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VIII. CONCLUSION

A bottom-up state-space size estimation technique for Petri nets (PN’s) has been described. The estimation relies on the computation of state counting functions, (sc-functions), for general subnets and interconnections. Provided that at some level of abstraction, the PN has a modular design and can be described with the subnets and interconnections, the computation of state-space size is exact. Otherwise, errors in the estimation may result from changes in the PN model to permit analysis or inaccuracies in the sc-functions. A manufacturing example demonstrated the complexity of a system that could be analyzed.

Benefits of this approach include accurate state-space size estimation, simple tabular representation of sc-functions that facilitate automation, and the ability to interject hand-computed results into the estimation. In addition to developing more interconnection models, other improvements, such as estimating the number of state-space graph links, are possible.

ACKNOWLEDGMENT

The authors thank G. Ciardo for the SPNP software.

REFERENCES


On Singular Time-Optimal Control Along Specified Paths

Zvi Shiller

Abstract—This paper presents a general necessary condition for singular time-optimal control of robotic manipulation moving along specified paths. Early work [1], [7] ignored the issue of singular control, assuming bang-bang acceleration along the path. Recent work [6] has shown that the time optimal control can be singular if one of the equations of motion reduces to a velocity constraint. This paper derives a more general necessary condition for singular control. It is also proven that singular control cannot exist if the set of admissible controls is strictly convex, as is demonstrated for a two-link planar manipulator with elliptical actuator constraints.

I. INTRODUCTION

The time optimal control problem of robotic manipulators moving along specified paths has been efficiently solved by reducing the task space to path coordinates [1], [5], [7]. The resulting reduction of the state space to two independent states transforms the control vector to a scalar control variable and the actuator constraints to state dependent control and state inequality constraints.

Early work has shown that the time optimal control is bang-bang, i.e., it either maximizes or minimizes the acceleration along the path [1], [5], [6]. This result was based on the assumption that the equations of motion are never singular in the acceleration. Pfeiffer and Johanni [5] showed that this assumption fails at the so called critical points, at which at least one equation of motion reduces to a pure velocity constraint. It was later discovered that the optimal control can be singular (not necessarily bang-bang) at the so called singular points, at which the maximum or minimum acceleration at the velocity limit is infeasible due to the slope of the velocity constraint [6].

The existence of singular points is of special importance since bang-bang control at such points tends to chatter and thus violate the actuator constraints [6]. In addition to undesirable vibrations, this chatter may result in unbounded computation times since the computational complexity of this problem depends on the number of switches along the trajectory [6]. Another motivation for considering singular points is the existence of singular arcs, defined as continuous segments consisting of singular points. Singular arcs are analogous to constrained arcs of state-constrained problems in that both slide along the velocity constraint [6]. The only difference between the two is that the former is inherent to the system, whereas the latter is due to arbitrarily specified state inequality constraints. The ability to consider singular points and arcs thus allows the extension of the original algorithms to treat state-constrained problems. The connection between state-constrained and singular control problems has been previously suggested in [4].

In this paper, we generalize the concept of singular points for time optimal motions along specified paths. It is shown that the existence of singular points is related to the convexity of the set of admissible controls. It is also shown that singular points do not exist if the feasible actuator set is strictly convex. Selecting a strictly convex set of the admissible controls would, therefore, guarantee...
nonsingular trajectories and, hence, better computational efficiency. This is demonstrated numerically for a two-link manipulator, using elliptical actuator constraints, which are strictly convex.

II. THE PROBLEM

The optimization of the motion time along a specified path can be stated as the following problem:

$$\min_{T \in W} J = \int_0^T |dt|$$

subject to system dynamics

$$M(\theta) \ddot{\theta} + \dot{\theta} C(\theta) \dot{\theta} + G(\theta) = T$$

state (position) constraints

$$p(\theta) = 0; p \in \mathbb{R}^{n-1}$$

and actuator constraints

$$T \in W \subset \mathbb{R}^n$$

where $$M(\theta) \in \mathbb{R}^{n \times n}$$ is the inertia matrix, $$C(\theta) \in \mathbb{R}^{n \times n \times n}$$ is an array of the coefficients of the centrifugal and Coriolis forces, $$G(\theta)$$ is the vector of the gravity forces, $$T \in \mathbb{R}^n$$ is the vector of actuator efforts, $$W$$ is the set of admissible controls, and $$n$$ is the number of degrees-of-freedom. The set of admissible controls most commonly considered in the control literature assumes constant and decoupled actuator limits, and is of the form

$$W = \{ T | T_{min} \leq T \leq T_{max}, i = 1, 2, \ldots, n \}$$

This form of the actuator constraints is mathematically simple and leads to elegant theoretical results by greatly simplifying the application of the Pontryagin maximum principle.

