On observation of time-delay systems with unknown inputs


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Consider the following nonlinear systems:

\[
\begin{align*}
\dot{x} &= f(x) \\
y &= h(x)
\end{align*}
\]

Condition: \( \text{rank} \begin{pmatrix} dh \\ dL_fh \\ dL^2_fh \\ \vdots \end{pmatrix} = n. \)
Observability for nonlinear systems without delays

Consider the following nonlinear systems:

\[ \begin{cases}
\dot{x} = f(x) \\
y = h(x)
\end{cases} \]

Condition: \( \text{rank} \begin{pmatrix}
dh \\
dL_f h \\
dL^2_f h \\
\vdots
\end{pmatrix} = n. \)

Ex:

\[ \begin{aligned}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= x_3 + x_1 x_2 \\
\dot{x}_3 &= x_1^2 + x_2 x_3 \\
y &= x_1
\end{aligned} \]

Calculate the differentiation of the output:

\[ \begin{aligned}
dy &= dx_1 \\
\dot{y} &= dx_1 + dx_2 \\
\ddot{y} &= (1 + x_2) dx_1 + (1 + x_1) dx_2 + dx_3
\end{aligned} \]
Observability for nonlinear systems with delays

\begin{align*}
\dot{x}_1(t) &= x_1(t - \tau) + x_2(t) \\
\dot{x}_2(t) &= x_3(t) + x_1(t)x_2(t - 2\tau) \\
\dot{x}_3(t) &= x_1^2(t - \tau) + x_3(t) \\
y(t) &= x_1(t)
\end{align*}
Observability for nonlinear systems with delays

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y(t) &= x_1(t)
\end{align*}
\]

Calculate the differentiation of the output:

\[
\begin{align*}
\dot{y}(t) &= \dot{x}_1(t) \\
\ddot{y}(t) &= \dot{x}_1(t - \tau) + \dot{x}_2(t) \\
\dddot{y}(t) &= x_2(t - 2\tau)\dot{x}_1(t) + \dot{x}_1(t - 2\tau) + \dot{x}_2(t - \tau) + x_1(t)\dot{x}_2(t - 2\tau) + \dot{x}_3(t)
\end{align*}
\]

Quite complicated to be analyzed.
Observability for nonlinear systems with delays

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t - \tau) + x_2(t) \\
\dot{x}_2(t) &= x_3(t) + x_1(t)x_2(t - 2\tau) \\
\dot{x}_3(t) &= x_1^2(t - \tau) + x_3(t) \\
y(t) &= x_1(t)
\end{align*}
\]

Calculate the differentiation of the output:

\[
\begin{align*}
dy(t) &= dx_1(t) \\
d\dot{y}(t) &= dx_1(t - \tau) + dx_2(t) \\
d\ddot{y}(t) &= x_2(t - 2\tau)dx_1(t) + dx_1(t - 2\tau) + dx_2(t - \tau) + x_1(t)dx_2(t - 2\tau) + dx_3(t)
\end{align*}
\]

Quite complicated to be analyzed. Introduce delay operator \(\delta\), then

\[
\begin{align*}
dy(t) &= dx_1(t) \\
d\dot{y}(t) &= d(\delta x_1(t)) + dx_2(t) = \delta dx_1(t) + dx_2(t) \\
d\ddot{y}(t) &= \delta^2 x_2(t)dx_1(t) + d(\delta^2 x_1(t)) + d(\delta x_2(t)) + x_1(t)d(\delta^2 x_2(t)) + dx_3(t) \\
&= \delta^2 x_2(t)dx_1(t) + \delta^2 dx_1(t) + \delta dx_2(t) + x_1(t)\delta^2 dx_2(t) + dx_3(t) \\
&= (\delta^2 x_2 + \delta^2)dx_1 + (\delta + x_1\delta^2)dx_2 + dx_3
\end{align*}
\]

Since the coefficients are polynomials of \(\delta\), we can try to establish a polynomial ring for TDS, which is not commutative.
Consider the following nonlinear time-delay system:

\[
\begin{align*}
\dot{x} &= f(x(t - i\tau)) + \sum_{j=0}^{s} g^j(x(t - i\tau))u(t - j\tau) \\
y &= h(x(t - i\tau)) \\
&\quad = [h_1(x(t - i\tau)), \ldots, h_p(x(t - i\tau))]^T \\
x(t) &= \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0]
\end{align*}
\]

where \( x \in W \subset \mathbb{R}^n \) denotes the state variables, \( u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m \) is the unknown admissible input, \( y \in \mathbb{R}^p \) is the measurable output. \( p \geq m \) and \( i \in S_- = \{0, 1, \ldots, s\} \) is a finite set of constant time-delays.
Non-commutative algebraic framework [4]

\( \mathcal{K} \): the field of functions of a finite number of the variables from 
\( \{x_j(t - i\tau), j \in [1, n], i \in S_-\} \).

