^1(\varphi'(t), dw(t))$ is actually a Riemann-Stieltjes integral and bounded on $\{w \in \Omega \mid \|w\|_{m,\theta/2} \leq R+1\}$. The same estimates hold for 1-forms. Let β be the 1-form which satisfies the assumptions of the theorem. Let R be a sufficiently large number and set $\bar{\beta} = (T_\varphi^*\beta)\hat{\chi}_R$. Then $\bar{\beta} \in \mathbb{D}^{\infty,q}(W^d, H^*) \cap L^2(W^d, H^*)$ and $d\bar{\beta} = 0$ on $U_{r,\varphi}$. Therefore by Lemma 3.13, there exists $\bar{g} \in \mathbb{D}^{\infty,q}(W^d, H^*) \cap \mathbb{D}^{1,2}(W^d, H^*)$ such that $d\bar{g} = \bar{\beta}$ on $U_{r',\varphi}$. Define $g = \left(T_{-\varphi}^*\bar{g}\right)\hat{\chi}_{R'}$, where R' is also a sufficiently large positive number. Then g satisfies the desired properties. □

Proof of Lemma 3.13. Let $N \in \mathbb{N}$ and set

$$\begin{aligned} R_N &= \Big\{ w(N)^\perp = (w(N)^{\perp,1}, \ldots, w(N)^{\perp,d}) \in \Omega_N^\perp \;\Big|\; \max_{1\leq i\leq d} \|w(N)^{\perp,i}\|_{m,\theta'/2} < r/4, \\ &\quad \max_{1\leq i<j\leq d} \|C(w(N)^{\perp,i}, w(N)^{\perp,j})\|_{m,\theta} < r/4, \max_{1\leq i\leq j\leq d} \|C(\varphi^i, w(N)^{\perp,j})\|_{m,\theta} < r/4, \\ &\quad \max_{1\leq i\leq j\leq d} \|C(w(N)^{\perp,i}, \varphi^j)\|_{m,\theta} < r/4 \Big\}. \end{aligned} \tag{3.62}$$

Let us define a subset $U_{r,\varphi,N}$ of $U_{r,\varphi}$ by

$$\begin{aligned} U_{r,\varphi}(w(N)^\perp) &= \Big\{ w(N) \in \Omega_N \;\Big|\; w \in U_{r,\varphi}, \max_{1\leq i<j\leq d} \|C(w(N)^i, w(N)^{\perp,j})\|_{m,\theta} < r/4, \\ &\quad \max_{1\leq i<j\leq d} \|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta} < r/4 \Big\}, \end{aligned} \tag{3.63}$$

$$U_{r,\varphi,N} = \Big\{ w \in \Omega \mid w(N) \in U_{r,\varphi}(w(N)^\perp), w(N)^\perp \in R_N \Big\}. \tag{3.64}$$

$U_{r,\varphi,N}$ is an approximate set of $U_{r,\varphi}$. $U_{r,\varphi}(w(N)^\perp)$ can be identified with a bounded open subset of the euclidean space of dimension $2^N d$. Using Lemma 3.8 and Lemma 3.10 and an induction, we prove the following Claims.

Claim 1 Let $w(N)^\perp \in R_N$. Poincaré's inequality holds on $U_{r,\varphi}(w(N)^\perp)$ in the following form:

$$\mathrm{Var}(g; U_{r,\varphi}(w(N)^\perp)) \leq C \int_{U_{r,\varphi}(w(N)^\perp)} |Dg(w(N))|_H^2 d\mu_{N,U_{r,\varphi}(w(N)^\perp)}(w(N)), \tag{3.65}$$

where C is a positive constant which depends only on $r, d, \varphi, m, \theta, \theta'$ and $\mu_{N,U_{r,\varphi}(w(N)^\perp)}$ is a normalized probability measure on $U_{r,\varphi}(w(N)^\perp)$.

Claim 2 There exists a measurable function g_N on $U_{r,\varphi,N}$ such that for almost all $w(N)^\perp \in R_N$, the function $w(N) \in U_{r,\varphi}(w(N)^\perp) \to g_N(w(N), w(N)^\perp)$ is a C^∞ function with

$$\max_{w(N)\in U_{r,\varphi}(w(N)^\perp)} |g_N(w(N), w(N)^\perp)| < \infty \tag{3.66}$$

$$\int_{U_{r,\varphi}(w(N)^\perp)} g_N(w(N), w(N)^\perp) d\mu(w(N)) = 0 \tag{3.67}$$

and $d_N g_N = \beta_N$ holds on $U_{r,\varphi,N}$. Here $d_N g_N$ is the exterior differential of g_N with respect to the variable $w(N)$ and $\beta_N = P_N\beta$ which is the projection of β onto $(\Omega_N \cap H)^*$.

To prove these claims, we introduce the following sets. First, we fix $w(N)^\perp \in R_N$. Let

$$\begin{aligned} B_{d,N}(w(N)^\perp) &= \Big\{ w(N)^d \mid \|w(N)^d + w(N)^{\perp,d}\|_{m,\theta'/2} < r, \max_{1\leq i\leq d} \|C(\varphi^i, w^d)\|_{m,\theta} < r, \\ &\quad \|C(w^d, \varphi^d)\|_{m,\theta} < r, \max_{1\leq l<d} \|C(w(N)^{\perp,l}, w(N)^d)\|_{m,\theta} < r/4 \Big\}. \end{aligned} \tag{3.68}$$

For $1 \le k \le d-1$, taking $w(N)^i \in B_{i,N}(w(N)^{i+1}, \dots, w(N)^d, w(N)^\perp)$ $(k+1 \le i \le d)$ inductively, we define

$$\begin{aligned}
&B_{k,N}(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp) \\
&\quad = \Big\{ w(N)^k \Big| \; \|w(N)^k + w(N)^{\perp,k}\|_{m,\theta'/2} < r, \;\; \max_{l>k} \|C(w^k, w^l)\|_{m,\theta} < r, \\
&\quad \max_{1\le i\le k} \|C(\varphi^i, w^k)\|_{m,\theta} < r, \max_{l\ge k} \|C(w^k, \varphi^l)\|_{m,\theta} < r, \\
&\quad \max_{l>k} \|C(w(N)^k, w(N)^{\perp,l})\|_{m,\theta} < r/4, \max_{1\le j<k} \|C(w(N)^{\perp,j}, w(N)^k)\|_{m,\theta} < r/4 \Big\}. \quad (3.69)
\end{aligned}$$

Note that $0 \in B_{k,N}(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp)$. We denote all $(w(N)^{k+1}, \dots, w(N)^d)$ which can be obtained in this way by $S_{k+1,d}(w(N)^\perp)$. Let $U_d(w(N)^\perp) = U_{r,\varphi}(w(N)^\perp)$. For $1 \le k \le d-1$ and $\big(w(N)^{k+1}, \dots, w(N)^d\big) \in S_{k+1,d}(w(N)^\perp)$ define

$$\begin{aligned}
&U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp) \\
&\quad = \Bigg\{ (w(N)^1, \dots, w(N)^k) \;\Big|\; \max_{1\le i\le k} \|w^i\|_{m,\theta'/2} < r, \\
&\quad \max_{1\le i<j\le k} \|C(w^i, w^j)\|_{m,\theta} < r, \max_{1\le i\le k<l\le d} \|C(w^i, w^l)\|_{m,\theta} < r, \\
&\quad \max_{1\le i\le j\le k} \|C(\varphi^i, w^j)\|_{m,\theta} < r, \max_{1\le i\le k, i\le j} \|C(w^i, \varphi^j)\|_{m,\theta} < r, \\
&\quad \max_{1\le i\le k, i<j\le d} \|C(w(N)^i, w(N)^{\perp,j})\|_{m,\theta} < r/4, \max_{1\le i<j, 1<j\le k} \|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta} < r/4 \Bigg\}.
\end{aligned} \quad (3.70)$$

Then

$$\begin{aligned}
&B_{k,N}(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp) \\
&\qquad = \Big\{ w(N)^k \;\Big|\; \mu_k \Big(U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp)^{w(N)^k} \Big) > 0 \Big\} \quad (3.71)
\end{aligned}$$

and for $w(N)^k \in E_{k,N}(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp)$,

$$U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp)^{w(N)^k} = U_{k-1}(w(N)^k, \dots, w(N)^d, w(N)^\perp). \quad (3.72)$$

Also

$$\begin{aligned}
&U_{k-1}(0, w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp) \\
&\quad = \Big\{ (w(N)^1, \dots, w(N)^{k-1}) \;\Big|\; \mu(U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp)_{(w(N)^1, \dots, w(N)^{k-1})}) > 0 \Big\}
\end{aligned}$$

and for $(w(N)^1, \dots, w(N)^{k-1}) \in U_{k-1}(0, w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp)$,

$$\begin{aligned}
&U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^\perp)_{(w(N)^1, \dots, w(N)^{k-1})} \\
&\quad = \Big\{ w(N)^k \;\Big|\; \|w(N)^k + w(N)^{\perp,k}\|_{m,\theta'/2} < r, \max_{1\le i<k} \|C(w^i, w^k)\|_{m,\theta} < r, \max_{l>k} \|C(w^k, w^l)\|_{m,\theta} < r \\
&\quad \max_{1\le i\le k} \|C(\varphi^i, w^k)\|_{m,\theta} < r, \;\; \max_{l\ge k} \|C(w^k, \varphi^l)\|_{m,\theta} < r, \\
&\quad \max_{l>k} \|C(w(N)^k, w(N)^{\perp,l})\|_{m,\theta} < r/4, \max_{1\le i<k} \|C(w(N)^{\perp,i}, w(N)^k)\|_{m,\theta} < r/4 \Big\}. \quad (3.73)
\end{aligned}$$

Note that $U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^{\perp})_{(w(N)^1, \dots, w(N)^{k-1})}$ is a convex set of $\mathbb{R}^{2^N}$ and contains 0. Further, by Lemma 3.10, we have for all $1 \le k \le d-1$,

$$\inf\Big\{\mu\Big(U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^{\perp})_x \cap U_k(w(N)^{k+1}, \dots, w(N)^d, w(N)^{\perp})_y\Big) \;\Big|\; x, y \in U_{k-1}\left(0, w(N)^{k+1}, \dots, w(N)^d, w(N)^{\perp}\right), (w(N)^{k+1}, \dots, w(N)^d) \in S_{k+1,d}(w(N)^{\perp}),\; w(N)^{\perp} \in R_N\Big\} > 0 \quad (3.74)$$

and the lower bound is given by the inverse of products of $C(l, r/4, r, m, \theta, \theta')$. Therefore, we see that Claim 1 holds with the constant C which depends only on the inverse of products of $C(l, r/4, r, m, \theta, \theta')$ by Lemma 3.8 and Lemma 3.11.

