Notes on “On the restudy of fuzzy complex analysis: Part I and Part II”

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Abstract

In this note, we show by counterexamples that some results of Qiu and colleagues [On the restudy of fuzzy complex analysis: Part I. The sequence and series of fuzzy complex numbers and their convergences, Fuzzy Sets and Systems 115 (2000) 445–450; On the restudy of fuzzy complex analysis: Part II. The continuity and differentiation of fuzzy complex functions, Fuzzy Sets and Systems 120 (2001) 517–521.] concerning the convergence of the series of fuzzy complex numbers and the differentiation of fuzzy complex functions are incorrect, and offer some of their modified versions. We also show that the derivative introduced by them is inappropriate for the functions mapping complex numbers into fuzzy complex numbers.

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1. Introduction

In [5] Qiu et al. studied the convergence of the sequence and series of fuzzy complex numbers. They proved a proposition [5, Theorem 4.2] as follows:

Let \{\tilde{z}_n\}∞n=1 be a sequence in \(\tilde{C}\) and \(\tilde{S} \in \tilde{C}\). Then \(\sum_{n=1}^{\infty} \tilde{z}_n\) converges to \(\tilde{S}\) if and only if for every \(\tilde{z}_n \in [\tilde{z}_n]^0\): \(\sum_{n=1}^{\infty} \tilde{z}_n\) converges to \(w \in [\tilde{S}]^0\).

By introducing a differentiation to fuzzy complex functions, they got two conclusions in [6, Theorem 3.1 and Corollary 3.1] as follows, respectively:

The fuzzy complex function \(f : E \rightarrow \tilde{C}\) is differentiable at \(z_0 \in E\) (or \(f'(z_0) = \tilde{A} \in \tilde{C}\)) if and only if for any \(\alpha \in I\) the support function \(s(w, \alpha, z) = s(w, [f(z)]^\alpha)\) of \(f(z)\) is differentiable at \(z_0\) (or \(\partial/\partial z s(w, \alpha, z_0) = s(w, [\tilde{A}]^\alpha)\)) uniformly for \(w \in J\);

\(f : E \rightarrow \tilde{C}\) is differentiable at \(z_0 \in E\) if and only if for any \(\alpha \in I\) the set-valued mappings \(f^\alpha : E \rightarrow P_k(C)\), \(f^\alpha(z) = [f(z)]^\alpha\) is differentiable at \(z_0\) uniformly for \(w \in J\) and \(\partial/\partial z s(w, [f(z)]^\alpha)|_{z=z_0} = s(w, [f'(z_0)]^\alpha)\).

However, we will show that none of these assertions hold by counterexamples, and correct them in the sequel.

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2. Preliminaries

For the sake of simplicity, we use the same notations as in [5,6]. The symbol $P_k(\mathbb{C})$ denotes the family of all nonempty compact convex subsets of complex numbers $\mathbb{C}$. Define addition and scalar multiplication in $P_k(\mathbb{C})$ as usual. Recall that the Hausdorff metric is defined as

$$h(A, B) = \max \left\{ \sup_{z \in A} \inf_{w \in B} |z - w|, \sup_{w \in B} \inf_{z \in A} |z - w| \right\},$$

where $A, B \in P_k(\mathbb{C})$. We will write $z = x + iy$ for complex numbers and $\tilde{z}$ (or $\tilde{Z}$) for fuzzy complex numbers. $\tilde{z}$ is defined by its membership function $\mu(z|\tilde{z})$ which is a mapping from the complex numbers $\mathbb{C}$ into $[0, 1]$. An $\alpha$-cut of $\tilde{z}$ is $[\tilde{z}]^\alpha = \{ z \mid \mu(z|\tilde{z}) \geq \alpha \}$, for $0 < \alpha \leq 1$. We separately specify $[\tilde{z}]^0$ to be the closure of the union of $[\tilde{z}]^\alpha$ for $0 < \alpha \leq 1$.