The multi-dimensional optimization problem (1) is reduced to a one dimensional optimization by parameterizing the path with a scalar variable, $$s$$, so that:

$$\theta = f(s) \in \mathbb{R}^n$$

Differentiating (6) twice with respect to time, substituting in (2), and ignoring gravity for simplicity, yields the equations of motion in terms of $$s$$ and its time derivatives [6]

$$m(s) \ddot{s} + b(s) \dot{s}^2 = T$$

where

$$m(s) = M(s) f_s(s)$$

$$b(s) = M(s) f_{ss}(s) + f_T(s) C(s)f_s(s)$$

The vectors $$f_s(s)$$ and $$f_{ss}(s)$$ are the partial derivatives of $$f(s)$$ with respect to $$s$$. If $$s$$ is selected as the arc length, then $$f_s(s)$$ is the tangent unit vector to the path, and the time derivatives $$\dot{s}$$ and $$\ddot{s}$$ are the speed and the acceleration along the path, respectively.

Using $$s$$ and $$\dot{s}$$ as the independent states, and $$\ddot{s}$$ as the independent control variable, we can reduce problem (1) to the following optimization problem [6]:

$$\min_{\ddot{s}} J = \int_0^T |dt|$$

subject to the double integrator

$$\ddot{x} = \{ \dot{s}, \ddot{s} \}^T$$

the boundary conditions

$$x(0) = \{0, \dot{s}(0)\}^T, x(t_f) = \{s_f, \dot{s}(t_f)\}^T$$

the state dependent control constraints

$$\dot{s}_d(s, \ddot{s}) \leq \ddot{s} \leq \dot{s}(s, \ddot{s})$$

and the state constraint

$$\ddot{s} - \ddot{s}_{max}(s) \leq 0$$

where, for the constant actuator limits (5), assuming $$m_i \neq 0$$ (the $$j$$th element of $$m$$) for all $$i = 1, \ldots, n$$.

$$\ddot{s}_d(s, \ddot{s}) = \max_{i \leq j \leq i} \left\{ \frac{T_i - b_i(s)}{m_i(s)} \right\}$$

$$\dot{s}_{max}(s) = \min_{i \leq j \leq i} \left\{ \frac{T_j - b_j(s)}{m_j(s)} \right\}$$

Equations (13) and (14) express the acceleration bounds at every point $$(s, \ddot{s})$$ in the reduced state space, whereas (15) expresses the velocity limit at every point $$s$$. Plotting the velocity limit as a function of $$s$$ forms the velocity limit curve, which serves as an upper bound for any feasible trajectory in the phase plane $$s - \dot{s}$$.

It is easy to show that if $$m_j(s) \neq 0$$ for all $$i = 1, \ldots, n$$, at some point $$s \in [0, s_f]$$, then the acceleration bounds (13) and (14) coincide at the velocity limit, i.e., $$\dot{s}_d(s, \ddot{s}_{max}) = \dot{s}_d(s, \ddot{s}_{max})$$ [6]. This greatly simplifies the optimization problem since it allows us to ignore the state constraint (12), which is active only whenever both control constraints (11) are active. It was shown that the solution to the optimization problem (8), when subject only to state dependent control constraints, is bang-bang in the acceleration [6]. The same result was derived differently in [1], [7], leading to efficient algorithms for computing the time optimal trajectory. Similar to other bang-bang control problems, the main computational effort of these algorithms is the computation of the switching points between the acceleration extremes.