\( \mathcal{E} \): the vector space over \( \mathcal{K} \): 
\( \mathcal{E} = \text{span}_\mathcal{K}\{d\xi : \xi \in \mathcal{K}\} \).

\( \delta \): backward time-shift operator, i.e. 
\( \delta^i \xi(t) = \xi(t - i\tau) \) and

\[
\delta^i (a(t)d\xi(t)) = \delta^i a(t)\delta^i d\xi(t)
\]

\( \mathcal{K}(\delta) \): the set of polynomials of the form

\[
a(\delta) = a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a}, a_i(t) \in \mathcal{K}
\]  \hspace{1cm} (2)

Addition in \( \mathcal{K}(\delta) \) is usual, but the multiplication is given as

\[
a(\delta)b(\delta) = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i \leq r_a, j \leq r_b} a_i(t)b_j(t - i\tau)\delta^k
\]  \hspace{1cm} (3)
\[
a(\delta) = \delta x_1 \delta, \quad b(\delta) = x_2 + x_1 \delta^2
\]
\[
a(\delta) + b(\delta) = \delta x_1 \delta + x_2 + x_1 \delta^2
\]
\[
a(\delta) b(\delta) = \delta x_1 \delta (x_2 + x_1 \delta^2) = \delta x_1 x_2 \delta + \delta x_1 x_1 \delta^3
\]
\[
b(\delta) a(\delta) = (x_2 + x_1 \delta^2) \delta x_1 \delta = x_2 \delta x_1 \delta + x_1 \delta^3 x_1 \delta^3
\]

\(\mathcal{K}(\delta)\) satisfies the associative law and it is a non-commutative ring (see [4]). However, it is proved that the ring \(\mathcal{K}(\delta)\) is a left Ore ring [2, 4], which enables to define the rank of a module over this ring. Let \(\mathcal{M}\) denote the left module over \(\mathcal{K}(\delta)\)

\[
\mathcal{M} = \text{span}_{\mathcal{K}(\delta)}\{d\xi, \xi \in \mathcal{K}\}
\]
Time-delay systems under non-commutative rings

With the definition of $\mathcal{K}(\delta]$, (1) can be rewritten in a more compact form as follows:

\[
\begin{align*}
\dot{x} &= f(x, \delta) + \sum_{i=1}^{m} G_i u_i(t) \\
y &= h(x, \delta) \\
x(t) &= \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0]
\end{align*}
\]

where $f(x, \delta) = f(x(t - i\tau))$ and $h(x, \delta) = h(x(t - i\tau))$ with entries belonging to $\mathcal{K}$, $G_i = \sum_{j=0}^{s} g_i^j \delta^j$ with entries belonging to $\mathcal{K}(\delta]$. It is assumed that $\text{rank}_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} = p$, which implies that $[h_1, \ldots, h_p]^T$ are independent functions of $x$ and its backward shifts.
Observability and Left invertibility

**Definition**

System (1) is locally observable if the state $x(t)$ can be expressed as:

$$x(t) = \alpha(y(t - j\tau), \ldots, y^{(k)}(t - j\tau))$$

(5)

for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. It is locally causally observable if (5) is satisfied for $j \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, and locally non-causally observable if (5) is satisfied for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. 

The unknown input $u(t)$ can be estimated if it can be written as follows:

$$u(t) = \beta(y(t - j\tau), \ldots, y^{(k)}(t - j\tau))$$

(6)

for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. It can be causally estimated if (6) is satisfied for $j \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, and non-causally estimated if (6) is satisfied for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. 

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Example

\[
\begin{align*}
\dot{x}_1 &= x_2 + \delta x_1, \quad \dot{x}_2 = \delta^2 x_2 - \delta x_3, \\
\dot{x}_3 &= \delta x_4 + \delta u_1 + \delta^4 u_2, \quad \dot{x}_4 = \delta u_2 \\
y_1 &= x_1, \quad y_2 = \delta x_4
\end{align*}
\]

(7)

A straightforward calculation gives

\[
\begin{align*}
x_1(t) &= y_1(t), \\x_2(t) &= \dot{y}_1(t) - y_1(t - \tau) \\
x_3(t) &= \ddot{y}_1(t - \tau) - y_1(t - 2\tau) - \ddot{y}_1(t + \tau) + \dot{y}_1(t) \\
x_4(t) &= y_2(t + \tau)
\end{align*}
\]

and

\[
\begin{align*}
u_1(t) &= \dddot{y}_1(t) - \dot{y}_1(t - \tau) - \dddot{y}_1(t + 2\tau) + \ddot{y}_1(t + \tau) \\
&\quad - y_2(t + \tau) - \dot{y}_2(t - \tau) \\
u_2(t) &= \dot{y}(t + 2\tau)
\end{align*}
\]
Unimodular matrix and change of coordinate

Definition
(Unimodular matrix) [3] Matrix $A \in \mathcal{K}^{n \times n}(\delta)$ is said to be unimodular over $\mathcal{K}(\delta)$ if it has a left inverse $A^{-1} \in \mathcal{K}^{n \times n}(\delta)$, such that $A^{-1}A = I_{n \times n}$. 