**Definition 1 (Qiu et al. [5]).** $\tilde{z}$ is a fuzzy complex number if and only if:

1. $\mu(z|\tilde{z})$ is upper semi-continuous;
2. $[\tilde{z}]^2$ is compact for $0 \leq \alpha \leq 1$;
3. $[\tilde{z}]^1$ is non-empty;
4. $\tilde{z}$ is fuzzy convex i.e. $\mu((1-\alpha)z_1 + \alpha z_2|\tilde{z}) \geq \min\{\mu(z_1|\tilde{z}), \mu(z_2|\tilde{z})\}$ for all $z_1, z_2 \in \mathbb{C}, \lambda \in [0, 1]$.

Let $\tilde{\mathbb{C}}$ denote the set of fuzzy complex numbers. Denote $\tilde{\mathbb{C}} = \{ \tilde{z} \in \tilde{\mathbb{C}} \mid \tilde{z}$ is concave $\}$. In other words, if $\tilde{z} \in \tilde{\mathbb{C}}$ then $\mu((1-\alpha)z_1 + \alpha z_2|\tilde{z}) \geq \alpha \mu(z_1|\tilde{z}) + (1 - \alpha)\mu(z_2|\tilde{z})$ for all $z_1, z_2 \in [\tilde{z}]^0, \alpha \in [0, 1]$.

Now define a metric $H$ in $\tilde{\mathbb{C}}$ by the equation [5] (which was first introduced for fuzzy sets in [3,4])

$$H(\tilde{z}, \tilde{w}) = \sup_{0 \leq \alpha \leq 1} h([\tilde{z}]^\alpha, [\tilde{w}]^\alpha).$$

**Definition 2 (Qiu et al. [6]).** Let $E \subseteq \mathbb{C}$, then we say that $f : E \to \tilde{\mathbb{C}}$ is differentiable at $z_0 \in E$ if there exists $\tilde{A} \in \tilde{\mathbb{C}}$ such that for all $z \in E$ with $z \to z_0$, $f(z) - f(z_0)/z - z_0$ converges metrically to $\tilde{A}$ in $\tilde{\mathbb{C}}$, $H$, i.e. $\lim_{z \to z_0, z \in E} H(f(z) - f(z_0)/z - z_0, \tilde{A}) = 0$ and now write

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \tilde{A}$$

and denote $f'(z_0) = \tilde{A}$.

**Definition 3 (Qiu et al. [6]).** Let $I = [0, 1], J = \{ w \in \mathbb{C} \mid |w| = 1 \}, E \subseteq \mathbb{C}, A \in P_k(\mathbb{C}), g : E \to P_k(\mathbb{C}), f : E \to \tilde{\mathbb{C}}$ and $\tilde{z} \in \tilde{\mathbb{C}}$ then

1. for any $w \in J$ we say that $s(w) = s(w|A) = \sup_{u \in A} \langle w, u \rangle$ is a support function of $A$ where $\langle \cdot, \cdot \rangle$ is the real inner product in $\mathbb{C}$;
2. for any $w \in J$ and $z \in E$ we say that $s(w, z) = s(w|g(z)) = \sup_{u \in g(z)} \langle w, u \rangle$ is a support function of $g(z)$;
3. for any $w \in J$ and $z \in E$ we say that $s_2(w, z) = s(w|[\tilde{z}]^2) = \sup_{u \in [\tilde{z}]^2} \langle w, u \rangle$ is a support function of $[\tilde{z}]^2$;
4. for any $w \in J, x \in I$ and $z \in E$ we say that $s(w, x, z) = s(w|[f(z)]^2) = \sup_{u \in [f(z)]^2} \langle w, u \rangle$ is a support function of $[f(z)]^2$.

3. Main results

Theorem 4.2 in [5] is as follows:

Let $\{z_n\}^\infty_1$ be a sequence in $\tilde{\mathbb{C}}$ and $\tilde{S} \subseteq \tilde{\mathbb{C}}$. Then $\sum_{n=1}^\infty z_n$ converges to $\tilde{S}$ if and only if for every $z_n \in [\tilde{z}_n]^0, \sum_{n=1}^\infty z_n$ converges to $w \in [\tilde{S}]^0$. 

However, this is not true as shown below.