We prove Claim 2. Let $w(N)^{\perp} \in R_N$. Then $\beta_N(\cdot, w(N)^{\perp}) \in \wedge^1 T^* U_{r,\varphi}(w(N)^{\perp})$ is also a closed C^∞-differential form and the supremum norm of all derivatives are finite for almost all $w(N)^{\perp}$ by the Sobolev embedding theorem. By Lemma 3.9, we can construct a bounded function $u_N(\cdot, \omega(N)^{\perp}) \in C^\infty(U_{r,\varphi}(w(N)^{\perp}))$ explicitly such that $d_N u_N = \beta_N$ and $u_N(w(N), w(N)^{\perp})$ is a measurable function on $U_{r,\varphi,N}$.

$$g_N = u_N - \mu\left(U_{r,\varphi}(w(N)^{\perp})\right)^{-1} \int_{U_{r,\varphi}(\omega(N)^{\perp})} u_N(w(N), w(N)^{\perp}) d\mu(w(N))$$

is the desired function.

Now, we prove the existence of g which satisfies the desired property in the Lemma. Let g_N be the function in the Claim 2. Then by the Poincaré inequality established in the Claim 1, it holds that

$$\|g_N\|^2_{L^2(U_{r,\varphi,N})} \le C\|\beta_N\|^2_{L^2(U_{r,\varphi,N})} \le C\|\beta\|^2_{L^2(U_{r,\varphi})}. \quad (3.75)$$

Let $\hat{g}_N(w) = g_N(w) 1_{U_{r,\varphi,N}}(w)$. Let us choose a positive numbers r_1, r_2 such that $0 < r' < r_1 < r_2 < r$. Let ρ be a smooth function on $\mathbb{R}^{3d(d+1)/2}$ such that $\max_y |\rho(y) - \max_i |y^i||$ is sufficiently small. It is easy to see the existence of such a function using a molifier. Then there exists a small positive number ε such that for any $r_1 \le s \le r_2$,

$$\left\{x = (x^i) \in \mathbb{R}^{3d(d+1)/2} \mid \max_i |x^i| < r' + \varepsilon\right\} \subset \left\{x = (x^i) \in \mathbb{R}^{3d(d+1)/2} \mid \rho(x^{(m)}) < s^m\right\} \subset \left\{x = (x^i) \in \mathbb{R}^{3d(d+1)/2} \mid \max_i |x^i| < r\right\}, \quad (3.76)$$

where $x^{(m)} = ((x^1)^m, \dots, (x^{3d(d+1)/2})^m)$. Note that the index j of $(x^i)^j$ is the power and i stands for the i-th element. Let $\hat{\rho}(w)$ be the composition of ρ and the $3d(d+1)/2$ random variables

$$\|w^i\|^m_{m,\theta'/2}\ (1 \le i \le d), \|C(w^j, w^k)\|^m_{m,\theta}\ (1 \le j < k \le d)$$
$$\|C(\varphi^i, w^j)\|^m_{m,\theta}\ (1 \le i \le j \le d), \|C(w^i, \varphi^j)\|^m_{m,\theta}\ (1 \le i \le j \le d). \quad (3.77)$$

Let ψ be the smooth decreasing function such that $\psi(u) = 1$ for $u \le \frac{r_1^m + r_2^m}{2}$ and $\psi(u) = 0$ for $u \ge \frac{r_1^m + 2r_2^m}{3}$. Let χ be the smooth decreasing function such that $\chi(u) = 1$ for $u \le (r/6)^m$

$\chi(u) = 0$ for $u \geq (r/5)^m$ and set

$$\begin{aligned}\hat{\chi}_N(w) &= \chi\Big(\sum_{i=1}^d \|w(N)^{\perp,i}\|_{m,\theta'/2}^m + \sum_{1\le j<k\le d} \|C(w(N)^{\perp,j}, w(N)^{\perp,k})\|_{m,\theta}^m \\ &\quad + \sum_{1\le i\le j\le d} \|C(\varphi^i, w(N)^{\perp,j})\|_{m,\theta}^m + \sum_{1\le i\le j\le d} \|C(w(N)^{\perp,i}, \varphi^j)\|_{m,\theta}^m \\ &\quad + \sum_{1\le i<j\le d} \|C(w(N)^i, w(N)^{\perp,j})\|_{m,\theta}^m + \sum_{1\le i<j\le d} \|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta}^m\Big).\end{aligned}$$

Let $h_N(w) = \hat{g}_N(w)\psi(\hat{\rho}(w))\,\hat{\chi}_N(w)$. Since $\sup_N \|\hat{g}_N\|_{L^2(W^d,\mu)} < \infty$, there exists a subsequence $\hat{g}_{N(k)}$ $(N(1) < N(2) < \cdots)$ such that $\hat{g}_{N(k)}$ converges weakly to some $\hat{g}_\infty \in L^2(W^d,\mu)$. Noting that $\|\hat{\chi}_N\|_\infty \le 1$ and $\lim_{N\to\infty} \hat{\chi}_N(w) = 1$ for all $w \in \Omega$, we see that $\hat{g}_{N(k)}(w)\psi(\hat{\rho}(w))\hat{\chi}_{N(k)}(w)$ also converges weakly to $\hat{g}_\infty(w)\psi(\hat{\rho}(w))$ which we denote by $h_\infty(w)$. We calculate the weak derivative of h_∞. Fix a natural number N_0 and let $\theta \in \mathbb{D}^\infty(W^d \to P_{N_0}H^*)$. Then

$$\begin{aligned}\int_{W^d} h_\infty(w) D^*\theta(w) d\mu(w) &= \lim_{k\to\infty} \int_{W^d} h_{N(k)}(w) D^*\theta(w) d\mu(w) \\ &= \lim_{k\to\infty} \int_{W^d} \big(d_{N(k)} h_{N(k)}(w), \theta(w)\big)\, d\mu(w). \qquad (3.78)\end{aligned}$$

Here

$$\begin{aligned}d_{N(k)}\left(\hat{g}_{N(k)}\psi(\hat{\rho})\hat{\chi}_{N(k)}\right) &= \beta_{N(k)}\psi(\hat{\rho})\hat{\chi}_{N(k)} + \hat{g}_{N(k)} d_{N(k)}\left(\psi(\hat{\rho}(w))\right)\hat{\chi}_{N(k)}(w) \\ &\quad + \hat{g}_{N(k)}\psi(\hat{\rho}(w)) d_{N(k)}\hat{\chi}_{N(k)}(w). \qquad (3.79)\end{aligned}$$

Noting that

$$\begin{aligned}\lim_{k\to\infty} \|d_{N(k)}\left(\psi(\hat{\rho})\right) - d\left(\psi(\hat{\rho})\right)\|_{L^4(\mu)} &= 0, \qquad (3.80)\\ \lim_{k\to\infty} \|d_{N(k)}\hat{\chi}_{N(k)}\|_{L^4(\mu)} &= 0, \qquad (3.81)\end{aligned}$$

we get

$$\begin{aligned}&\int_{W^d} h_\infty(w) D^*\theta(w) d\mu(w) \\ &\quad = \int_{W^d} \Big(\beta(w)\psi(\hat{\rho}(w)) + \hat{g}_\infty(w) d\left(\psi(\hat{\rho}(w))\right), \theta(w)\Big) d\mu(w). \qquad (3.82)\end{aligned}$$

This implies $dh_\infty = \beta\psi(\hat{\rho}) + \hat{g}_\infty d\left(\psi(\hat{\rho})\right)$ in weak sense. By Lemma 3.5 and Lemma 3.6, $d\left(\psi(\hat{\rho})\right)$ is a bounded function. Hence $dh_\infty \in L^2(W^d,\mu)$ which implies $h_\infty \in \mathbb{D}^{1,2}(W^d,\mathbb{R})$. Also h_∞ satisfies that $dh_\infty = \beta$ on $U_{r',\varphi}$. Finally we need to show the regularity of the higher order derivatives of h_∞. Choosing a smooth function ψ_1 on $\mathbb{R}$ such that $\psi_1(u) = 1$ for $u \le \frac{r_1^m + 3r_2^m}{4}$ and $\psi_1(u) = 0$ for $u \ge \frac{r_1^m + 4r_2^m}{5}$, we have

$$\hat{g}_\infty \psi_1(\hat{\rho}) d\left(\psi(\hat{\rho})\right) = \hat{g}_\infty d\left(\psi(\hat{\rho})\right).$$

We see that $\hat{g}_\infty \psi_1(\hat{\rho}) \in \mathbb{D}^{1,2}(W^d,\mathbb{R})$ by the same argument as the above. Iterating this procedure, we get $h_\infty \in \mathbb{D}^{\infty,q}(W^d,\mathbb{R})$. □

Remark 3.15. We can prove the Poincaré inequality on $U_r(\varphi)$ in the same way as Claim 2. For any $g \in \mathbb{D}^{1,2}(W^d)$,

$$\mathrm{Var}(g; U_{r,\varphi}) \leq C \int_{U_{r,\varphi}} |Dg(w)|_H^2 d\mu_{U_{r,\varphi}}(w). \tag{3.83}$$

We can define a local Sobolev space $W^1(U_{r,\varphi})$. It is not clear that $W^1(U_{r,\varphi})$ coincides with the restriction of $\mathbb{D}^{1,2}(W^d)$ to $U_{r,\varphi}$ at the moment. The extension property of functions on convex sets were studied in [16], but $U_{r,\varphi}$ is not an H-convex set.

We introduce the following notions.