The simplest optimal trajectories are those that do not reach the velocity limit curve, having only one switch from maximum acceleration to maximum deceleration. Trajectories with more switches (usually an odd number) touch the limit curve at one or more points, each being a switch from maximum deceleration to maximum acceleration [1], [7]. An important class of such switching points are those ignored in the original works of Bobrow [1] and Shin and McKay [7]. Pfeiffer and Johanni [5] showed that the points where $$m_j(s) = 0$$, for some $$j \in [1, n]$$ are potential switching points from deceleration to acceleration. Typical to these points, called critical, is that the velocity constraint (12) is active independently of the acceleration constraints (11), since $$m_j(s) = 0$$ reduces the $$j$$th equation of motion to a pure velocity constraint. Hence, the existence of critical points calls for a more complete theory of the time optimal control problem that accounts for state inequality constraints [6].

Fig. 1 shows schematically examples of two critical points, $$a$$ and $$c$$, and one tangency point, $$b$$, along a typical time optimal trajectory. Unlike at the tangency point, the acceleration bounds at the critical points do not coincide, as is indicated by the arrows at each point. Point $$c$$ is different from point $$a$$ in that the maximum acceleration at $$c$$ is infeasible. At point $$a$$, the optimal trajectory switches from maximum deceleration to maximum acceleration, whereas at $$c$$ it must switch to some interior (singular) acceleration, determined by the slope of the velocity limit curve at that point [6]. In [6], point $$c$$ was called singular. In this paper, we extend the definition of singularity
The significance of critical points (henceforth to be called the velocity limit curve at a critical point than at a tangency point.

The optimal trajectory along an arbitrary path is more likely to reach a singular point parallel to motion (7) for a two-dimensional system.

This leads to the following definition:

Definition 1: The point $s_1 \in [0, s_f]$ along the optimal trajectory is said to be singular if $s_d(s, s_{\text{max}}) \neq s_d(s, s_{\text{max}})$.

Using this definition, we now proceed to investigate the cause of singular points, using a geometric interpretation of the equations of motion (7) for a two-dimensional system.

For constant actuator limits (5), the set of admissible controls, $W$, is bounded by a rectangle, as shown in Fig. 2. At a given point $s$, (7) represents a vector summation of the vectors $m\dot{s}$ and $b\dot{s}^2$, also shown in Fig. 2. At a given $s$, the controls satisfying (7) reduce to the line, $\Gamma$, drawn from the point $b\dot{s}^2$ parallel to $m$. Each point along $\Gamma$ corresponds to some acceleration value, $\ddot{s}$, obtained by projecting this point parallel to $b$ onto $m$. The line $\Gamma'$ represents the line parallel to $m$ that intersects only the boundary $\partial W$. The intersections of $\Gamma'$ with the boundary of $W$ correspond to the acceleration bounds, $\ddot{s}_a$ and $\ddot{s}_b$, as shown in Fig. 2. Note that for higher dimensional systems, Fig. 2 represents the intersection of the plane, defined by $m$ and $b$, with $W$. If $m$ is not parallel to any of the major axes, then the intersection between $\Gamma$ and $W$ reduces to a point, from which it follows that the acceleration bounds at the velocity limit coincide, i.e., $\ddot{s}_a(\dot{s}_{\text{max}}) = \ddot{s}_b(\dot{s}_{\text{max}})$. Otherwise, $\ddot{s}_a(\dot{s}_{\text{max}}) \neq \ddot{s}_b(\dot{s}_{\text{max}})$, which is typical to a singular point. Note that $m$ is parallel to one of the major axes only if $m_i = 0$ for some $i = 1, \ldots, n$, which is consistent with the definition of critical points in [5] and [6].

Thus, the condition that one of the equations of motion must be singular ($m_i = 0$) in order for a singular point to occur holds only for systems with a polyhedral set of admissible controls having boundaries parallel to the major axes and planes. This condition does not hold for admissible controls with any other geometry.

We now show that singular points can occur only if the set of admissible controls is convex, but not strictly convex. We will make use of the following definitions:

Definition 2: A point $x$ in a convex set $C$ is an extreme point if there do not exist points $x_1, x_2 \in C$ such that

$$x = \alpha x_1 + (1 - \alpha)x_2, \quad 0 < \alpha < 1$$

Definition 3: A strictly convex set is a convex set with every point on its boundary being an extreme point.