(Change of coordinate) [3] For system (1), $z = \phi(\delta, x) \in \mathcal{K}^{n \times 1}$ is a causal change of coordinate over $\mathcal{K}(\delta)$ for (1) if there exists locally a function $\phi^{-1} \in \mathcal{K}^{n \times 1}$ and some constants $c_1, \ldots, c_n \in \mathbb{N}$ such that $\text{diag}\{\delta c_i\} x = \phi^{-1}(\delta, z)$. The change of coordinate is bicausal over $\mathcal{K}$ if $\max\{c_i\} = 0$, i.e. $x = \phi^{-1}(\delta, z)$. 

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$$\text{diag}\{\delta^{c_i}\}x = \phi^{-1}(\delta, z).$$

The change of coordinate is bicausal over $\mathcal{K}$ if $\max\{c_i\} = 0$, i.e. $x = \phi^{-1}(\delta, z)$. 
Lie derivative for TDS

Let \( f(x(t - j\tau)) \) and \( h(x(t - j\tau)) \) for \( 0 \leq j \leq s \) respectively be an \( n \) and \( p \) dimensional vector with entries \( f_r \in \mathcal{K} \) for \( 1 \leq r \leq n \) and \( h_i \in \mathcal{K} \) for \( 1 \leq i \leq p \). Let

\[
\frac{\partial h_i}{\partial x} = \left[ \frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n} \right] \in \mathcal{K}^{1 \times n}(\delta)
\]

(8)

where for \( 1 \leq r \leq n \):

\[
\frac{\partial h_i}{\partial x_r} = \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t - j\tau)} \delta^j \in \mathcal{K}(\delta)
\]

then the Lie derivative for TDS can be defined as follows

\[
L_f h_i = \frac{\partial h_i}{\partial x} (f) \quad \text{and} \quad L_{G_i} h_i = \frac{\partial h_i}{\partial x} (G_i)
\]
Relative degree for TDS

Definition

(Relative degree) System (4) has relative degree \((\nu_1, \cdots, \nu_p)\) in an open set \(W \subseteq \mathbb{R}^n\) if, for \(1 \leq i \leq p\), the following conditions are satisfied:

1. For all \(x \in W\), \(L^r G_j L^r f h_i = 0\), for all \(1 \leq j \leq m\) and \(0 \leq r < \nu_i - 1\);

2. There exists \(x \in W\) such that \(\exists j \in [1, m], L^r G_j L^{\nu_i - 1} f h_i \neq 0\).

If for \(1 \leq i \leq p\), (1) is satisfied for all \(r \geq 0\), then we set \(\nu_i = \infty\).
Observability indices for TDS

Let $\mathcal{F}_k := \text{span}_{K(\delta)} \left\{ dh, dL_f h, \ldots, dL_f^{k-1} h \right\}$ for $1 \leq k \leq n$, satisfying $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$. Then we define

$$d_1 = \text{rank}_{K(\delta)} \mathcal{F}_1, \text{ and } d_k = \text{rank}_{K(\delta)} \mathcal{F}_k - \text{rank}_{K(\delta)} \mathcal{F}_{k-1}$$

for $2 \leq k \leq n$. Let $k_i = \text{card} \{ d_k \geq i, 1 \leq k \leq n \}$ then $(k_1, \cdots, k_p)$ are the observability indices, and $\sum_{i=1}^{p} k_i = n$ since it is assumed that (4) is observable with $u = 0$. Reordering, if necessary, the output components of (4), such that

$$\text{rank}_{K(\delta)} \frac{\partial \left[ h, L_f h, \ldots, L_f^{n-1} h \right]^T}{\partial x} = \text{rank}_{K(\delta)} \frac{\partial \left[ h_1, L_f h_1, \ldots, L_f^{k_1-1} h_1, \ldots, h_p, L_f h_p, \ldots, L_f^{k_p-1} h_p \right]^T}{\partial x} = k_1 + \cdots + k_p = n$$
Canonical form and causal observability