**Example 1.** Define

\[
\mu(z | \tilde{S}) = \begin{cases} z & \text{if } z \in (0, 1], \\ 0 & \text{elsewhere,} \end{cases} \quad \mu(z | \tilde{Z}) = \begin{cases} 1 - z & \text{if } z \in [0, 1], \\ 0 & \text{elsewhere.} \end{cases}
\]

and define the n-term of \([\tilde{z}_n]_1^\infty\) as

\[
\mu(z | \tilde{z}_n) = \begin{cases} 2^n z & \text{if } z \in (0, 2^{-n}], \\ 0 & \text{elsewhere.} \end{cases}
\]

Obviously, \(\tilde{S}, \tilde{Z}, \tilde{z}_n \in \tilde{\mathcal{C}}\). By a simple calculation we have \(\tilde{S} = \sum_{n=1}^{\infty} \tilde{z}_n \neq \tilde{Z}\) but \([\tilde{S}]^0 = [\tilde{Z}]^0 = [0, 1]\), which contradicts with Theorem 4.2 in [5]. We state below the correct form of this theorem.

**Theorem 1.** Let \([\tilde{z}_n]_1^\infty\) be a sequence in \(\tilde{\mathcal{C}}\) and \(\tilde{S} \in \tilde{\mathcal{C}}\). Then \(\sum_{n=1}^{\infty} \tilde{z}_n\) converges to \(\tilde{S}\) if and only if for every \(z_n \in [\tilde{z}_n]^2\), \(\sum_{n=1}^{\infty} z_n\) converges to \(w \in [\tilde{S}]^2\) for \(z \in [0, 1]\).

**Proof.** The proof of the sufficiency is just the original proof in [5]. Thus, we only prove the necessity.

Let \(\sum_{n=1}^{\infty} \tilde{z}_n = \tilde{S}\) i.e. \(\lim_{n \to \infty} H(\tilde{z}_n, \tilde{S}) = 0\), where \(\tilde{S}_n = \sum_{i=1}^{n} \tilde{z}_i\). By Theorem 3.1 in [5] we have \(h([\tilde{S}_n]^2, [\tilde{S}]^2)\) converges uniformly to zero for \(z \in [0, 1]\). Then by the conclusion in [4, p. 412], for all \(z \in [0, 1]\), we have

\[
[\tilde{S}]^2 = \liminf [\tilde{S}_n]^2 = \left\{ z \in \mathbb{C} \mid z = \sum_{n=1}^{\infty} z_n, \quad z_n \in [\tilde{z}_n]^2, \quad \text{for all } n = 1, 2, \ldots \right\}.
\]

We complete the whole proof here. \(\square\)

Just like Dubois and Prade have pointed out that the usual definition of the derivative is useless for fuzzy real functions [2], we will also point out that Definition 2 (that is, Definition 3.1 in [6]) is inappropriate for fuzzy complex functions.

Let \(A \in P_h(C)\), then define the diameter of \(A\) as \(diam A = \sup\{\mid z_1 - z_2\mid, \quad z_1, z_2 \in A\}\). Obviously, \(diam A = 0\) if and only if \(A\) is a singleton.

**Lemma 1.** Let \(A, B \in P_h(C)\), then \(diam (A \pm B) \geq\ max\{diam A, diam B\}\).

**Proof.** Write \(|A| = \{a \in \mathbb{R} \mid a = |z_1 - z_2|, \quad z_1, z_2 \in A\}\). For every \(a \in |A|\), that is, \(a = |z_1 - z_2|, \quad z_1, z_2 \in A\), we have \(z_1 \pm z, \quad z_2 \pm z \in (A \pm B)\), where \(z \in B\), which implies \(a = |z_1 - z_2| = \|z_1 \pm z - (z_2 \pm z)\| \in |A \pm B|\). Consequently, we have \(diam A = \sup_{a \in |A|} a \leq \sup_{a \in |A \pm B|} a = diam (A \pm B)\). The same is true for \(B\). \(\square\)

**Theorem 2.** If \(f : E \to \tilde{\mathbb{C}}\) is differentiable on \(E\) according to Definition 2, then \(f\) is an ordinary complex function, that is, \(f : E \to \mathbb{C}\).