We are now ready for the following Lemmas.

Lemma 1: A singular point, as defined in Definition 1, can occur if and only if there exist two distinct points $x_2, x_1 \in \partial V$ such that $\gamma(x_2 - x_1) \equiv m$ for some $\gamma \in \mathbb{R}$, i.e., a segment of the boundary $\partial V$ consists of a line parallel to $m$.

Proof: The velocity limit, $\dot{s}_{\text{max}}$, at some point $s \in [0, s_f]$, corresponds to the solution of the linear problem

$$\max_{s} \{ \dot{s}^2 \}$$

subject to

$$x = \dot{s}(s) + \dot{s}^2(s)$$

$$x \in V$$

If $V$ is convex, then $x^*$, corresponding to the solution $\dot{s}_{\text{max}}$, must lie at the intersection of the line $\Gamma\parallel m$ (parallel to $m$) with $\partial V$, as shown in Fig. 3. If $V$ is strictly convex, then the point $x^* \in \partial V \cap \Gamma$ is unique, and $x^*$ maps to a unique $\dot{s}(\dot{s}_{\text{max}})$, assuming $b$ and $m$, defined in (7), are independent.

If the boundary $\partial V$ includes a line segment parallel to $m$, then there exist two distinct points $x_2, x_1 \in \partial V \cap \Gamma$ such that $\gamma(x_2 - x_1) \equiv m$ for some $\gamma \in \mathbb{R}$. These points map to $\dot{s}(\dot{s}_{\text{max}})$, and $\dot{s}(\dot{s}_{\text{max}})_1, \dot{s}(\dot{s}_{\text{max}})_2$, respectively, as shown in Fig. 4. Since the mapping between $x$ and $\dot{s}(s)$ is unique and defined for all $x \in V$ for nonzero $b, m$, and $b \neq m$, then $x_1 \neq x_2$ follows $\dot{s}(\dot{s}_{\text{max}})_1 \neq \dot{s}(\dot{s}_{\text{max}})_2$, and vice versa. It follows that if there exist two distinct interior
IDENTIFYING SINGULAR POINTS

Based on the necessary conditions stated in Lemma 1 and the geometric interpretation of singular points, we can determine if a given point along the path is singular for general polyhedral sets of admissible controls.

Let the set of admissible controls be defined by the intersections of the planes \( \mathcal{B}_i \subset \mathbb{R}^n, i = 1, \ldots, m \), each satisfying

\[
x^T \mathbf{a}_i - c_i = 0, x \in \mathbb{R}^n
\]

where \( \mathbf{a}_i \) is a unit vector normal to the plane, and \( c_i \) is the perpendicular distance of the plane from the origin. By Lemma 1, some point \( s \) may be singular if the vector \( \mathbf{m}(s) \) is parallel to the intersection \( \mathbf{V} \cap \mathcal{B}_i \) for some \( i \), or equivalently, that \( \mathbf{m}(s) \) be normal to \( \mathbf{a}_i \):

\[
\mathbf{m}^T(s) \mathbf{a}_i = 0 \quad \text{for some } i = 1, \ldots, m
\]

where \( \mathbf{V} \cap \mathcal{B}_i \) be of nonzero measure

\[
(22)
\]

If (22) is satisfied for some \( i = 1, \ldots, m \), then the velocity limit at the corresponding singular point is computed by substituting (7) for \( x \) in (21) to yield

\[
\mathbf{m}^T(s) \mathbf{a}_i + \mathbf{b}^T \mathbf{z}_{\text{max}} \mathbf{a}_i = c_i.
\]

Then, substituting (22) into (23) we solve for \( \mathbf{z}_{\text{max}} \) at the intersection of \( \mathbf{b} \) with \( \mathcal{B}_i \):

\[
\mathbf{z}_{\text{max}} = \frac{\mathbf{c}_i}{\mathbf{b}^T \mathbf{a}_i}
\]

If there exists more than one such plane satisfying (22), then the velocity limit is the minimum of the values obtained from (24). If \( \mathbf{b} \) is normal to \( \mathbf{a}_i \) or \( \mathbf{b}^T \mathbf{a}_i = 0 \), then the boundary of \( \mathbf{W} \) that coincides with the plane \( \mathcal{B}_i \) has no effect on the velocity limit.