**Theorem 1**

For $1 \leq i \leq p$, denote $k_i$ the observability indices and $\nu_i$ the relative degree index for $y_i$ of (4), and note $\rho_i = \min \{\nu_i, k_i\}$. Then there exists a change of coordinate $\phi(x, \delta) \in \mathcal{K}^{n \times 1}$, such that (4) can be transformed into the following form:

$$
\dot{z}_i = A_i z_i + B_i V_i \tag{9}
$$

$$
\dot{\xi} = \alpha(z, \xi, \delta) + \beta(z, \xi, \delta) u \tag{10}
$$

$$
y_i = C_i z_i \tag{11}
$$

where $A_i \in \mathbb{R}^{\rho_i \times \rho_i}$ is in the Brounovsky form and $z_i = (h_i, \ldots, L_f^{\rho_i-1} h_i)^T \in \mathcal{K}^{\rho_i \times 1}, B_i = (0, \ldots, 0, 1)^T \in \mathbb{R}^{\rho_i \times 1}$.

$$
V_i = L_f^{\rho_i} h_i(x, \delta) + \sum_{j=1}^m L_G_j L_f^{\rho_i-1} h_i(x, \delta) u_j \in \mathcal{K}, \alpha \in \mathcal{K}^{l \times 1}
$$

$$
\beta \in \mathcal{K}^{l \times 1}(\delta) \text{ with } l = n - \sum_{j=1}^p \rho_j, C_i = (1, 0, \ldots, 0) \in \mathbb{R}^{1 \times \rho_i}
$$

Moreover if $k_i < \nu_i$, one has $V_i = L_f^{\rho_i} h_i = L_f^{k_i} h_i$. 
For (9), note
\[ H(x, \delta) = \Psi(x, \delta) + \Gamma(x, \delta)u \] (12)
with
\[
H(x, \delta) = \begin{pmatrix}
    h_1^{(\rho_1)} \\
    \vdots \\
    h_p^{(\rho_p)}
\end{pmatrix},
\Psi(x, \delta) = \begin{pmatrix}
    L_{f}^{\rho_1} h_1 \\
    \vdots \\
    L_{f}^{\rho_p} h_p
\end{pmatrix}
\]
and
\[
\Gamma(x, \delta) = \begin{pmatrix}
    L_{G_1} L_{f}^{\rho_1-1} h_1 & \cdots & L_{G_m} L_{f}^{\rho_1-1} h_1 \\
    \vdots & \ddots & \vdots \\
    L_{G_1} L_{f}^{\rho_p-1} h_p & \cdots & L_{G_m} L_{f}^{\rho_p-1} h_p
\end{pmatrix}
\]
where \( H(x, \delta) \in \mathcal{K}^{p \times 1} \), \( \Psi(x, \delta) \in \mathcal{K}^{p \times 1} \) and \( \Gamma(x, \delta) \in \mathcal{K}^{p \times m}(\delta) \).

And for (4), let denote \( \Phi \) the observable space from its outputs:
\[
\Phi = \{ dh_1, \cdots, dL_{f}^{\rho_1-1} h_1, \cdots, dh_p, \cdots, dL_{f}^{\rho_p-1} h_p \} \] (13)
Main theorem

Theorem 2
For system (4) with outputs \((y_1, \ldots, y_p)\) and corresponding \((\rho_1, \ldots, \rho_p)\) with \(\rho_i = \min\{k_i, \nu_i\}\) where \(k_i\) and \(\nu_i\) are respectively the associated observability indices and the relative degree, if

\[
\text{rank}_{\mathcal{K}(\delta)} \Phi = n
\]

where \(\Phi\) defined in (13), then there exists a change of coordinate \(\phi(x, \delta)\) such that (4) can be transformed into (9-11) with \(\text{dim} \xi = 0\). Moreover, if the change of coordinate is locally bicausal over \(\mathcal{K}\), then the state \(x(t)\) of (4) is locally causally observable.

For the matrix \(\Gamma \in \mathcal{K}^{p \times m}(\delta)\) where \(m \leq p\), if \(\text{rank}_{\mathcal{K}(\delta)} \Gamma = m\), then there exists a matrix \(Q \in \mathcal{K}^{p \times p}(\delta)\) such that \(Q \Gamma = \begin{bmatrix} \bar{\Gamma} \\ 0 \end{bmatrix}\) where \(\bar{\Gamma} \in \mathcal{K}^{m \times m}(\delta)\) has full row rank \(m\). Moreover, if \(\bar{\Gamma} \in \mathcal{K}^{m \times m}(\delta)\) is also unimodular over \(\mathcal{K}(\delta)\), then the unknown input \(u(t)\) of (4) can be causally estimated.
Example

\[
\begin{aligned}
\dot{x}_1 &= -\delta x_1 + x_2, \quad \dot{x}_2 = -\delta x_3 + u_1 \\
\dot{x}_3 &= \delta x_1 + \delta u_1 + u_2, \quad \dot{x}_4 = -x_4 + 2\delta x_4/3 \\
y_1 &= x_1, \quad y_2 = x_3, \quad y_3 = x_4
\end{aligned}
\]