Definition 3.16. *Let D be a measurable subset of Ω with $\mu(D) > 0$.*
(1) D is called an H-connected set if $w, w+h \in D$ then there exists a C^∞ curve $h : [0,1] \to H$ such that $h(0) = 0$ and $h(1) = h$ and $w + h(\tau) \in D$ for all $0 \leq \tau \leq 1$.
(2) D is called an H-simply connected set if the following holds: Let w be any point of D. Let $\{h(0,\tau) \mid 0 \leq \tau \leq 1\}$ and $\{h(1,\tau) \mid 0 \leq \tau \leq 1\}$ be C^∞ curves on H such that $h(0,0) = h(1,0) = 0$, $h(0,1) = h(1,1)$ and $\{w + h(i,\tau) \mid 0 \leq \tau \leq 1\} \subset D$ for $i = 0,1$. Then there exists a C^∞ map $\mathcal{H} : (\sigma,\tau) \in [0,1]^2 \to H$ such that $\mathcal{H}(0,\tau) = h(0,\tau)$, $\mathcal{H}(1,\tau) = h(1,\tau)$ for all $0 \leq \tau \leq 1$ and $w + \mathcal{H}(\sigma,\tau) \in D$ for all $(\sigma,\tau) \in [0,1]^2$.

Lemma 3.17. (1) *Let $\varphi_i \in H$ and $r_i > 0$ $(i = 1,2)$. The following three conditions* (i), (ii), (iii) *are equivalent.*

(i) $\mu\left(U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2)\right) > 0$.

(ii) $U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2) \neq \emptyset$.

(iii) $U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2) \cap H \neq \emptyset$.

(2) *Let $D_i = U_{r_i}(\varphi_i)$ $(1 \leq i \leq n)$. Assume that $\left(\cup_{i=1}^k D_i\right) \cap D_{k+1} \neq \emptyset$. Then $D = \cup_{i=1}^n D_i$ is an H-connected set.*

Proof. The fact (i) $\Longrightarrow$ (ii) is trivial. (ii) $\Longrightarrow$ (iii) follows from that $\lim_{N\to\infty} d_\Omega(w(N), w) = 0$ for any $w \in \Omega$. We prove (iii) implies (i). By the assumption, there exists $h \in U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2) \cap H$. Let ε be a sufficiently small positive number. Let $w \in U_\varepsilon(0)$. Then $w + h \in U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2)$ and $\mu\left(U_\varepsilon(0) + h\right) > 0$. This proves (i).
(2) Let $w, w+h \in D$. Without loss of generality, we may assume that $w \in D_1$, $w + h \in D_i$ and $D_k \cap D_{k+1} \neq \emptyset$ for all $1 \leq k \leq i-1$. Let $\psi_i \in D_i \cap D_{i+1} \cap H$. Let $\varphi_{i,w(N)^\perp} = \varphi_i + w(N)^\perp$ and $\psi_{i,w(N)^\perp} = \psi + w(N)^\perp$ Then for sufficiently large N, it holds that

$$\{(1-\tau)\varphi_{k,w(N)^\perp} + \tau\psi_{k,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_k \quad (k = 1,\ldots,i-1), \tag{3.84}$$
$$\{(1-\tau)\psi_{k-1,w(N)^\perp} + \tau\varphi_{k,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_k \quad (k = 2,\ldots,i) \tag{3.85}$$
$$\{(1-\tau)w + \tau\varphi_{1,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_1, \tag{3.86}$$
$$\{(1-\tau)(w+h) + \tau\varphi_{i,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_i. \tag{3.87}$$

This follows from Theorem 3.1. Hence, we have proved the existence of a piecewise linear path $h = h(\tau)$ $(0 \leq \tau \leq 1)$ such that $h(0) = 0$, $h(1) = h$ and $w + h(\tau) \subset D$ for all $0 \leq \tau \leq 1$. Note that if $\sup_\tau \|\tilde{h}(\tau) - h(\tau)\|_H$ is sufficiently small, then $\{w + \tilde{h}(\tau) \mid 0 \leq \tau \leq i+1\} \subset D$. Thus we see the existence of a smooth path connecting w and $w + h$. □

We use Theorem 3.14 to prove Theorem 2.1. To this end, we fix a version of the solution of SDE (2.1). For $h \in H$, let $X(t,a,h)$ be the solution to the following ODE:

$$\begin{aligned} \dot{X}(t,a,h) &= \left(L_{X(t,a,h)}\right)_* \dot{h}_t, \\ X(0,a,h) &= a \in G. \end{aligned}$$

For $w \in \Omega$, define

$$X(t,a,w) = \lim_{N\to\infty} X(t,a,w(N)). \tag{3.88}$$

The limit (3.88) exists by Theorem 3.1 and the universal limit theorem [24, 25]. We have the following.

Theorem 3.18. *The measurable map $X : [0,\infty) \times G \times \Omega \to G$ in (3.88) which satisfies the following.*
(1) *$X(t,a,w)$ is a version of the solution to the SDE* (2.1)*.*
(2) *For any t,a, the map $w \to X(t,a,w)$ is continuous in the sense that there exists an increasing function F on $\mathbb{R}$ such that for all $w,\eta \in \Omega$,*

$$\sup_{0\le t\le 1} d(X(t,a,w), X(t,a,\eta)) \le F(\max\{d_\Omega(0,w), d_\Omega(0,\eta)\}) d_\Omega(w,\eta).$$

Moreover $w \to X(t,a,w)$ is ∞-quasi-continuous with respect to the supremum norm of W^d for any t,a.
(3) *For all t,a,w, $X(t,a,w) = aX(t,e,w)$. In particular, $a \to X(t,a,w)$ is a C^∞-diffeomorphism.*
(4) *For any $\phi \in H^1([0,1] \to G \mid \phi_0 = e)$, it holds that*

$$X(t,\phi_t,w) = X(t,e,w+\xi(\phi)), \tag{3.89}$$

where $\xi(\phi)$ is the solution to

$$\dot{\xi}(\phi)_t = Ad\left(X(t,e,w)^{-1}\right)\left(\phi_t^{-1}\dot{\phi}_t\right) \qquad t>0 \tag{3.90}$$

$$\xi(\phi)_0 = 0. \tag{3.91}$$

(5) *For $h \in H$, let $Z(t,h,w)$ be the smooth path on G which satisfies the ODE:*

$$Z(t,h,w)^{-1}\dot{Z}(t,h,w) = Ad\left(X(t,e,w)\right)\dot{h}_t \qquad t>0 \tag{3.92}$$

$$Z(0,h,w) = e. \tag{3.93}$$

Then it holds that $X(t,Z(t,h,w),w) = X(t,e,w+h)$.
(6) *For any $h \in H$*

$$\xi\left(Z(\cdot,h,w)\right) = h. \tag{3.94}$$

Let $B_\varepsilon(e) = \{a \in G \mid d(a,e) < \varepsilon\}$. We assume that ε is sufficiently small and $B_\varepsilon(e)$ is diffeomorphic to a standard ball in an euclidean space. Let

$$\mathcal{D}_\varepsilon = \{w \in \Omega \mid X(1,e,w) \in B_\varepsilon(e)\}.$$

We construct a covering of $\mathcal{D}_\varepsilon$ by a countable sets of $U_r(\varphi)$. To this end, let $\varepsilon_n = \varepsilon(1-\frac{1}{n})$ $(n = 1, 2, \ldots)$ and

$$\begin{aligned} A_K &= \{w \in \Omega \mid d_\Omega(0,w) < K\} && (3.95)\\ B_{N,\kappa} &= \Big\{w \in \Omega \;\Big|\; \max_i \|w(N)^{\perp,i}\|_{m,\theta'/2} < \kappa, \max_{1\le i<j\le d} \|C(w(N)^{\perp,i}, w(N)^{\perp,j})\|_{m,\theta} < \kappa, \\ &\qquad \max_{1\le i\le j\le d} \|C(w(N)^{i}, w(N)^{\perp,j})\|_{m,\theta} < \kappa, \max_{1\le i\le j\le d} \|C(w(N)^{\perp,i}, w(N)^{j})\|_{m,\theta} < \kappa, \Big\} && (3.96) \end{aligned}$$

where $0 < \kappa < 1$. For $w \in A_K \cap B_{N,\kappa}$, $\max_i \|w(N)^i\|_{m,\theta'/2} < K + 1$. Let

$$\mathcal{D}_{\varepsilon_n,K,N,\kappa} = \mathcal{D}_{\varepsilon_n} \cap A_K \cap B_{N,\kappa} \tag{3.97}$$

(3.98)

For any $\kappa > 0, n, K$, we have

$$\liminf_{N\to\infty} \mathcal{D}_{\varepsilon_n,K,N,\kappa} = \mathcal{D}_{\varepsilon_n} \cap A_K. \tag{3.99}$$

For fixed n, K we can find a positive number $\kappa(n,K)$ such that there exists a finite cover of $\mathcal{D}_{\varepsilon_n,K,N,\kappa(n,K)}$ by $U_r(\varphi)$ which satisfies $U_r(\varphi) \subset \mathcal{D}_{\varepsilon_{2n}}$. Since (3.99) holds, this implies that there exists a countable cover of $\mathcal{D}_{\varepsilon_n} \cap A_K$ by $U_r(\varphi)$ which are included in $\mathcal{D}_{\varepsilon_{2n}}$ and so does for $\mathcal{D}_\varepsilon$ too. More precisely we prove the following.

Lemma 3.19. (1) *Let*

$$\kappa < \min\left(\frac{\varepsilon}{4nF\left(K + 3\left(C_{m,\theta}(K+2)+1\right)\right)\left(C_{m,\theta}(K+2)+1\right)}, 1\right). \tag{3.100}$$

Let $w \in \mathcal{D}_{\varepsilon_n,K,N,\kappa}$. *We take* $\varphi \in H$ *such that*

$$\|\varphi - w(N)\|_H \le \frac{\kappa}{3\left(6\kappa + 2(C_{m,\theta,\theta'}+1)(K+3)\right)}. \tag{3.101}$$

Then

$$w \in U_{4\kappa/3}(\varphi) \subset U_{\sqrt{2}\kappa}(\varphi) \subset \mathcal{D}_{\varepsilon_{2n}}. \tag{3.102}$$

(2) *Let* κ *be the positive number satisfying* (3.100). *Then for any* $N \in \mathbb{N}$, *there exists* $L = L(n,K,N,\kappa)$ *and finite number piecewise linear paths* $\{\varphi_i\}_{i=1}^L \subset \Omega_N$ *such that*

$$\mathcal{D}_{\varepsilon_n,K,N,\kappa} \subset \cup_{i=1}^L U_{4\kappa/3}(\varphi_i) \subset \cup_{i=1}^L U_{\sqrt{2}\kappa}(\varphi_i) \subset \mathcal{D}_{\varepsilon_{2n}}. \tag{3.103}$$

(3) *Let* $\{\varphi_i\}_{i=1}^\infty$ *and* $\kappa_i > 0$ *be all piecewise linear paths and positive numbers which are defined in* (2) *for all* N, K, n. *Then it holds that*

$$\mathcal{D}_\varepsilon = \cup_{i=1}^\infty U_{4\kappa_i/3}(\varphi_i) = \cup_{i=1}^\infty U_{\sqrt{2}\kappa_i}(\varphi_i). \tag{3.104}$$

(4) *Assume that* G *is simply connected. Then we can reorder the covering* (3.104) *such that*

$$\mu\left(\left(\cup_{i=1}^n U_{4\kappa_i/3}(\varphi_i)\right) \cap U_{4\kappa_{n+1}/3}(\varphi_{n+1})\right) > 0$$

for all $n \ge 1$.