To compute the acceleration range at the singular point, we substitute (24) back into (7) to yield an expression for the acceleration at the velocity limit

\[
\mathbf{a}_i = 0 \quad \text{at singular points}
\]

Equation (25) defines the feasible acceleration range at the singular point. The extreme points on the line \( \mathbf{V} \cap \mathcal{B}_i \) map to the acceleration bounds.
The set consists of a hyper-ellipsoid, defined by

$$\sum_{i=1}^{n} \left( \frac{T_i}{T_{i,\max}} \right)^2 \leq 1, i = 1, \ldots, n$$  \hspace{1cm} (28)$$

where $T_{i,\max}$ are the constant actuator limits. This set accounts for the coupling between the individual actuators, which is common if the actuators are driven by a single power supply. A similar approach to modeling the actuator constraints for computing time optimal trajectories has been presented in [3]. It was motivated to save computation time, with no reference to the singularity issue.

Substituting (7) into (28) yields the scalar inequality constraint

$$a_1 \dot{s}^2 + 2a_2 s \dot{s} + a_3 \dot{s}^2 \leq 1$$  \hspace{1cm} (29)$$

where

$$a_1 = \sum_{i=1}^{n} \left( \frac{m_i^2}{T_{i,\max}^2} \right)$$  
$$a_2 = \sum_{i=1}^{n} \frac{m_i b_i}{T_{i,\max}^2}$$  
$$a_3 = \sum_{i=1}^{n} \frac{k_i^2}{T_{i,\max}^2}$$

Note that $a_1 \neq 0$ since $M$ is generally positive definite and $f_i$ is never zero. Therefore, we can solve (29) for the upper and lower bounds on $\dot{s}$:

$$\dot{s}_u(s, \dot{s}) = -a_2 \dot{s}^2 + \sqrt{(a_2^2 - a_1 a_3) \dot{s}^4 + a_1}$$  \hspace{1cm} (30)$$

$$\dot{s}_l(s, \dot{s}) = -a_2 \dot{s}^2 - \sqrt{(a_2^2 - a_1 a_3) \dot{s}^4 + a_1}$$  \hspace{1cm} (31)$$

The speed along the path $\dot{s}$ is then constrained by the inequality

$$\left( a_2^2 - a_1 a_3 \right) s^4 + a_1 \geq 0$$  \hspace{1cm} (32)$$

The velocity limit is obtained from the equality part of (32):

$$\dot{s}_{\text{max}} = \sqrt{\frac{a_1}{a_1 a_3 - a_2^2}}$$  \hspace{1cm} (33)$$

The radicand in (33) is positive since from (29) $a_1 > 0$, and it is easy to show that $a_1 a_3 > a_2^2$.

Given the acceleration bounds (30), (31) and the velocity limit (33), we can compute the optimal trajectory using any of the existing algorithms [1], [5]-[X] since singularity is no longer an issue. From (30) and (31), we can see that

$$\phi(s) = 2a_1 a_2 a_3 s + 4a_1 a_2 a_3 - a_1 a_3 - 4a_2^3 = 0$$  \hspace{1cm} (34)$$

It is easy to verify that $\phi(s) < 0$ if $\dot{s}_u > \dot{s}$, and $\phi(s) > 0$ if $\dot{s}_l < \dot{s}$. Since a tangency point is feasible only where the limit curve switches from a source ($\phi(s) < 0$) to a sink ($\phi(s) > 0$) [5], a potential tangency point is located at points where $\phi(s) = 0$ and where $\phi_s(s) > 0$.

The optimal trajectory computed with the elliptical actuator constraints (28) uses less actuator effort than the one computed with the widely used rectangular set since it might represent a realistic coupling between the individual actuators, caused by the power limits of a common power supply.

### VII. Examples

The method described in Section VI was implemented for a two-link planar manipulator with the parameters given in Table I.

