Characterization of Cobweb Posets as KoDAGs

Ewa Krot-Sieniawska
Institute of Computer Science, Białystok University
PL-15-887 Białystok, st.Sosnowa 64, POLAND
e-mail: ewakrot@wp.pl, ewakrot@ii.uwb.edu.pl

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Abstract

The characterization of the large family of cobweb posets as DAGs and oDAGs is given. The *dim* 2 poset such that its Hasse diagram coincide with digraf of arbitrary cobweb poset Π is constructed.

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1 Introduction

The family of the so called cobweb posets Π has been invented by A.K.Kwaśniewski in [1, 3]. These structures are such a generalization of the Fibonacci tree growth that allows joint combinatorial interpretation for all of them under the admissibility condition (for references to Kwaśniewski papers see recent [4, 5]).
Let $\{F_n\}_{n\geq 0}$ be a natural numbers valued sequence with $F_0 = 1$ (with $F_0 = 0$ being exceptional as in case of Fibonacci numbers). Any sequence satisfying this property uniquely designates cobweb poset defined as follows.
For $s \in \mathbf{N_0} = \mathbf{N} \cup \{\mathbf{0}\}$ let

$$\Phi_s = \{\langle j, s\rangle,\ \ 1 \leq j \leq F_s\},$$

Then corresponding cobweb poset is an infinite partially ordered set $\Pi = (V, \leq)$, where

$$V = \bigcup_{0\leq s} \Phi_s$$

are the elements (vertices) of Π and the partial order relation $\leq$ on V for $x = \langle s, t\rangle, y = \langle u, v\rangle$ being elements of cobweb poset Π is defined by formula

$$(x \leq_P y) \Longleftrightarrow [(t < v) \vee (t = v \wedge s = u)].$$

Obviously as any poset a cobweb poset can be represented, via its Hasse diagram and here it is an infinite directed graf $\Pi = (V, E)$, where set V of its vertices is defined as above and where

$$E = \{(\langle j, p\rangle, \langle q, (p+1)\rangle)\} \cup \{(\langle 1, 0\rangle, \langle 1, 1\rangle)\},$$

and where $1 \le j \le F_p$ and $1 \le q \le F_{(p+1)}$ stays for set of (directed) edges.
For example the Hasse diagram of Fibonacci cobweb poset designated by the famous Fibonacci sequence looks as follows.

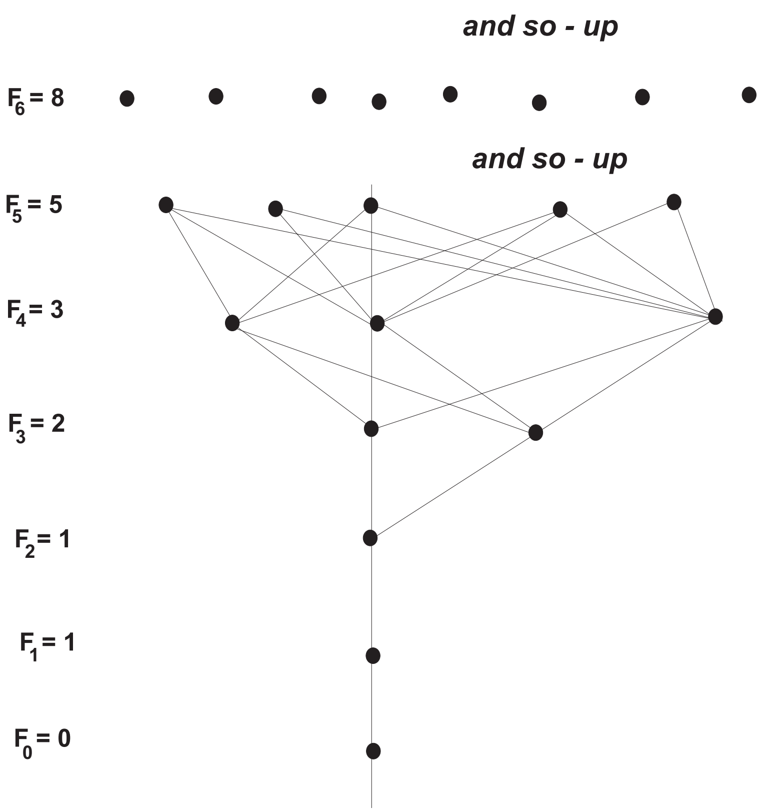

Fig. 1. The construction of the Fibonacci "cobweb" poset

The Kwaśniewski cobweb posets under consideration represented by graphs are examples of oderable directed acyclic graphs (oDAG) which we start to call from now in brief: KoDAGs. These are structures of universal importance for the whole of mathematics - in particular for discrete "'mathemagics"' [http://ii.uwb.edu.pl/akk/] and computer sciences in general (quotation from [4, 5]):