(14)

\[\Rightarrow \nu_1 = k_1 = 2, \quad \nu_2 = k_2 = 1, \quad \nu_3 = \infty \text{ and } k_3 = 1 \Rightarrow \rho_1 = 2, \quad \rho_2 = 1 \text{ and } \rho_3 = 1 \Rightarrow \]

\[\Phi = \{dh_1, dL_f h_1, dh_2, dh_3\} = \{dx_1, -\delta dx_1 + dx_2, dx_3, dx_4\}\]

\[\Rightarrow \text{rank}_{K(\delta)} \Phi = 4 \Rightarrow \]

\[z = \phi(x, \delta) = (x_1, x_2 - \delta x_1, x_3, x_4)^T\]
Example

Since

\[ x = \phi^{-1} = (z_1, \delta z_1 + z_2, z_3, z_4)^T \]

⇒ the change of coordinate is bicausal over \( \mathcal{K} \), thus the state of (14) is locally causally observable:

\[
\begin{align*}
&x_1(t) = y_1(t), \quad x_2(t) = y_1(t - \tau) + \dot{y}_1(t) \\
&x_3(t) = y_2(t), \quad x_4(t) = y_3(t)
\end{align*}
\]

Moreover, since \( \Gamma = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \) ⇒ \( \Gamma^{-1} = \begin{pmatrix} 1 & 0 \\ -\delta & 1 \end{pmatrix} \) s.t. \( \Gamma^{-1} \Gamma = l_{2 \times 2} \)

⇒ \( \Gamma \) is unimodular over \( \mathcal{K}(\delta) \)⇒ the unknown inputs are causally observable as well:

\[
\begin{align*}
&u_1(t) = \ddot{y}_1(t - \tau) + \dot{y}_1(t) + y_2(t - \tau) \\
&u_2(t) = \ddot{y}_2(t) - y_1(t - \tau) - \ddot{y}_1(t - 2\tau) - \dot{y}_1(t - \tau) - y_2(t - 2\tau)
\end{align*}
\]
Remark

- The condition of $\text{rank}_{K(\delta)} \Phi = n$ is sometimes hard to be satisfied.
- When $\text{rank}_{K(\delta)} \Phi < n$, is it still possible to estimate the state and the unknown inputs?

In [1], a constructive algorithm to solve this problem for nonlinear systems without delays has been proposed, which in fact can be generalized to treat the same problem for nonlinear time-delay systems.
Illustrative example

Ex:

\[
\begin{aligned}
\dot{x}_1 &= -\delta x_1 + \delta x_4 u_1, \\
\dot{x}_2 &= -\delta x_3 + x_4, \\
\dot{x}_3 &= x_2 - \delta x_4 u_1, \\
\dot{x}_4 &= u_1.
\end{aligned}
\]  

(15)

\Rightarrow \nu_1 = k_1 = 1, \nu_2 = 1, k_2 = 3 \Rightarrow \rho_1 = \rho_2 = 1 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3\}

\Rightarrow \text{rank}_{\mathcal{K}(\delta)} \Phi = 2 < n \Rightarrow \text{Theorem 2 cannot be applied.}

Precisely,

\[
\dot{y}_1 = -\delta x_1 + \delta x_4 u_1
\]

and \(\dot{y}_2 = -\delta^2 x_1 + \delta^2 x_4 \delta u_1 + x_2 - \delta x_4 u_1 \Rightarrow \text{derivative impossible.}\)
Illustrative example

Ex:

\[
\begin{align*}
\dot{x}_1 &= -\delta x_1 + \delta x_4 u_1, \quad \dot{x}_2 = -\delta x_3 + x_4 \\
\dot{x}_3 &= x_2 - \delta x_4 u_1, \quad \dot{x}_4 = u_2 \\
y_1 &= x_1, \quad y_2 = \delta x_1 + x_3
\end{align*}
\]

(15)

\[\Rightarrow \nu_1 = k_1 = 1, \quad \nu_2 = 1, \quad k_2 = 3 \Rightarrow \rho_1 = \rho_2 = 1 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3\} \]

\[\Rightarrow \text{rank}_{K(\delta)}\Phi = 2 < n \Rightarrow \text{Theorem 2 cannot be applied.}\]

Precisely,

\[\dot{y}_1 = -\delta x_1 + \delta x_4 u_1\]

and \[\dot{y}_2 = -\delta^2 x_1 + \delta^2 x_4 \delta u_1 + x_2 - \delta x_4 u_1 \Rightarrow \text{derivative impossible.}\] However,

\[\dot{y}_2 - (\delta - 1)\dot{y}_1 \Rightarrow \dot{y}_2 - (\delta - 1)\dot{y}_1 = -\delta x_1 + x_2 \Rightarrow x_2 = \dot{y}_2 - (\delta - 1)\dot{y}_1 + \delta y_1\]

Note \(y_3 = x_2 \Rightarrow \nu_3 = k_3 = 2 \Rightarrow \rho_3 = 2 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3, dx_2, -\delta dx_3 + dx_4\} \Rightarrow \text{rank}_{K(\delta)}\Phi = 4\]
Notation and Definition