**Proof.** If there exists a \(z_0 \in E\) such that \(f(z_0) \notin \mathbb{C}\) and \(f\) is differentiable at \(z_0\), then there exists an \(z_0\) such that \([f(z_0)]^2_0\) is not a singleton and there is an \(A \in \tilde{\mathbb{C}}\) such that \(\lim_{z \to z_0, z \in E} H(f(z) - f(z_0)/z - z_0, A) = 0\). By Lemma 1 and the conclusions in [5, p. 446], we have

\[
diam \left[ f(z) - f(z_0) \right]^{z_0}_{z - z_0} = \frac{diam([f(z)]^{2}_0 - [f(z_0)]^{2}_0)}{|z - z_0|} \geq\ diam[f(z_0)]^{2}_0/|z - z_0|.
\]

Since \(\lim_{z \to z_0} \diam[f(z_0)]^{2}_0/|z - z_0| = \infty\), we have \(\lim_{z \to z_0} diam[f(z) - f(z_0)/z - z_0]^{2}_0 = \infty\), which implies \(\lim_{z \to z_0} h([f(z) - f(z_0)/z - z_0]^{2}_0, [A]^{2}_0) = \infty\). This is a contradiction. \(\square\)
The following example shows that Theorem 3.1 and Corollary 3.1 in [6] do not hold in general.

**Example 2.** Define a fuzzy complex number $\tilde{z} \in \tilde{C}$ as

$$\mu(z, \tilde{z}) = \begin{cases} 1 - \alpha & \text{if } z = x e^{i\beta}, \text{ where } \alpha \in [0, 1], \beta \in [0, 2\pi], \\ 0 & \text{elsewhere} \end{cases}$$

and define the function $f : \mathbb{C} \to \tilde{C}$ as $f(z) \equiv \tilde{z}$ for all $z \in \mathbb{C}$. Then by a routine calculation we have $s(w, x, z) = s(w|f(z)|^2) \equiv 1 - \alpha$ for all $z \in \mathbb{C}, w \in J$ and $\alpha \in [0, 1]$. Thus $\partial/\partial z s(w, x, z) \equiv 0$. However, since $\tilde{z} - \bar{\tilde{z}} \neq 0$, by a simple calculation or by Theorem 2, we have $f$ is not differentiable everywhere. This contradicts with Theorem 3.1 and Corollary 3.1 in [6].

Finally, we show that applying $H$-differentiation in [3] to fuzzy complex function, Theorem 3.1 in [6] will be recovered. If $\tilde{u}, \tilde{v} \in \tilde{C}$, and if there exists a fuzzy complex number $\tilde{z} \in \tilde{C}$ such that $\tilde{z} + \tilde{u} = \tilde{v}$, then $\tilde{z}$ is called the Hukuhará difference of $\tilde{v}$ and $\tilde{u}$ and is denoted by $\tilde{v} \otimes \tilde{u}$ [3].

**Definition 4.** Let $E \subseteq \mathbb{C}$, then we say that $f : E \to \tilde{C}$ is $H$-differentiable at $z_0 \in E$ if there exists $f'(z_0) \in \tilde{C}$ such that for all $z \in E$ with $z \to z_0$, $f(z) \otimes f(z_0) / z - z_0$ converges metrically to $f'(z_0)$ in $(\tilde{C}, H)$, i.e., $\lim_{z \to z_0, z \in E} H(f(z) \otimes f(z_0))/z - z_0, f'(z_0)) = 0$ and now write

$$\lim_{z \to z_0} \frac{f(z) \otimes f(z_0)}{z - z_0} = f'(z_0).$$

**Lemma 2.** Let $\tilde{u}, \tilde{v} \in \tilde{C}$, then $s(\tilde{v} \otimes \tilde{u})(w, x) = s(\tilde{v}(w, x)) - s(\tilde{u}(w, x))$ for any $w \in J$ and $x \in I$.

**Proof.** For any $w \in J$ and $x \in I$, we have $s(\tilde{v}(w, x)) = \sup_{z_1 + z_2 \in \tilde{v}(w, x)} \langle \tilde{v}, \langle w, z_1 + z_2 \rangle \rangle = \sup_{z_1 + z_2 \in \tilde{v}(w, x)} \langle \langle w, z_1 \rangle \rangle + \langle \langle w, z_2 \rangle \rangle$, where $z_1 \in [\tilde{u}]^2$ and $z_2 \in [\tilde{v} \otimes \tilde{u}]^2$, which implies that $s(\tilde{v}(w, x)) \leq s(\tilde{v}(w, x)) + s(\tilde{u}(w, x))$.