**TABLE I**

<table>
<thead>
<tr>
<th>Parameters of the Two-Link Manipulator</th>
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<tr>
<td>$l_1 = 1.0$ m</td>
</tr>
<tr>
<td>$l_2 = 1.0$ m</td>
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</table>

Using (30), the acceleration at the velocity limit, $\ddot{s}_{\text{m}} = \ddot{s}(s, \dot{s}_{\text{max}})$ is

$$\ddot{s}_{\text{m}} = -\frac{a_2 \dot{s}_{\text{max}}^2}{a_1}$$  \hspace{1cm} (36)$$

Equating (35) and (36) and substituting $\dot{s}_{\text{max}}$ from (33) yields a necessary condition for a tangency point:

$$
\phi(s) = 2a_1 a_2 a_3 s + 4a_1 a_2 a_3 - a_1 a_3 - 4a_2^3 = 0
$$

The optimal trajectory computed with the elliptical actuator constraints (28) uses less actuator effort than the one computed with the rectangular set (5). It is not necessarily less optimal than the trajectory computed with the widely used rectangular set since it might represent a realistic coupling between the individual actuators, caused by the power limits of a common power supply.

![Fig. 5. A two-link manipulator at the end points of a path in Cartesian space.](image)

**Fig. 5.** A two-link manipulator at the end points of a path in Cartesian space.

![Fig. 6. Velocity limit curves and time optimal trajectories for the rectangular and the elliptical actuator constraints.](image)

**Fig. 6.** Velocity limit curves and time optimal trajectories for the rectangular and the elliptical actuator constraints.
mathematical model selected to represent the actuator constraints, and become singular at singular points, where the velocity constraint boundary of the ellipse except at the switching points, as shown in optimal control is nonsingular if the set of admissible controls tational efficiency of the existing algorithm 

actuator constraints. The elliptical set was demonstrated to produce a strictly convex set that avoids singular points is formed by elliptical specified paths has been addressed. Singularity is used in the control limit. 

is strictly convex. Singular control is, therefore, the result of the non-strict convexity of the convex set of admissible controls. It was proven that the optimal control is nonsingular if the set of admissible controls is strictly convex. Singular control is, therefore, the result of the mathematical model selected to represent the actuator constraints, and is not necessarily characteristic of the physical system. One choice of a strictly convex set that avoids singular points is formed by elliptical actuator constraints. The elliptical set was demonstrated to produce singularity free trajectories for a two link manipulator. 

This paper provides new insights into the singular control problem of the specified path optimization, helping us to identify potential singular points and arcs, or, alternatively, avoid singular control. Recognizing singular points facilitates smooth optimal trajectories, whereas avoiding singular control significantly improves the computational efficiency of the existing algorithm [1], [7]. The treatment of singular control also extends the theory of the specified path optimization to cases with general state inequality constraints.

REFERENCES


PD Control with Computed Feedforward of Robot Manipulators: A Design Procedure

Rafael Kelly and Ricardo Salgado

Abstract—In the present paper, we propose an explicit design procedure for selecting the proportional and derivative gains of the PD control with computed feedforward of robot manipulators. The proposed procedure, which is easily obtained from the robot dynamics and desired trajectory, assures that the closed-loop system has an unique equilibrium point. In addition, this equilibrium turns out to be (locally) exponentially stable. Simulation tests are included, with reference to a robot having two degrees of freedom.

I. INTRODUCTION

One of the most important goals in control of robot manipulators is motion control or trajectory control. Motion control is used when the robot arm moves in a free space following a desired trajectory without interacting with the environment. Among the model-based robot controllers reported in the literature that have been shown to be effective for motion control, we can list the following: Computed torque control, PD control with feedforward (nonadaptive version of controller proposed by [1]), PD+ suggested in [2]. PD control with computed feedforward and nonlinear feedback (Desired Compensation Control Law) presented in [3], and PD control with computed feedforward [4], [6], [7]. Excluding the latter, the remaining controllers have been shown to hold a globally asymptotically (or exponentially) stable closed-loop system.

The PD control with computed feedback is, perhaps, the simplest controller which can be employed for motion control of robots. This control scheme consists of a linear PD feedback plus a feedforward of the nominal robot dynamics computed (possibly off-line) along the desired joint trajectory. The only design parameters of this controller are the proportional and derivative gains. The PD control

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