For the case where \( \text{rank}_{\mathcal{K}(\delta)} \Phi = j < n \), select \( j \) linearly independent vector over \( R[\delta] \) from \( \Phi \), where \( R[\delta] \) means the set of polynomials of \( \delta \) with coefficients belonging to \( R \), noted as

\[
\Phi = \{dz_1, \cdots, dz_j\}
\]

Note

\[
\mathcal{L} = \text{span}_{R[\delta]} \{z_1, \cdots, z_j\}
\]

and let \( \mathcal{L}(\delta) \) be the set of polynomials of \( \delta \) with coefficients over \( \mathcal{L} \), define the module spanned by element of \( \Phi \) over \( \mathcal{L}(\delta) \) as follows

\[
\Omega = \text{span}_{\mathcal{L}(\delta)} \{\xi, \xi \in \Phi\} \tag{16}
\]

Define \( \mathcal{G} = \text{span}_{R[\delta]} \{G_1, \ldots, G_m\} \) and its left annihilator

\[
\mathcal{G}^\perp = \text{span}_{R[\delta]} \{\omega \in \Omega \mid \omega g = 0, \forall g \in \mathcal{G}\}.
\]
Theorem 3
For (4) with outputs $y = (y_1, \cdots, y_p)^T$ and corresponding $(\rho_1, \ldots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where $k_i$ and $\nu_i$ are respectively the associated observability indices and the relative degree which yields (12) with $\text{rank}_{\mathcal{K}} \Phi < n$ where $\Phi$ is defined in (13), there exists $l$ new independent outputs over $\mathcal{K}$ which are functions of $y$ and its time derivatives and backwards time shifts, if and only if $\text{rank}_{\mathcal{K}} \mathcal{H} = l$ where

$$\mathcal{H} = \text{span}_{R[\delta]} \{ \omega \in \mathcal{G}^\perp \cap \Omega \mid \omega f \notin \mathcal{L} \}$$

(17)

with $\Omega$ defined in (16).
Moreover, the new outputs, noted $\bar{y}_i$ for $1 \leq i \leq l$, are given as follows:

$$\bar{y}_i = \omega_i f \pmod{\mathcal{L}}$$

where $\omega_i \in \mathcal{H}$.  

Theorem for general case
Remarks

• Roughly speaking, for

\[ H(x, \delta) = \Psi(x, \delta) + \Gamma(x, \delta)u \]

if there exists a $1 \times p$ vector $Q$ with entries $q_i \in L(\delta]$, such that $Q\Gamma = 0$ and $Q\Psi \notin L$, then we denote

\[ y_{p+1} = Q\Psi \mod L \]

a new output since it is not affected by the unknown input $u$, and it does not belong to the current measurable vector $L$.

• Theorem 3 gives a constructive way to treat the case where $\text{rank}_{\mathcal{K}(\delta]} \Phi < n$.

• A 'Check-Extend' procedure is iterated until one obtains $\text{rank}_{\mathcal{K}(\delta]} \Phi = n$. 
Routine to deduce the new outputs

Input: DTS with $x \in R^n, y \in R^p, u \in R^m$
Output: $\Phi$ or failed
Initialization: Compute $\nu_i, k_i, \rho_i, \Phi, \text{rank}_{K(\delta)} \Phi = j$
Loop:
   While $j < n$
      $\Phi = \{dz_1, \cdots, dz_j\}$
      $\mathcal{L} = \text{span}_{R[\delta]} \{z_1, \cdots, z_j\}$
      $\Omega = \text{span}_{\mathcal{L}(\delta)} \{\xi, \xi \in \Phi\}$
      $\mathcal{H} = \text{span}_{R[\delta]} \{\omega \in G^\perp \cap \Omega \mid \omega \notin \mathcal{L}\}$
      $\text{rank}_{K(\delta)} \mathcal{H} = l$
      If $l > 0$
         $\exists l$ 1-forms, s.t. $\mathcal{H} = \text{span}_{R[\delta]} \{\omega_1, \cdots, \omega_l\}$
         $y = y \cup \{\omega_i f \mod \mathcal{L}, 1 \leq i \leq l\}$
         Reorder $y$
         For each $y_i \in y$, calculate $\nu_i, k_i, \rho_i$
         $\phi = \{\cdots, dh_i, \cdots, dL^f_i h_i, \cdots\}$
         $\text{rank}_{K(\delta)} \Phi = j$
      Else
         Return(failed)
   End
Return($\Phi$)
\[
\begin{cases}
    \dot{x}_1 = -\delta x_1 + \delta x_4 u_1, & \dot{x}_2 = -\delta x_3 + x_4 \\
    \dot{x}_3 = x_2 - \delta x_4 u_1, & \dot{x}_4 = u_2 \\
    y_1 = x_1, & y_2 = \delta x_1 + x_3
\end{cases}
\]  
\Rightarrow \rho_1 = \rho_2 = 1 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3\} \Rightarrow \text{rank}_{\mathcal{K}(\delta)} \Phi = 2 < n. 
\] 