Proof. (1) Suppose that $w \in \mathcal{D}_{\varepsilon_n,K,N,\kappa}$. Here κ is a positive number with $\kappa < 1$. Then $\|w(N)\|_{m,\theta'/2} < K+1$. By Lemma 3.12 (2), $d_\Omega(w(N),w) < (C_{m,\theta}(K+2)+1)\,\kappa$. Hence $d_\Omega(w(N),0) \le K + (C_{m,\theta}(K+2)+1)\,\kappa$. By Theorem 3.18 (1),

$$\begin{aligned} d(X(1,e,w(N)),e) &\le d(X(1,e,w(N)),X(1,e,w)) + d(X(1,e,w),e) \\ &< F\left(K+(C_{m,\theta}(K+2)+1)\,\kappa\right)(C_{m,\theta}(K+2)+1)\,\kappa + \varepsilon_n. \end{aligned} \quad (3.105)$$

Consequently, if

$$\kappa < \kappa(n,p,K,\varepsilon) := \min\left(\frac{\varepsilon}{npF\left(K+(C_{m,\theta}(K+2)+1)\right)(C_{m,\theta}(K+2)+1)}, 1\right)$$

then $X(1,e,w(N)) \in B_{\varepsilon(1-\frac{1}{n}(1-\frac{1}{p}))}(e)$. Let $\eta \in U_{2\kappa}(w(N))$. Then $d_\Omega(w(N),\eta) < 2\,(C_{m,\theta}(K+2)+1)\,\kappa$. Thus

$$\begin{aligned} d_\Omega(0,\eta) &< K + (C_{m,\theta}(K+2)+1)\,\kappa + 2\,(C_{m,\theta}(K+2)+1)\,\kappa \\ &= K + 3\,(C_{m,\theta}(K+2)+1)\,\kappa. \end{aligned} \quad (3.106)$$

Therefore

$$\begin{aligned} &d\left(X(1,e,\eta),e\right) \\ &\quad \le d(X(1,e,\eta),X(1,e,w(N))) + d(X(1,e,w(N)),e) \\ &\quad < 2F\left(K+3\,(C_{m,\theta}(K+2)+1)\,\kappa\right)(C_{m,\theta}(K+2)+1)\kappa + \varepsilon\left(1-\frac{1}{n}(1-\frac{1}{p})\right). \end{aligned} \quad (3.107)$$

Hence if

$$\kappa < \min\left(\kappa(n,p,K,\varepsilon), \frac{\varepsilon}{2nqF\left(K+3\,(C_{m,\theta}(K+2)+1)\right)(C_{m,\theta}(K+2)+1)}\right),$$

$d(X(1,e,\eta),e) < \varepsilon\left(1-\frac{1}{n}(1-\frac{1}{p}-\frac{1}{q})\right)$ holds. Now we set $p=q=4$ and κ to be a positive number such that

$$\kappa < \min\left(\frac{\varepsilon}{4nF\left(K+3\,(C_{m,\theta}(K+2)+1)\right)(C_{m,\theta}(K+2)+1)}, 1\right). \quad (3.108)$$

For such a κ, it holds that if $w \in \mathcal{D}_{\varepsilon_n,K,N,\kappa}$ then $\eta \in \mathcal{D}_{\varepsilon_{2n}}$ for any $\eta \in U_{2\kappa}(w(N))$. That is, $w \in U_\kappa(w(N)) \subset U_{2\kappa}(w(N)) \subset \mathcal{D}_{\varepsilon_{2n}}$. Applying Lemma 3.12 (1) to the case where $\varphi_1 = w(N)$, $\varphi_2 = \varphi$, $r = \kappa$, $\delta = \sqrt{2}-1, 1/3$, we have if

$$\|w(N)-\varphi\|_H < \frac{\kappa}{3\left(6\kappa + C_{m,\theta,\theta'} + 2(2K+3)\right)}$$

then

$$w \in U_\kappa(w(N)) \subset U_{4\kappa/3}(\varphi) \subset U_{\sqrt{2}\kappa}(\varphi) \subset U_{2\kappa}(w(N)) \subset \mathcal{D}_{\varepsilon_{2n}}.$$

This completes the proof of (1). (2),(3) follows from (1). We prove (4). We denote $U_{r_i,H}(\varphi_i) = U_{r_i}(\varphi_i) \cap H$. If G is simply connected, then $\mathcal{D}_\varepsilon \cap H = \cup_{i=1}^\infty U_{r_i,H}(\varphi_i)$ is a connected set in H. Hence there exists a permutation $\{n(i)\}_{i=1}^\infty$ of $\mathbb{N}$ such that

$$\left(\cup_{i=1}^k U_{r_{n(i)},H}(\varphi_{n(i)})\right) \cap U_{r_{n(i+1)},H}(\varphi_{n(i+1)}) \ne \emptyset \quad \text{for all } i \in \mathbb{N}.$$

By Lemma 3.17, this completes the proof. $\square$

Proposition 3.20. *Assume that G is a simply connected compact Lie group. Let V be an open set of G which is diffeomorphic to a small ball. Let*

$$H^1_V = \{\gamma \in H^1([0,1] \to G) \mid \gamma_0 = e, \gamma_1 \in V\}.$$

Let $\{\gamma(i,\tau) \mid 0 \le \tau \le 1\} \subset H^1_V$ $(i = 0,1)$ be two C^∞-curves with the same starting point and the end point in H^1_V, that is, we assume

$$\gamma(0,0) = \gamma(1,0) \in H^1_V, \gamma(0,1) = \gamma(1,1) \in H^1_V.$$

Then there exists a C^∞-map $\mathcal{M} : (\sigma,\tau)(\in [0,1]^2) \to \mathcal{M}(\sigma,\tau) \in H^1_V$ such that $\mathcal{M}(0,\tau) = \gamma(0,\tau)$ and $\mathcal{M}(1,\tau) = \gamma(1,\tau)$ for all τ.

Proof. This follows from that $\pi_2(G) = 0$ and so $\pi_1(L_e(G)) = 0$. See [9] and [27]. This is the result in continuous category. In the case of H^1-paths, it suffices to approximate the continuous homotopy by a smooth homotopy. □

Lemma 3.21. *Assume that G is a simply connected compact Lie group. Then $\mathcal{D}_\varepsilon$ is an H-connected and H-simply connected set for any sufficiently small ε.*

Proof. First we prove that $\mathcal{D}_\varepsilon$ is an H-connected set. Assume that $w, w + h \in \mathcal{D}_\varepsilon$. Then $X(1, e, w + h), X(1, e, w) \in B_\varepsilon(e)$. Since $X(1, e, w + h) = X(1, Z(1, e, h), w) = X(1, e, w)$, $t \to Z(t, h, w)$ is a H^1-curve on G starting at e and $Z(1, h, w) \in X^{-1}(1, \cdot, w)(B_\varepsilon(e))$. Also $e \in X^{-1}(1, \cdot, w)(B_\varepsilon(e))$ holds. Since G is simply connected and $B_\varepsilon(e)$ is a contractive set, there exists a map $(\tau, t) \in [0, 1]^2 \to \gamma^{h,w}(\tau)_t \in G$ such that

(i) $\gamma^{h,w}(0)_t = e$ and $\gamma^{h,w}(1)_t = Z(t, h, w)$ for all $0 \le t \le 1$,

(ii) $\tau \in [0, 1] \to \gamma^{h,w}(\tau)$ is a C^∞-map with values in $H^1_{X^{-1}(1,\cdot,w)(B_\varepsilon(e))}$.

Now we define $h(\tau) = \xi(\gamma^{h,w}(\tau))$. $\tau \to h(\tau)$ is a C^∞-curve on H. Also $X(t, \gamma^{h,w}(\tau)_t, w) = X(t, e, w + h(\tau))$ $((\tau, t) \in [0, 1]^2)$ holds by the definition. Therefore $h(0) = 0$, $h(1) = h$ and $X(1, e, w+h(\tau)) \in B_\varepsilon(e)$ for all $0 \le \tau \le 1$. This proves that $\mathcal{D}_\varepsilon$ is an H-connected set. Next we prove the H-simply connectedness of $\mathcal{D}_\varepsilon$. Let $\tau \in [0, 1] \to h(i, \tau) \in H$ $(i = 0, 1)$ be C^∞-curves on H such that

(i) $w + h(i, \tau) \in \mathcal{D}_\varepsilon$ for all $0 \le \tau \le 1$ and $i = 0, 1$.

(ii) $h(0, 0) = h(1, 0) = 0$, $h(0, 1) = h(1, 1)$.

Then $Z(t, h(0, 0), w) = Z(t, h(1, 0), w) = e$ and $Z(t, h(0, 1), w) = Z(t, h(1, 1), w)$ hold for all $0 \le t \le 1$. Also $t \to Z(t, h(i, \tau), w)$ is a H^1-curve on G starting at e and the end point $Z(1, h(i, \tau), w) \in X^{-1}(1, \cdot, w)(B_\varepsilon(e))$ for all $0 \le \tau \le 1$ and $i = 0, 1$. Therefore $\tau \to Z(\cdot, h(i, \tau), w)$ is a C^1-map from $[0, 1]$ to $H^1_{X^{-1}(1,\cdot,w)(B_\varepsilon(e))}$. Since $B_\varepsilon(e)$ is a contractive set, $H^1_{X^{-1}(1,\cdot,w)(B_\varepsilon(e))}$ is also a simply connected set by Proposition 3.20. Therefore there exists a C^∞ map

$$(\sigma, \tau) \in [0, 1]^2 \to \mathcal{M}^{h,w}(\sigma, \tau) \in H^1_{X^{-1}(1,\cdot,w)(B_\varepsilon(e))} \tag{3.109}$$

such that $\mathcal{M}^{h,w}(i, \tau)_t = Z(t, h(i, \tau), w)$ for all $0 \le \tau, t \le 1$ and $i = 0, 1$. Let

$$\mathcal{H}(\sigma, \tau) = \xi\left(\mathcal{M}^{h,w}(\sigma, \tau)\right). \tag{3.110}$$

Then

(i) $\mathcal{H}(i,\tau)=h(i,\tau)$ for all $0\le\tau\le 1$ and $i=0,1$,

(ii) $(\sigma,\tau)\in[0,1]^2\to\mathcal{H}(\sigma,\tau)\in H$ is a C^∞ map,

(iii) $w+\mathcal{H}(\sigma,\tau)\in\mathcal{D}_\varepsilon$ for all (σ,τ).