> DAGs considered as a generalization of trees have a lot of applications in computer science, bioinformatics, physics and many natural activities of humanity and nature. For example in information categorization systems, such as folders in a computer or in Serializability

Theory of Transaction Processing Systems and many others. Here we introduce specific DAGs as generalization of trees being inspired by algorithm of the Fibonacci tree growth. For any given natural numbers valued sequence the graded (layered) cobweb posets' DAGs are equivalently representations of a chain of binary relations. Every relation of the cobweb poset chain is biunivocally represented by the uniquely designated **complete** bipartite digraph-a digraph which is a di-biclique designated by the very given sequence. The cobweb poset is then to be identified with a chain of di-bicliques i.e. by definition - a chain of complete bipartite one direction digraphs. Any chain of relations is therefore obtainable from the cobweb poset chainof complete relations via deleting arcs (arrows) in di-bicliques. Let us underline it again : *any chain of relations is obtainable from the cobweb poset chain of complete relations via deleting arcs in di-bicliques of the complete relations chain.* For that to see note that any relation R_k as a subset of $A_k \times A_{k+1}$ is represented by a one-direction bipartite digraph D_k. A "complete relation" C_k by definition is identified with its one direction di-biclique graph $d - B_k$. Any R_k is a subset of C_k. Correspondingly one direction digraph D_k is a subgraph of an one direction digraph of $d - B_k$.
The one direction digraph of $d - B_k$ is called since now on **the di-biclique** i.e. by definition - a complete bipartite one direction digraph. Another words: cobweb poset defining di-bicliques are links of a complete relations' chain.

2 DAG $\longrightarrow$ oDAG problem

In [6] Anatoly D. Plotnikov considered the so called "DAG $\longrightarrow$ oDAG problem". He had determined condition when a digraph G may may be presented by the corresponding *dim* 2 poset R and he had established the algorithm for how to find it.
Before citing Plotnikov's results let us however recall some indispensable definitions following thesource reference [6].
Let P and Q be some partial orders on the same set A. Then Q is said to be an **extension** of P if $a \leq_P b$ implies $a \leq_Q b$, for all $a, b \in A$. A poset L is a **chain**, or a **linear order** if we have either $a \leq_L b$ or $b \leq_L a$ for any $a, b \in A$. If Q is a linear order then it is a **linear extension** of P.
The **dimension** *dim* R of R being a partial order is the least positive integer s for which there exists a family $F = (L_1, L_2, \dots, L_s)$ of linear extensions of R such that $R = \bigcap_{i=1}^{s} L_i$. A family $F = (L_1, L_2, \dots, L_s)$ of linear orders on A is called a **realizer** of R on A if

$$R = \bigcap_{i=1}^{s} L_i.$$

We shall use D_n to denote the set of all acyclic directed n-vertex graphs without loops and multiple edges. Each digraph $\vec{G} = (V, \vec{E}) \in D_n$ will be called **DAG**.

A digraph $\vec{G} \in D_n$ shall be called **orderable (oDAG)** if there exists are *dim* 2 poset such that its Hasse diagram coincides with the digraph $\vec{G}$.
Let $\vec{G} \in D_n$ be a digraph, which does not contain the arc (v_i, v_j) if there exists the directed path $p(v_i, v_j)$ from the vertex v_i into the vertex v_j for any v_i, $v_j \in V$. Such digraph is called **regular**. Let $D \subset D_n$ is the set of all regular graphs.
Let there be given a regular digraph $\vec{G} = (V, E) \in D$, and let the chain $\vec{X}$ has three elements x_{i_1}, x_{i_2}, $x_{i_3} \in X$ such that $i_1 < i_2 < i_3$, while simultaneously there are not paths $p(v_{i_1}, v_{i_2})$,$p(v_{i_2}, v_{i_3})$ in the digraph $\vec{G}$ but there exists a path $p(v_{i_1}, v_{i_3})$. Such representation of graph vertices by elements of the chain $\vec{X}$ is called the representation in **inadmissible form**. Another words- the chain $\vec{X}$ Constitutes the graph vertices in **admissible form**.
Anatoly Plotnikov then shows that:

Lemma 1. [6] *A digraph $\vec{G} \in D_n$ may be represented by a dim 2 poset if:*

(1) there exist two chains $\vec{X}$ and $\vec{Y}$, each of which is a linear extension of $\vec{G}_t$;

(2) the chain $\vec{Y}$ is a modification of $\vec{X}$ with inversions, which remove the ordered pairs of $\vec{X}$ that there do not exist in $\vec{G}$.

The above lemma results in the algorithm for finding *dim* 2 representation of a given DAG (i.e. corresponding oDAG) while the following theorem establishes the conditions for constructing it.