\[ G = \text{span}_{\mathbb{R}[\delta]} \{(\delta x_4, 0, -\delta x_4, 0)^T, (0, 0, 0, 1)^T\} \Rightarrow \text{span}_{\mathbb{R}[\delta]} \{dx_1 + dx_3, dx_2\} 
\] 

\[ \text{rank}_{\mathcal{K}(\delta)} \Phi = 2 \Rightarrow \mathcal{L} = \text{span}_{\mathbb{R}[\delta]} \{x_1, \delta x_1 + x_3\} \Rightarrow \Omega = \text{span}_{\mathcal{L}(\delta)} \{dx_1, dx_3\} 
\] 

\[ \Omega \cap \mathcal{G}^\perp = \text{span}_{\mathcal{L}(\delta)} \{dx_1, dx_3\} \cap \text{span}_{\mathbb{R}[\delta]} \{dx_1 + dx_3, dx_2\} 
\] 

\[ = \text{span}_{\mathcal{L}(\delta)} \{dx_1 + dx_3\} 
\] 

\[ \Rightarrow \forall \omega \in \Omega \cap \mathcal{G}^\perp, \omega f \notin \mathcal{L} \text{ since } \omega f = -\delta x_1 + x_2 \Rightarrow \text{new output } h_3: 
\] 

\[ y_3 = h_3 = \omega f \mod \mathcal{L} = x_2 = \delta y_1 + (1 - \delta)\dot{y}_1 + \dot{y}_2 \]
Example

$\rho_1 = \rho_2 = 1$ and $\rho_3 = 2 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3, dx_2, -\delta dx_3 + dx_4\}$

$\Rightarrow \text{rank}_{\mathcal{K}(\delta)} \Phi = 4 = n$, thus we find the following change of coordinate

$$z = \phi(x, \delta) = (x_1, \delta x_1 + x_3, x_2, -\delta x_3 + x_4)^T$$

it is easy to check that it is bicausal over $\mathcal{K}(\delta)$, since

$$x = \phi^{-1} = (z_1, z_3, z_2 - \delta z_1, z_4 + \delta z_2 - \delta^2 z_1)$$

and one gets

$$\begin{cases} 
    x_1 = y_1, & x_2 = y_3, & x_3 = y_2 - \delta y_1, \\
    x_4 = -\delta^2 y_1 + \delta y_2 + \dot{y}_3 
\end{cases}$$

where the new output $y_3$ is defined in (19).

$$\begin{cases} 
    u_1 = \frac{\dot{y}_1}{-\delta^3 y_1 + \delta^2 y_2 + \delta \dot{y}_3} \\
    u_2 = -\delta \dot{y}_1 + \dot{\dot{y}}_3
\end{cases}$$
Remark

- It is the locally bicausal change of coordinate which makes the state of the system locally causally observable.
- It is the unimodular characteristic of $\Gamma$ over $\mathcal{K}(\delta]$ which guarantees the causal reconstruction of unknown inputs.

The following is devoted to dealing with the non-causal case.
Non-causal observability

\( \nabla: \) the forward time-shift operator, such that for \( i, j \in \mathbb{N}, \)

\[
\nabla f(t) = f(t + \tau), \quad \nabla^i \delta^j f(t) = \delta^j \nabla^i f(t) = f(t - (j - i)\tau)
\]

\( \bar{K}: \) the field of functions of a finite number of variables from
\( \{x_j(t - i\tau), j \in [1, n], i \in S\} \) where \( S = \{-s, \cdots, 0, \cdots, s\} \)
\( \bar{K}(\delta, \nabla): \) the set of polynomials of the following form:

\[
a(\delta, \nabla) = \bar{a}_r \nabla^r + \cdots + \bar{a}_1 \nabla + a_0(t) + a_1(t)\delta + \cdots + a_r(t)\delta^r
\]

where \( a_i(t) \) and \( \bar{a}_i(t) \) belonging to \( \bar{K}. \)

Usual addition + the following multiplication:

\[
a(\delta, \nabla)b(\delta, \nabla) = \sum_{i=0}^{r_a} \sum_{j=0}^{r_b} a_i \delta^i b_j \delta^{i+j} + \sum_{i=1}^{r_a} \sum_{j=0}^{r_b} a_i \delta^i b_j \nabla^{i+j} + \sum_{i=0}^{r_a} \sum_{j=1}^{r_b} \bar{a}_i \nabla^i b_j \nabla^{i+j} + \sum_{i=1}^{r_a} \sum_{j=1}^{r_b} \bar{a}_i \nabla^i b_j \nabla^{i+j}
\]