These complete the proof. □

Lemma 3.22 (Stokes theorem in H-direction)**.** (1) *Let $f\in\mathbb{D}^{1,q}(W^d)$, where $q>1$. Then for any C^1-curve $h=h(\tau)$ $(0\le\tau\le 1)$ on H, we have*

$$f(w+h(1))=f(w+h(0))+\int_0^1\Big((Df)(w+h(t)),\dot h(t)\Big)_H dt\quad \mu\text{-almost all } w. \tag{3.111}$$

(2) *Let $\beta\in\mathbb{D}^{1,q}(W^d,H^*)$, where $q>1$. Let $\mathcal{H}=\mathcal{H}(\sigma,\tau)\in H$ $(\sigma,\tau)\in[0,1]^2$ be a C^2-map. We assume that $\mathcal{H}(\sigma,0)=\mathcal{H}(0,0)$ and $\mathcal{H}(\sigma,1)=\mathcal{H}(0,1)$ for all $0\le\sigma\le 1$. Then it holds that*

$$\begin{aligned}&\int_0^1(\beta(w+\mathcal{H}(1,\tau)),\partial_\tau\mathcal{H}(1,\tau))\,d\tau-\int_0^1(\beta(w+\mathcal{H}(0,\tau)),\partial_\tau\mathcal{H}(0,\tau))\,d\tau\\&\quad=\iint_{(\sigma,\tau)\in[0,1]^2}(d\beta)(w+\mathcal{H}(\sigma,\tau))\,(\partial_\sigma\mathcal{H}(\sigma,\tau),\partial_\tau\mathcal{H}(\sigma,\tau))\,d\sigma d\tau\quad \mu\text{-almost all } w.\end{aligned}\tag{3.112}$$

Proof. (1) This is trivial for $f\in\mathfrak{F}C_b^\infty(W^d)$. General cases follows from the limiting argument.
(2) First we assume that $\beta\in\mathfrak{F}C_b^\infty(W^d,H^*)$. By the definition of the exterior differential, we have

$$d\beta(w)(X,Y)=((D\beta)(w)[X],Y)-((D\beta)(w)[Y],X),$$

where $X,Y\in H$. Here $(D\beta)(w)[X]$ denotes the derivative in the direction to X. Let $\phi(\sigma)=\int_0^1(\beta(w+\mathcal{H}(\sigma,\tau)),\partial_\tau\mathcal{H}(\sigma,\tau))\,d\tau$. We have

$$\begin{aligned}\dot\phi(\sigma)&=\int_0^1((D\beta)(w+\mathcal{H}(\sigma,\tau))[\partial_\sigma\mathcal{H}(\sigma,\tau)],\partial_\tau\mathcal{H}(\sigma,\tau))\,d\tau+\int_0^1(\beta(w+\mathcal{H}(\sigma,\tau)),\partial_\sigma\partial_\tau\mathcal{H}(\sigma,\tau))\,d\tau\\&=\int_0^1(d\beta)(w+\mathcal{H}(\sigma,\tau))(\partial_\sigma\mathcal{H}(\sigma,\tau),\partial_\tau\mathcal{H}(\sigma,\tau))d\tau\\&\quad+\int_0^1((D\beta)(w+\mathcal{H}(\sigma,\tau))[\partial_\tau\mathcal{H}(\sigma,\tau)],\partial_\sigma\mathcal{H}(\sigma,\tau))\,d\tau+\int_0^1(\beta(w+\mathcal{H}(\sigma,\tau)),\partial_\sigma\partial_\tau\mathcal{H}(\sigma,\tau))\,d\tau\end{aligned}$$

and

$$\begin{aligned}&\int_0^1((D\beta)(w+\mathcal{H}(\sigma,\tau))[\partial_\tau\mathcal{H}(\sigma,\tau)],\partial_\sigma\mathcal{H}(\sigma,\tau))\,d\tau+\int_0^1(\beta(w+\mathcal{H}(\sigma,\tau)),\partial_\sigma\partial_\tau\mathcal{H}(\sigma,\tau))\,d\tau\\&\quad=(\beta(w+\mathcal{H}(\sigma,1)),\partial_\sigma\mathcal{H}(\sigma,1))-(\beta(w+\mathcal{H}(\sigma,0)),\partial_\sigma\mathcal{H}(\sigma,0))=0.\end{aligned}\tag{3.113}$$

Therefore we get

$$\begin{aligned}\phi(1)-\phi(0)&=\int_0^1\dot\phi(\sigma)d\sigma\\&=\iint_{(\sigma,\tau)\in[0,1]^2}(d\beta)(w+\mathcal{H}(\sigma,\tau))\,(\partial_\sigma\mathcal{H}(\sigma,\tau),\partial_\tau\mathcal{H}(\sigma,\tau))\,d\sigma d\tau.\end{aligned}\tag{3.114}$$

By the limiting argument, we complete the proof. □

4 A retract map in a Wiener space

Let $X(t,a,w)$ be the solution of the SDE which is defined in Theorem 3.18. Let

$$S = \{w \in \Omega \mid X(1,e,w) = e\}.$$

Proposition 4.1. (1) *It holds that* $X_*\mu_e = \nu_e$.
(2) *For* $\alpha \in \mathfrak{F}C_b^\infty(\wedge^p T^* L_e(G))$*, define*

$$X^*\alpha(w) = \alpha(X(w))(DX(w),\cdots,DX(w)).$$

It holds that

$$(R_{X(t,e,w)})_*^{-1} DX(t,e,w)[h] = \int_0^t Ad\left(X(s,e,w)\right)\dot{h}(s)ds$$

and

$$\begin{aligned}
\|(X^*\alpha)(w)\|_{\otimes^p T_w^* S} &= \|\alpha(X(w))\|_{\otimes^p T^*_{X(w)} L_e(G)} && (4.1)\\
\nabla_S (X^*\alpha)(w) &= X^*(\nabla\alpha)(w) && (4.2)\\
\|\nabla_S^k (X^*\alpha)(w)\|_{\otimes^{p+k} T_w^* S} &= \|\nabla^k \alpha(X(w))\|_{\otimes^{p+k} T^*_{X(w)} L_e(G)} && (4.3)\\
\|X^*\alpha\|_{k,q} &= \|\alpha\|_{k,q}. && (4.4)
\end{aligned}$$

We construct a map $\Psi : \mathcal{D}_\varepsilon \to S$ using flows which are defined by a family of ODE on Ω. Let $a \in B_\varepsilon(e)$. We consider a right-invariant vector field $A_a = -\log a \in \mathfrak{g}$ and define a vector field on Ω:

$$V_a(w) = -\int_0^{\cdot} Ad\left(X(s,e,w)^{-1}\right)(\log a)ds.$$

Let $h(\tau,a,w)$ be the solution to

$$\begin{aligned}
\frac{d}{d\tau} h(\tau,a,w) &= V_a\left(w + h(\tau,a,w)\right) && (4.5)\\
h(0,a,w) &= 0. && (4.6)
\end{aligned}$$

Let $\Psi(\tau,a,w) = w + h(\tau,a,w)$ and $\Psi_a(w) = \Psi(1,a,w) = w + h(1,a,w)$. If $X(1,e,w) = a$, then $X(1,e,\Psi(\tau,a,w)) = e^{-\tau \log a} a$ $(0 \le \tau \le 1)$ holds. We define

$$\Psi(w) = \Psi_{X(1,e,w)}(w) \qquad w \in \mathcal{D}_\varepsilon. \tag{4.7}$$

For $\theta \in \mathfrak{F}C_b^\infty(\wedge^p T^* S)$, we define

$$(\Psi^*\theta)(w) = \theta(\Psi(w))(D\Psi(w)\cdot,\cdots,D\Psi(w)\cdot).$$

Note that $\|D\Psi\|_{L^\infty(H,T_w S)} < \infty$.

Lemma 4.2. (1) *Let* $q > 1$. *For any* $\eta \in \mathbb{D}^\infty(W^d)$*, it holds that*

$$\begin{aligned}
&\int_{\mathcal{D}_\varepsilon} |\Psi^*\theta(w)|^q \eta(w) d\mu(w)\\
&= \int_{B_\varepsilon(e)} da \int_S d\mu_e(w) |\theta(w)\left(D\Psi(\Psi(-1,a,w))\cdot,\cdots,D\Psi(\Psi(-1,a,w))\cdot\right)|^q\\
&\quad \times \eta\left(\Psi(-1,a,w)\right)\exp\left(\int_0^1 \xi_a\left(\Psi(-\tau,a,w)\right)d\tau\right),
\end{aligned} \tag{4.8}$$

where

$$\xi_a(w) \quad = \quad -\left(\int_0^1 Ad(X(s,e,w)dw_s, \log a\right). \tag{4.9}$$

In particular $\|\Psi^*\theta\|_{L^q(\mathcal{D}_\varepsilon,\mu)} \le C_{q,r}\|\theta\|_{L^r(S,\mu_e)}$ *for any* $1<q<r$.
(2) *Let* χ *be a smooth function on* G *such that* $\chi=1$ *near* e *and* supp $\chi \subset B_\varepsilon(e)$. *Set* $\hat{\chi}(w)=\chi(X(1,e,w))$. *Define* $T_\chi\theta=\hat{\chi}\Psi^*\theta$ *for* $\theta\in\mathfrak{F}C_b^\infty(\wedge^pT^*S)$. *Then* T_χ *can be extended uniquely to a bounded linear operator from* $\mathbb{D}^{k,r}(\wedge^pT^*S)$ *to* $\mathbb{D}^{k,q}(\wedge^pT^*W^d)$ *for any* $1<q<r$ *and* $k\in\mathbb{N}\cup\{0\}$. *The pull-back* $\iota^*\theta\in\mathbb{D}^{k,q}(\wedge^pT^*S)$ *is well-defined for sufficiently regular p-form* θ *on* W^d *with* $\|\theta\|_{k,r}<\infty$ *for sufficiently large* k *and any* $1<q<r$. *Moreover it holds that*

$$d_S\iota^*\theta=\iota^*d\theta. \tag{4.10}$$

(3) *For sufficiently large* k *and* $q>1$, *it holds that for any* $\theta\in\mathbb{D}^{k,q}(\wedge^pT^*S)$

$$\iota^*T_\chi\theta=\theta. \tag{4.11}$$

(4) *Let* $\varphi\in H$ *and* $U_r(\varphi)\subset\mathcal{D}_\varepsilon$. *Then there exists a constant* C *which depends only on* r,φ *such that*

$$\|\Psi^*\theta\|_{L^2(U_r(\varphi))}\le C\|\theta\|_{L^2(\mu_e)}. \tag{4.12}$$

Proof. (1) follows from [10]. (2) follows from (1). (3) follows from $D\Phi(w)=P(w)$ on S. (4) follows from (1) and Theorem 3.18 (2). □

5 Proof of the main theorem

The following ergodicity of the Wiener measure is used to construct f in Theorem 2.1 by the local datum on $U_r(\varphi)$.