Theorem 2. [6] *A digraph $\vec{G} = (V, \vec{E}) \in D_n$ can be represented by dim 2 poset iff it is regular and its vertices can be presented by the chain $\vec{X}$ in admissible form.*

3 Kwaśniewski Cobweb Posets as KoDAGs

In this section we show that Kwaśniewski cobweb posets are orderable Directed Acyclic Graphs (oDAGs) hence: KoDAGs.
Obviously, arbitrary cobweb poset $\Pi = (V, E)$ defined as above is a DAG (it is directed acyclic graph without loops and multiple edges). One can also verify that

Proposition 3. *$\Pi = (V, E)$ is a regular digraph.*

Proof. For two elements $\langle i, n\rangle, \langle j, m\rangle \in V$ a directed path $p(\langle i, n\rangle, \langle j, m\rangle) \notin E$ will exist iff $n < m + 1$ but then $(\langle i, n\rangle, \langle j, m\rangle) \notin E$ i.e. Π does not contain the edge $(\langle i, n\rangle, \langle j, m\rangle)$. ∎

It is also possible to verify that vertices of cobweb poset Π can be presented in admissible form by the chain $\vec{X}$ being a linear extension of cobweb P as follows:

$$\vec{X} = \Big(\langle 1,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,4\rangle, \langle 1,5\rangle, \langle 2,5\rangle, \langle 3,5\rangle,$$
$$\langle 4,5\rangle, \langle 5,5\rangle, ...\Big),$$

where

$$(\langle s,t\rangle \leq_{\vec{X}} \langle u,v\rangle) \Longleftrightarrow [(t < v) \vee (t = v \wedge s \leq u)]$$

for $1 \leq s \leq F_t,\ 1 \leq u \leq F_v, \quad t,v \in \mathbf{N} \cup \{0\}$.
Then the cobweb poset Π satisfies conditions of the Theorem 2, so it is oDAG. In order to find the chain $\vec{Y}$ being a linear extension of cobweb P one uses the Lemma 1 thus arriving at:

$$\vec{Y} = \Big(\langle 1,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,4\rangle, \langle 2,4\rangle, \langle 1,4\rangle, \langle 5,5\rangle, \langle 4,5\rangle, \langle 3,5\rangle,$$
$$\langle 2,5\rangle, \langle 1,5\rangle, ...\Big),$$

where

$$(\langle s,t\rangle \leq_{\vec{Y}} \langle u,v\rangle) \Longleftrightarrow [(t < v) \vee (t = v \wedge s \geq u)]$$

for $1 \leq s \leq F_t,\ 1 \leq u \leq F_v, \quad t,v \in \mathbf{N} \cup \{0\}$ and finally

$$(P, \leq_P) = \vec{X} \cap \vec{Y}.$$

So we have the announced charcterization Theorem 4 being proved now.

Theorem 4. *An arbitrary cobweb poset $\Pi = (V,E)$ is an example od orderable directed acyclic digraph (oDAG) and $\Pi = (V, \leq) = X \cap Y$, for X, Y being linear extensions of partial order Π defined as above.*

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References

[1] A.K. Kwaśniewski, Cobweb posets as noncommutative prefabs, *Adv. Stud. Contemp. Math.*, **14**, 1 (2007), 37-47, ArXiv:math/0503286, cs.DM http://arxiv.org/abs/math/0503286

[2] A.K. Kwaśniewski, Prefab posets' Whitney numbers, *Bull. Soc. Sci. Lett. Lodz.*, **60** (2005), 25-33, ArXiv: math.CO/0510027, cs.DM http://arxiv.org/abs/math/0510027

[3] A.K. Kwaśniewski, First observations on Prefab posets‘ Whitney numbers, *Advances in Applied Clifford Algebras*, Volume 18, Number 1 / February, 2008, 57-73. ONLINE FIRST, Springer Link Date, August 10, 2007, arXiv:0802.1696, cs.DM http://arxiv.org/abs/0802.1696

[4] A.K. Kwaśniewski, On cobweb posets and their combinatorially admissible sequences, ArXiv:math.Co/0512578v4 21 Oct 2007, submitted to G&C Japan [in part presented in INDIA 15-17 December 2007 ISRAMA], cs.DM http://arxiv.org/abs/math/0512578

[5] A.K. Kwaśniewski, M. Dziemiaczuk, Cobweb posets - Recent Results, ISRAMA 2007, December 1-17 2007 Kolcata, INDIA, arXiv:0801.3985, cs.DM http://arxiv.org/abs/0801.3985

[6] A.D. Plotnikov, About presentation of a digraph by dim 2 poset, *Adv. Stud. Contemp. Math.* **12** (1) (2006), 55-60.