It is clear that \( \mathcal{K} \subseteq \bar{K} \) and \( \mathcal{K}(\delta) \subseteq \bar{K}(\delta, \nabla). \)
Theorem 4
For system (4) with outputs \((y_1, \ldots, y_p)\) and corresponding \((\rho_1, \ldots, \rho_p)\), if \(\text{rank}_{\mathcal{K}(\delta)} \Phi = n\), where \(\Phi\) defined in (13), then there exists a change of coordinate \(z = \phi(x, \delta)\) such that (4) can be transformed into (9-11) with \(\text{dim}\xi = 0\).
Moreover, if the change of coordinate \(z = \phi(x, \delta)\) is locally bicausal over \(\bar{\mathcal{K}}\), then the state \(x(t)\) of (4) is at least non-causally observable.
For the deduced matrix \(\Gamma\) with \(\text{rank}_{\mathcal{K}(\delta)} \Gamma = m\), one can obtain a matrix \(\bar{\Gamma} \in \mathcal{K}^{m \times m}(\delta)\) which has full row rank \(m\). If \(\bar{\Gamma}\) is unimodular over \(\bar{\mathcal{K}}(\delta, \nabla)\), then the unknown input \(u(t)\) of (4) can be at least non-causally estimated as well. \(\blacksquare\)
Example

\[
\begin{cases}
\dot{x}_1 = \delta x_1 + x_2 \delta u_1, & \dot{x}_2 = -x_1 + u_1 + x_3 \delta u_2 \\
\dot{x}_3 = x_4 - x_1 \delta x_2 \delta^2 u_1, & \dot{x}_4 = \delta x_1 + \delta^3 x_2 \\
y_1 = x_1, & y_2 = \delta x_3
\end{cases}
\]  
(21)

\Rightarrow \nu_1 = k_1 = 1, \nu_2 = 1, k_2 = 3 \Rightarrow \rho_1 = \rho_2 = 1 \Rightarrow \Phi = \{dx_1, \delta dx_3\} \Rightarrow

\text{rank}_{K(\delta)} \Phi = 2 < n.

\mathcal{G} = \text{span}_{R[\delta]} \{G_1, G_2\} \Rightarrow \mathcal{G}^\perp = \text{span}_{R[\delta]} \{x_1 \delta dx_1 + dx_3, dx_4\}

\text{rank}_{K(\delta)} \Phi = 2 \Rightarrow \mathcal{L} = \text{span}_{R[\delta]} \{x_1, \delta x_3\} \Rightarrow \Omega = \text{span}_{\mathcal{L}(\delta)} \{dx_1, \delta dx_3\} \Rightarrow

\Omega \cap \mathcal{G}^\perp = \text{span}_{\mathcal{L}(\delta)} \{dx_1, \delta dx_3\} \cap \text{span}_{R[\delta]} \{x_1 \delta dx_1 + dx_3, dx_4\}

= \text{span}_{\mathcal{L}(\delta)} \{\delta x_1 \delta^2 dx_1 + \delta dx_3\}

\text{Since } \omega f = \delta x_1 \delta^3 x_1 + \delta x_4 \notin \mathcal{L} \Rightarrow

y_3 = h_3 = \omega f \mod \mathcal{L} = \delta x_4 = \delta y_1 \delta^2 \dot{y}_1 + \dot{y}_2 - \delta y_1 \delta^3 y_1
\]  
(22)
Example

$\Rightarrow \rho_1 = \rho_2 = 1, \nu_3 = k_3 = 2 \Rightarrow \rho_3 = 2 \Rightarrow \Phi = \{dx_1, \delta dx_3, \delta dx_4, \delta^3 dx_2\}$

$\Rightarrow \text{rank}_{\mathcal{K}(\delta)} \Phi = 4 = n \Rightarrow \mathcal{L} = \text{span}_{\mathbb{R}[\delta]} \{x_1, \delta x_3, \delta x_4, \delta^3 x_2\} \Rightarrow$ the following change of coordinate

$$z = \phi(x, \delta) = (x_1, \delta x_3, \delta x_4, \delta x_1 + \delta^3 x_2)^T$$

which is not bicausal over $\mathcal{K}$, but bicausal over $\bar{\mathcal{K}}$, since one has

$$x = \phi^{-1} = (z_1, -\nabla^2 z_1 + \nabla^3 z_4, \nabla z_2, \nabla z_3)^T$$

which gives

$$\begin{cases} 
  x_1 = y_1, x_2 = -\nabla^2 y_1 + \nabla^3 \dot{y}_3 \\
  x_3 = \nabla y_2, x_4 = \nabla y_3
\end{cases}$$

where $y_3$ is given in (22). Thus $x$ of (21) is observable, but non causally observable. The calculation for $u$ is omitted (see [5] for more details).