Conversely, Since $[\tilde{u}]^2$ and $[\tilde{v} \otimes \tilde{u}]^2$ are compact, using the continuity of $\langle \cdot, \cdot \rangle$, we have that there are a $z_1 \in [\tilde{u}]^2$ and a $z_2 \in [\tilde{v} \otimes \tilde{u}]^2$ such that $s(\tilde{u}(w, x)) \geq \langle \langle w, z_1 \rangle \rangle + \langle \langle w, z_2 \rangle \rangle$ and $s(\tilde{u}(w, x)) \leq s(\tilde{v}(w, x))$. Thus we have $\langle \langle w, z_1 + z_2 \rangle \rangle = s(\tilde{v}(w, x)) - s(\tilde{u}(w, x)) = s(\tilde{v}(w, x)) - s(\tilde{u}(w, x))$. □

**Remark.** Let $A, B \in P_k(\mathbb{C})$. The equation $s(w|A - B) = s(w|A) - s(w|B)$ does not hold in general, which is the reason why Theorem 3.1 and Corollary 3.1 in [6] are incorrect.

**Theorem 3.** The fuzzy complex function $f : E \to \tilde{C}$ is differentiable at $z_0 \in E$ (or $f'(z_0) = \tilde{A} \in \tilde{C}$) if and only if for any $x \in I$ the support function $s(w, x, z) = s(w|f(z)|^2)$ of $f(z)$ at $z_0$ satisfies

$$\lim_{\Delta z \to 0} \frac{s(w|[f(z_0 + \Delta z)]^2) - s(w|[f(z_0)]^2)}{|\Delta z|} = s(w|\tilde{A}|^2),$$

uniformly for $w \in J$.

**Proof.** If $f'(z_0) = \tilde{A} \in \tilde{C}$ then $\lim_{\Delta z \to 0} H(f(z) \otimes f(z_0)/\Delta z, \tilde{A}) = 0$ iff (by Theorem 3.1 in [5])

$$\lim_{\Delta z \to 0} h \left( \frac{|f(z_0 + \Delta z)| \otimes |f(z_0)|}{|\Delta z|}, |\tilde{A}|^2 \right) = 0$$

for all $x \in I$, iff (by the conclusions in [5, p. 446])

$$\lim_{\Delta z \to 0} h \left( \frac{|f(z_0 + \Delta z)|^2 \otimes |f(z_0)|^2}{|\Delta z|}, |\tilde{A}|^2 \right) = 0$$

for all $x \in I$, iff (by Lemma 3.1 in [6])

$$\lim_{\Delta z \to 0} \sup_{w \in J} |s \left( w \left| \frac{|f(z_0 + \Delta z)|^2 \otimes |f(z_0)|^2}{|\Delta z|} \right| - s(w|\tilde{A}|^2) \right| = 0,$$
iff (by Lemma 2)

\[
\lim_{{\Delta z \to 0}} \sup_{{w \in J}} \left| s \left( w \left| \frac{[f(z_0 + \Delta z)]^2}{\Delta z} \right| - s \left( w \left| \frac{[f(z_0)]^2}{\Delta z} \right| - s(w|\tilde{A}|^2) \right| = 0,
\right.\right.
\]

iff (by the property of the real inner product in \( \mathbb{C} \))

\[
\lim_{{\Delta z \to 0}} \sup_{{w \in J}} \left| s(w|\tilde{A}|^2) - s(w|\tilde{A}|^2) \right| = 0,
\]

iff

\[
\lim_{{\Delta z \to 0}} \frac{s(w|\tilde{A}|^2) - s(w|\tilde{A}|^2)}{|\Delta z|} = s(w|\tilde{A}|^2),
\]

for all \( z \in I \) and uniformly for \( w \in J \). \( \square \)

4. Conclusion

In this short paper we correct some results of [5,6] about the convergence of the series of fuzzy complex numbers and the differentiability of fuzzy complex functions. We find that the concept of differentiability of [6] is inappropriate and the Hukuhara-type differential (first introduced for fuzzy real functions in [3]) is the appropriate one for fuzzy complex functions although this type differential operator has its own imperfection (for instance, see [1,7]). In our further work, we will try to lay bare a more appropriate notion of differentiability for fuzzy complex functions.

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