Lemma 5.1. *Let* A,B *be measurable subsets of* W^d *with* $\mu(A)>0$ *and* $\mu(B)>0$. *Then there exists* $h\in H$ *and a measurable subset* $A_0\subset A$ *such that* $\mu(A_0)>0$ *and* $A_0+h\subset B$.

Let χ be a nonnegative function such that $\chi(u)=1$ for $u\le 4\varepsilon^2/9$ and $\chi(u)=0$ for $u\ge 9\varepsilon^2/16$. Let $\hat{\chi}(w)=\chi\left(d(X(1,e,w),e)^2\right)$.

Lemma 5.2. *Let* θ *be the* 1*-form on* S *in Theorem* 2.1. *Let* $\beta=\Psi^*\theta$. *Let* $1<q<p$. *Then there exists a measurable function* F *on* $\mathcal{D}_\varepsilon$ *and* ρ_n $(n\in\mathbb{N})$ *on* Ω *such that the following hold.*
(1) ρ_n *is a bounded non-negative* ∞*-quasi-continuous function and* $\rho_n\in\mathbb{D}^\infty(W^d)$ *holds.*
(2) *For any* $r>1$ *and* $k\in\mathbb{N}$, $\lim_{n\to\infty}C_r^k(\{w\in\Omega\mid\rho_n(w)=1\}^c)=0$ *and* $\lim_{n\to\infty}\|\rho_n-1\|_{r,k}=0$.
(3) *There exists* $F_n\in\mathbb{D}^{1,2}(W^d)\cap\mathbb{D}^{\infty,q}(W^d)$ *such that* $F(w)=F_n(w)$ *and* $dF_n(w)=\beta(w)$ *for* μ*-almost all* w *of* $\{w\in\Omega\mid\rho_n(w)\ne 0\}\cap\mathcal{D}_{\varepsilon/2}$.
(4) *Let* $\hat{F}_n=\tilde{F}_n\rho_n\hat{\chi}$, *where* $\tilde{F}_n$ *is a* (q,∞)*-quasi-continuous version of* F_n. *It holds that* $\hat{F}_n\in\mathbb{D}^{1,2}(W^d)\cap\mathbb{D}^{\infty,q}(W^d)$ *and*

$$d\hat{F}_n=\beta\rho_n\hat{\chi}+\tilde{F}_nd\rho_n\hat{\chi}+\tilde{F}_n\rho_nd\hat{\chi}. \tag{5.1}$$

Proof. Let χ_0 be a smooth decreasing function on $\mathbb{R}$ such that $\chi_0(u) = 1$ for $u \le 9\varepsilon^2/4$ and $\chi_0(u) = 0$ for $u \ge 4\varepsilon^2$. Let $\gamma = T_{\chi_0}\theta$. Then $\gamma \in \mathbb{D}^{\infty,q}(W^d, H^*)$. Also note that $\gamma = \beta$ on $\mathcal{D}_\varepsilon$. Let $U_{\sqrt{2}\kappa_i}(\varphi_i)$ $(i = 1, 2, \ldots)$ be the covering of $\mathcal{D}_\varepsilon$ in Lemma 3.19 (4). Let us choose r_i such that $4\kappa_i/3 < r_i < \sqrt{2}\kappa_i$. Since $d\gamma = 0$ on $U_{\sqrt{2}\kappa_i}(\varphi_i)$ and $\gamma \in L^2(U_{\sqrt{2}\kappa_i}(\varphi_i))$, by Theorem 3.14, we see that there exist $g_i \in \mathbb{D}^{\infty,q}(W^d) \cap \mathbb{D}^{1,2}(W^d)$ such that $dg_i = \gamma$ on $U_{r_i}(\varphi_i)$. g_i on $U_{r_i}(\varphi_i)$ is not determined uniquely, in fact, there is an ambiguity of additive constant. Actually we prove that there are constants c_i and a measurable function F on $\mathcal{D}_\varepsilon$ such that $F(w) = g_i(w) + c_i$ almost all $w \in U_{r_i}(\varphi_i)$ for any i and r_i. First set $c_1 = 0$. We define c_i $(i \ge 2)$ inductively in the following way. Suppose that there exist $c_1, \ldots, c_i$ and a measurable function G_i on $\cup_{j=1}^i U_{r_j}(\varphi_j)$ such that $G_i(w) = g_j(w) + c_j$ almost all $w \in U_{r_j}(\varphi_j)$ for all $1 \le j \le i$. By Theorem 3.14, there exist $G_{i,j} \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ such that $G_{i,j}(w) = G_i(w)$ on $U_{r_j}(\varphi_j)$. We prove that for any $\{r'_j\}$ with $4r_j/3 < r'_j < r_j$ $(1 \le j \le i)$ there exists $H_i \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ such that $H_i(w) = G_i(w)$ on $\cup_{j=1}^i U_{r'_j}(\varphi_j)$ and $dH_i = \beta$ on $U_{r'_j}(\varphi_j)$ for all $1 \le j \le i$.

Note that there exist $\phi_j \in \mathbb{D}^\infty(W^d)$ $(1 \le j \le i+2)$ such that the following identity holds. For $1 \le j \le i$

$$\phi_j(w) = \begin{cases} 1 & w \in U_{r'_j+\varepsilon_j}(\varphi_j), \\ 0 & w \in U_{r'_j+\varepsilon'_j}(\varphi_j)^c \end{cases}$$

and

$$\phi_{i+1}(w) = \begin{cases} 0 & w \in \cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta'_j}(\varphi_j), \\ 1 & w \in \left(\cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta_j}(\varphi_j)\right)^c, \end{cases}$$

$$\phi_{i+2}(w) = \begin{cases} 1 & w \in \cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta'_j-\tau'_j}(\varphi_j), \\ 0 & w \in \left(\cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta'_j-\tau_j}(\varphi_j)\right)^c. \end{cases}$$

Here we choose positive numbers such that $0 < \delta_j < \delta'_j < \varepsilon_j < \varepsilon'_j$, $\varepsilon_j - \delta'_j - \tau'_j > 0$, $0 < \tau_j < \tau'_j$ and $r'_j + \varepsilon'_j < r_j$. These functions can be constructed explicitly in a similar way to $\tilde{\rho}(w)$ in the proof of Theorem 3.14 using molifiers. Since $\sum_{j=1}^{i+1} \phi_j(w) \ge 1$ for any $w \in \Omega$,

$$\tilde{\phi}_j(w) = \frac{\phi_j(w)}{\sum_{j=1}^{i+1} \phi_j(w)}$$

belongs to $\mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ and $\sum_{j=1}^{i+1} \tilde{\phi}_j(w) = 1$ for all $w \in \Omega$. This is a partition of unity associated with the covering of Ω:

$$U_{r'_j+\varepsilon'_j}(\varphi_j)\ (1 \le j \le i), \quad \left(\cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta'_j}(\varphi_j)\right)^c$$

Since $\phi_{i+2}(w)\phi_{i+1}(w) = 0$ for all $w \in \Omega$, we have

$$\begin{aligned} G_i(w)\phi_{i+2}(w) &= \sum_{j=1}^{i+1} G_i(w)\phi_{i+2}(w)\tilde{\phi}_j(w) \\ &= \sum_{j=1}^{i} G_{i,j}(w)\phi_{i+2}(w)\tilde{\phi}_j(w). \end{aligned} \tag{5.2}$$

Therefore $H_i = G_i\phi_{i+2}$ is the desired function. By using the existence of H_i and the H-simply connectedness of $\mathcal{D}_\varepsilon$, we prove the existence of a measurable function G_{i+1} on $\cup_{j=1}^{i+1} U_{r'_j}(\varphi_j)$ and a constant c_{i+1} such that $G_{i+1}(w) = G_i(w)$ for almost all $w \in \cup_{j=1}^{i} U_{r'_j}(\varphi_j)$ and $G_{i+1}(w) = g_{i+1}(w) + c_{i+1}$ for almost all $w \in U_{r'_{i+1}}(\varphi_{i+1})$. Since $\mu\Big((\cup_{j=1}^{i} U_{r'_j}(\varphi_j)) \cap U_{r'_{i+1}}(\varphi_{i+1})\Big) > 0$, there exists a piecewise linear path $\varphi \in H$, $\delta > 0$ and $1 \le i_0 \le i$ such that $U_\delta(\varphi) \subset U_{r'_{i+1}}(\varphi_{i+1}) \cap U_{r'_{i_0}}(\varphi_{i_0})$. Because $d(g_{i+1} - g_{i_0}) = 0$ on $U_\delta(\varphi)$, $g_{i+1}(w) - g_{i_0}(w)$ is equal to a constant almost all w on $U_\delta(\varphi)$. We choose c_{i+1} such that $g_{i+1}(w) + c_{i+1} = g_{i_0}(w) + c_{i_0} (= G_i(w))$ almost all $w \in U_\delta(\varphi)$. It suffices to prove that

$$g_{i+1}(w) + c_{i+1} = G_i(w) \qquad \text{for almost all } w \in \Big(\cup_{j=1}^{i} U_{r'_j}(\varphi_j)\Big) \cap U_{r'_{i+1}}(\varphi_{i+1}). \tag{5.3}$$

Suppose that there exists a positive measure set $B \subset U_{r_{i_1}}(\varphi_{i_1}) \cap U_{r_{i+1}}(\varphi_{i+1})$ for some $1 \le i_1 \le i$ and $c' > 0$ such that

$$|g_{i+1}(w) + c_{i+1} - G_i(w)| > c' \qquad \text{for all } w \in B.$$

By the ergodicity of the Wiener measure, there exists a positive measure subset $A \subset U_\delta(\varphi)$ and $h \in H$ such that $A + h \subset B$. Choose a point $\eta \in A$ such that $\mu(V_r(\eta) \cap A) > 0$ for all $r > 0$. By the H-connectivity of $\cup_{j=1}^{i} U_{r'_j}(\varphi_j)$ and $U_{r_{i+1}}(\varphi_{i+1})$, there exists two C^∞-curves $h(i,\tau)$ $(0 \le \tau \le 1)$ such that $h(i,0) = 0$, $h(i,1) = h$ $(i = 0,1)$ and $\eta + h(0,\tau) \subset \cup_{j=1}^{i} U_{r'_j}(\varphi_j)$ $\eta + h(1,\tau) \subset U_{r'_{i+1}}(\varphi_{i+1})$ for all $0 \le \tau \le 1$. By choosing δ to be a sufficiently small positive number, we have for all $0 \le \tau \le 1$,

$$V_\delta(\eta) + h(0,\tau) \quad \subset \quad \cup_{j=1}^{i} U_{r'_i}(\varphi_i) \tag{5.4}$$
$$V_\delta(\eta) + h(1,\tau) \quad \subset \quad U_{r'_{i+1}}(\varphi_{i+1}) \tag{5.5}$$

By the H-simply connectedness of $\mathcal{D}_\varepsilon$, there exists a C^∞-map $\mathcal{H} = \mathcal{H}(\sigma,\tau)$ $(0 \le \sigma, \tau \le 1)$ such that $\mathcal{H}(0,\tau) = h(0,\tau)$, $\mathcal{H}(1,\tau) = h(1,\tau)$ and $\eta + \mathcal{H}(\sigma,\tau) \subset \mathcal{D}_\varepsilon$ for all $(\sigma,\tau) \in [0,1]^2$. Using the continuity of $X(1,e,\cdot)$ in the topology of d_Ω, we see that there exists $0 < \delta' < \delta$ such that for all $0 \le \sigma, \tau \le 1$ $V_{\delta'}(\eta) + \mathcal{H}(\sigma,\tau) \subset \mathcal{D}_\varepsilon$. Note that $dg_{i+1} = \beta$ on $U_{r'_{i+1}}(\varphi_{i+1})$ and $dH_i = \beta$ on $\cup_{j=1}^{i} U_{r'_i}(\varphi_i)$. By applying Lemma 3.22 and noting that $d\beta = 0$ on $\mathcal{D}_\varepsilon$, we obtain

$$(g_{i+1}(w+h) + c_{i+1}) - (g_{i+1}(w) + c_{i+1}) = G_i(w+h) - G_i(w) \qquad \text{for almost all } w \in A \cap V_{\delta'}(\eta).$$

This is a contradiction. This implies (5.3). Inductively, we obtain a measurable function F on $\mathcal{D}_\varepsilon$ such that for any i $F(w) = g_i(w) + c_i$ for some c_i and there exists $H_i \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ such that $F(w) = H_i(w)$ for almost all $w \in \cup_{j=1}^{i} U_{r_j}(\varphi_j)$. Let χ_1 be a non-negative smooth non-increasing function such that $\chi_1(u) = 1$ for $u \le (1/2)^m$ and $\chi_1(u) = 0$ for $u \ge (2/3)^m$. Let

$$\begin{aligned}
\chi_{n,2}(w) &= \chi_1\left(n^{-m}\left(\sum_{1\le i,j\le d} \|C(w^i,w^j)\|_{m,\theta}^m + \sum_{1\le k\le d} \|w^k\|_{m,\theta'/2}^m\right)\right),\\
\chi_{\kappa,N,3}(w) &= \chi_1\Bigg(\kappa^{-m}\Bigg(\sum_{k=1}^{n} \|w(N)^{\perp,k}\|_{m,\theta'/2}^m + \sum_{1\le i<j\le d} \|C(w(N)^{\perp,i}, w(N)^{\perp,j})\|_{m,\theta}^m\\
&\qquad + \sum_{1\le i\le j\le d} \|C(w(N)^i, w(N)^{\perp,j})\|_{m,\theta}^m + \sum_{1\le i\le j\le d} \|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta}^m\Bigg)\Bigg),
\end{aligned}$$

and set $\chi_{n,\kappa,N,4}(w) = \chi_{n,2}(w)\chi_{\kappa,N,3}(w)$. Then we have $\{\chi_{n,\kappa,N,4}(w) \neq 0\} \cap \mathcal{D}_{\varepsilon_2} \subset \mathcal{D}_{\varepsilon_2,n,N,\kappa}$. Now choosing $\kappa = \kappa(n)$ to be sufficiently small according to n as in Lemma 3.19, we have for sufficiently large $L_0 \in \mathbb{N}$,

$$\mathcal{D}_{\varepsilon_2,n,N,\kappa(n)} \subset \cup_{i=1}^{L_0} U_{4\kappa_i/3}(\varphi_i).$$

Therefore letting $N = a(\kappa(n))$ to be a sufficiently large natural number according to $\kappa = \kappa(n)$, $\rho_n(w) = \chi_{n,\kappa(n),a(\kappa(n)),4}(w)$ satisfies the properties (1), (2). As for (3), it suffices to set $F_n = H_i$ for sufficiently large i. (4) follows from (3). □

We prove main theorems.

Proof of Theorem 2.1. Let ρ_n be the function in Lemma 5.2. Then (1) holds. Let $f_n = \hat{F}_n$. We construct f on S. Let $C_n = \{\rho_n \neq 0\} \cap \mathcal{D}_{\varepsilon/2}$. By Lemma 5.2 (2), $\lim_{n\to\infty} \mu_e(C_n^c) = 0$. For $n, n' \in \mathbb{N}$, we have

$$\tilde{F}_n(w) = \tilde{F}_{n'}(w) = F(w) \quad \text{for } \mu\text{-almost all } w \text{ of } C_n \cap C_{n'}. \tag{5.6}$$

Hence there exists a Borel measurable subset $B_{n,n'}$ such that $C_q^k(B_{n,n'}) = 0$ and

$$\tilde{F}_n(w) = \tilde{F}_{n'}(w) \qquad \text{for all } w \in C_n \cap C_{n'} \cap B_{n,n'}^c. \tag{5.7}$$

This implies that $\tilde{F}_n(w) = \tilde{F}_{n'}(w)$ for μ_e-almost all $w \in C_n \cap C_{n'} \cap S$. Therefore there exists a measurable function f on S

$$f(w) = \tilde{F}_n(w) \qquad \text{for } \mu_e\text{-almost all } w \in C_n \cap S. \tag{5.8}$$

For this f and f_n, (2) (i), (ii) holds. Here we have used Lemma 5.2 (2). We prove (2) (iii). Note that $f\rho_n\eta = f_n\rho_n\eta \in \mathbb{D}^{\infty,q-}(W^d)$. Hence by Theorem 4.3 in [32], we have $f\rho_n\eta \in L^1(S,\mu_e)$. The equation in (2) (iv) is equivalent to

$$\int_S f_n\rho_n d_S^*(\rho_n\eta)\, d\mu_e = \int_S \left(d_S(f_n\rho_n), \rho_n\eta\right) d\mu_e$$

which follows from the integration by parts formula on S. We prove (2) (v). By the integration by parts formula on S, we have

$$\int_S \psi_K'(\hat{F}_n(w)) \Big(d_S\hat{F}_n(w), \eta(w)\Big)\, d\mu_e(w) = \int_S \psi_K(\hat{F}_n(w)) d_S^*\eta(w) d\mu_e(w). \tag{5.9}$$

By Lemma 5.2 (4), we get

$$d_S\hat{F}_n = \theta\rho_n + \tilde{F}_n d\rho_n. \tag{5.10}$$

Substituting (5.10) into (5.9) and replacing η by $\rho_n\eta$, we have

$$\int_S \psi_K'\left(f(w)\rho_n(w)\right)\Big(\theta(w)\rho_n(w) + f(w)d\rho_n(w), \rho_n(w)\eta(w)\Big) d\mu_e(w)$$
$$= \int_S \psi_K\left(f(w)\rho_n(w)\right) d_S^*\left(\rho_n\eta\right)(w) d\mu_e(w). \tag{5.11}$$

Here we have used that $f(w) = \tilde{F}_n(w)$ μ_e-almost all w on $\{\rho_n \neq 0\}$. Letting $n \to \infty$, we obtain

$$\int_S \psi_K'(f(w))(\theta(w), \eta(w)) d\mu_e(w) \quad = \quad \int_S \psi_K(f(w)) d_S^*\eta(w) d\mu_e(w). \tag{5.12}$$

This implies that the weak derivative of $\psi_K(f)$ is $\psi_K'(f)\theta$. Since $\left(d_S^*d_S, \mathfrak{F}C_b^\infty(W^d)\right)$ is essentially self-adjoint (see [1], [2]), $\psi_K(f) \in \mathbb{D}^{1,2}(S)$ and $d_S\psi_K(f) = \psi_K'(f)\theta$. $\square$

We need the Weitzenböck formula for $\square$ to prove Theorem 2.3.

Lemma 5.3. *Let $\mathbf{\Delta} = \sum_{i=1}^d (\mathrm{ad}\varepsilon_i)^2$, where $\{\varepsilon_i\}$ denotes an orthonormal system of $\mathfrak{g}$. Then*

$$
\begin{aligned}
(\square\alpha, h) &= \left(\nabla_{\mu_e}^*\nabla\alpha + \alpha + T_{b_1}\alpha, h\right) \\
&\quad + \int_0^1 ((\mathbf{\Delta}\alpha)_t, h_t)\, dt - \int_0^1\int_0^1 (\mathbf{\Delta}\alpha_t, h_s)\, dtds,
\end{aligned} \tag{5.13}
$$

where $(T_v\alpha)_t = \int_0^t [\alpha_s, v]ds - t\int_0^1 [\alpha_s, v]ds$ and $b_t = \int_0^t Ad(X(s,e,w)) \circ dw_s$. $[\cdot,\cdot]$ denotes the Lie bracket.

For simplicity we denote

$$
\square = \nabla_{\mu_e}^*\nabla + I + T_{b(1)} + T_2 + T_3,
$$

where T_2, T_3 is 0-order operator acting on 1-forms corresponding to the terms $\int_0^1 ((\mathbf{\Delta}\alpha)_t, h_t)\, dt$ and $-\int_0^1\int_0^1 (\mathbf{\Delta}\alpha_t, h_s)\, dtds$ respectively.

Proof of Theorem 2.3. Let $\alpha \in L^2(\wedge^1 T^*L_e(G))$ and assume that $\square\alpha = 0$ in weak sense. We need to show that $\alpha \in \cap_{1<p<2}\mathbb{D}^{\infty,p}(\wedge^1 T^*L_e(G))$. Let $\theta \in \mathfrak{F}C_b^\infty(\wedge^1 T^*S)$. Then

$$
\begin{aligned}
\left(\alpha, \nabla_{\mu_e}^*\nabla\theta\right) &= \left(\alpha, (\square - I - T_{b(1)} - T_2 - T_3)\theta\right) \\
&= -\left((I + T_{b(1)}^* + T_2^* + T_3^*)\alpha, \theta\right).
\end{aligned} \tag{5.14}
$$

Since $b_1 \in \cap_{p>1} L^p(S, d\mu_e)$, this implies the weak derivative $\nabla_{\mu_e}^*\nabla\alpha$ belongs to $\cap_{1<p<2}L^p(\wedge^1 T^*S)$. Hence by Theorem 2.16 in [2], $\alpha \in \cap_{1<p<2}\mathbb{D}^{2,p}(\wedge^1 T^*S)$. This implies $\alpha \in \cap_{1<p<2}\mathbb{D}^{\infty,p}(\wedge^1 S)$. Also note that $d\alpha = 0$. Thus $\bar{\alpha} = X^*\alpha \in \mathbb{D}^{1,2}(\wedge^1 S) \cap_{1<p<2} \mathbb{D}^{\infty,p}(\wedge^1 S)$ and $d_S\bar{\alpha} = 0$. Let f and f^K be the function in Theorem 2.1 which is obtained by replacing θ by $\bar{\alpha}$. Then $d_S f^K = \psi_K'(f)\bar{\alpha}$ on S. Note that $\bar{\alpha}$ satisfies the equation $d_S^*\bar{\alpha} = 0$. Hence we have

$$
\begin{aligned}
\int_S \|\bar{\alpha}(w)\|_{T_wS}^2 d\mu_e(w) &= \lim_{K\to\infty}\int_S \left(\bar{\alpha}(w), \psi_K'(f)\bar{\alpha}(w)\right)_{T_wS} d\mu_e(w) \\
&= \lim_{K\to\infty}\int_S d_S^*\bar{\alpha}(w) f^K(w) d\mu_e(w) \\
&= 0.
\end{aligned} \tag{5.15}
$$

This implies $\bar{\alpha} = 0$ and $\alpha = 0$ and completes the proof. $\square$

We give the proof of Weitzenböck formula. Note that this calculation is essentially similar to that of Γ_2 of the Dirichlet form in [13, 29]. We recall the basic results in [2].

Lemma 5.4. *Let X_h be the right-invariant vector field corresponding to $h \in H$.*
(1) *We have*

$$
\int_{L_e(G)} X_h f g d\nu_e = \int_{L_e(G)} f\left(-X_h g + (h,b)g\right) d\nu_e. \tag{5.16}
$$

Here $(h,b) = \int_0^1 (\dot{h}(s), db(s))$ is the restriction of the stochastic integral.

(2) *For any $h, k \in H$,*

$$\nabla_{X_h} X_k = X_{-P_0 \int_0^{\cdot} [h_s, \dot{k}_s] ds}, \tag{5.17}$$

where $P_0 h = h_t - t h_1$.
(3) *For any $h, k \in H_0$,*

$$[X_h, X_k] = X_{[k,h]},$$

where $[X_h, X_k]$ is the Lie bracket of the vector field on $L_e(G)$.

Proof of Lemma 5.3 We fix a complete orthonormal system $\{e_i\}$ of H_0. By Lemma 5.4, for any smooth 1-form α on $L_e(G)$,

$$d^*\alpha = \sum_i \left(-X_{e_i}\alpha(e_i) + (e_i, b)\alpha(e_i)\right). \tag{5.18}$$

Let β be a smooth 2-form on $L_e(G)$. By Lemma 5.4,

$$(d^*\beta)(e_k) = -\sum_i X_{e_i}\beta(e_i, e_k) + \sum_i (e_i, b)\beta(e_i, e_k) - \sum_{i<j} \beta(e_i, e_j)\left([e_j, e_i], e_k\right). \tag{5.19}$$

Using these, we have for $h \in H_0$

$$\begin{aligned} \left(\left(d^*d + dd^*\right)\alpha\right)(h) &= -\sum_i X_{e_i} X_{e_i}\alpha(h) + \sum_i (e_i, b) X_{e_i}\alpha(h) + \alpha(h) \\ &\quad + \alpha\left(P_0 \int_0^{\cdot} [h_s, db_s]\right) - \sum_i (e_i, b)\alpha([e_i, h]) \\ &\quad + \sum_i X_{[h,e_i]}\alpha(e_i) + \sum_i X_{e_i}\alpha([h, e_i]) - \sum_{i<j}\left(X_{e_i}\alpha(e_j) - X_{e_j}\alpha(e_i)\right)\left([e_j, e_i], h\right) \\ &\quad + \sum_{i<j} \alpha\left([e_j, e_i]\right)\left([e_j, e_i], h\right). \end{aligned} \tag{5.20}$$

By the definition of the covariant derivative, we have

$$\begin{aligned} (\nabla^*_{\nu_e}\nabla\alpha)(h) &= -\sum_i X_{e_i} X_{e_i}\alpha(h) + \sum_i (e_i, b) X_{e_i}\alpha(h) - \sum_i (e_i, b)\alpha(\nabla_{e_i} h) \\ &\quad + 2\sum_i X_{e_i}\alpha(\nabla_{e_i} h) - \sum_i \alpha(\nabla_{e_i}\nabla_{e_i} h). \end{aligned} \tag{5.21}$$

Hence

$$\begin{aligned} \left(\left(d^*d + dd^*\right)\alpha\right)(h) &= \left(\nabla^*_{\nu_e}\nabla\alpha\right)(h) + \alpha(h) + \sum_i \alpha(\nabla_{e_i}\nabla_{e_i} h) \\ &\quad + \frac{1}{2}\sum_{i,j} \alpha\left([e_j, e_i]\right)\left([e_j, e_i], h\right) + I_1 + I_2. \end{aligned} \tag{5.22}$$

Here

$$I_1 = \alpha\left(P_0 \int_0^{\cdot} [h_s, db_s]\right) - \sum_i (e_i, b)\alpha([e_i, h]) + \sum_i (e_i, b)\alpha(\nabla_{e_i} h), \tag{5.23}$$

$$
\begin{aligned}
I_2 &= \sum_i X_{[h,e_i]}\alpha(e_i) + \sum_i X_{e_i}\alpha([h,e_i]) - \sum_{i<j}\big(X_{e_i}\alpha(e_j) - X_{e_j}\alpha(e_i)\big)([e_j,e_i],h) \\
&\quad -2\sum_i X_{e_i}\alpha(\nabla_{e_i}h).
\end{aligned}
\tag{5.24}
$$

By the explicit calculation, $I_1 = (T_{b(1)}\alpha)(h)$ and $I_2 = 0$. We calculate $\frac{1}{2}\sum_{i,j}\alpha([e_j,e_i])([e_j,e_i],h)$ and $\sum_i \alpha(\nabla_{e_i}\nabla_{e_i}h)$.

$$
\begin{aligned}
\sum_i \alpha(\nabla_{e_i}\nabla_{e_i}h) &= \sum_i \int_0^1 \left(\dot{\alpha}_t, -\left[e_i(t), [e_i(t), \dot{h}_t] - \int_0^1 [e_i(s), \dot{h}_s]ds\right]\right) dt \\
&= -\sum_i \int_0^1 \Big([\dot{\alpha}_t, e_i(t)], [\dot{h}_t, e_i(t)]\Big)\, dt \\
&\quad + \sum_i \left(\int_0^1 [\dot{\alpha}_t, e_i(t)]\, dt, \int_0^1 \Big[\dot{h}_s, e_i(s)\Big]\, ds\right).
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,j}\alpha([e_j,e_i])([e_j,e_i],h) \\
&= \sum_i \int_0^1 \left([\dot{\alpha}_t, e_i(t)] - \int_0^1 [\dot{\alpha}_t, e_i(t)]dt, [\dot{h}_t, e_i(t)] - \int_0^1 [\dot{h}_t, e_i(t)]dt\right) dt \\
&- \sum_i \int_0^1 \left([\dot{\alpha}_t, e_i(t)] - \int_0^1 [\dot{\alpha}_t, e_i(t)]dt, \int_0^t [\dot{h}_s, \dot{e}_i(s)]ds - \int_0^1 \left(\int_0^u [\dot{h}_s, \dot{e}_i(s)]ds\right) du\right) dt.
\end{aligned}
\tag{5.26}
$$

Thus

$$
\begin{aligned}
&\sum_i \alpha(\nabla_{e_i}\nabla_{e_i}h) + \frac{1}{2}\sum_{i,j}\alpha([e_j,e_i])([e_j,e_i],h) \\
&= -\sum_i \int_0^1 \left([\dot{\alpha}_t, e_i(t)] - \int_0^1 [\dot{\alpha}_t, e_i(t)]dt, \int_0^t [\dot{h}_s, \dot{e}_i(s)]ds\right) dt \\
&= -\sum_{i=1}^d \left(\int_0^1 [[\alpha_t, \varepsilon_i], \varepsilon_i]\, dt, \int_0^1 h_t dt\right) - \sum_{i=1}^d \int_0^1 ([\alpha_t, \varepsilon_i], [h_t, \varepsilon_i])\, dt \\
&= -\left(\int_0^1 (\boldsymbol{\Delta}\alpha)_t dt, \int_0^1 h_t dt\right) + \int_0^1 (\boldsymbol{\Delta}\alpha_t, h_t)\, dt.
\end{aligned}
\tag{5.27}
$$

This completes the proof. □

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