Example Midterm #2 Problems

Here are a collection of questions that I deem suitable to ask on midterm exams in the future. They can be used to get a general feel for what sort of questions one might expect on a midterm. Note, however, that my experience has been that students almost always feel that the actual midterm questions are harder than those shown here. By and large, this is not usually the case (but there have been exceptions), questions often seem more difficult when they are new and when there is a time limit! In general it is safest to use the following questions to expose to yourself which areas you need more work on.

The questions are largely arranged by concept, progressing from issues of bias to confidence intervals to hypothesis tests. If you are going to look at a handful of questions, make sure you do some from each concept. The second midterm will cover all of these concepts.

Solutions are provided for some of the following questions - the later questions are missing solutions because I haven't had time to get around to solving them.

1) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from an exponential distribution with parameter λ. Show that $\bar{X}$ is an unbiased estimator of $1/\lambda$ given that $\bar{X}$ is the average of the sample.

2) If $X_1, X_2, \ldots, X_{16}$ is a random sample of size $n = 16$ from a normal distribution with mean 50 and variance 100, determine

$$\mathrm{P}\left[796.2 \leq \sum_{i=1}^{16} (X_i - 50)^2 \leq 2630\right]$$

3) Suppose that X is a random variable following some distribution and that $X_1,\ X_2, \ldots,\ X_n$ is a random sample from this distribution. One possible estimator of $\mathrm{E}^2[X]$ is $\bar{X}^2$ where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Show whether $\bar{X}^2$ is a biased estimator of $\mathrm{E}^2[X]$ or not.

4) Suppose that $\{X_1, X_2, \ldots, X_n\}$ form a random sample from a normal distribution for which both the mean and the variance are unknown. Estimate the point q such that $\mathrm{P}[X < q] = 0.95$.

5) The mass in milligrams of a certain pharmaceutical product is distributed normally with a variance of 0.0025 (mg^2). Find the minimum number of samples which must be taken in order to estimate the mean mass to within 0.025 mg at a 95% confidence level.

6) A metal tube is produced to have a mean wall thickness of 0.35 inches and a standard deviation of 0.025 inches. If 4 samples of the tube are taken and the wall thickness measured, find a lower limit below which the sample average will not fall more than 2 times out of a 1000 on average.

7) Let X denote the number of hours per week that a fourth year TUNS student spends studying probability and statistics. Suppose that $X \sim N(\mu, 4)$ and the unknown mean μ is to be estimated by $\bar{X}$, the mean of a random sample of size n.

   a) Determine the minimum number of samples required to ensure that the true value of μ is estimated to within 0.5 hours with probability 0.95.

   b) If a random sample of size 10 is to be taken, $\{X_1, X_2, \ldots, X_{10}\}$, what type of distribution does the statistic

$$\frac{\sum_{i=1}^{10}(X_i - \bar{X})^2}{\sigma^2}$$

(where $\sigma^2 = 4$) follow?

8) A light bulb manufacturer sells a light bulb that has a mean life of 1450 hours and a standard deviation of 33.7 hours. A new manufacturing process is being tested and there and there is interest in knowing the mean life μ of the new bulbs. How large a sample is required if μ is to be estimated to within 5 hours with 95% confidence? You may assume that the standard deviation is unaffected by the new process.

9) By letting $\theta = \lambda t$, the Poisson distribution can be expressed as

$$\mathrm{P}\left[X = x\right] = \frac{\theta^x}{x!}\, e^{-\theta} \qquad \text{for } x = 0, 1, 2, \ldots$$

Find the maximum likelihood estimator of the parameter θ, based on a sample $\{x_1, x_2, \ldots, x_n\}$ of size n.

10) Find the estimator of the parameter λ in the exponential distribution by the method of moments, based on a random sample of size n. State any necessary assumptions.

11) An industrial engineer is interested in estimating the mean time required to assemble a printed circuit board. How large a sample is required if the engineer wishes to to 95% confident that the error in estimating the mean is less than 0.25 minutes. Assume that the standard deviation of assembly time is 0.45 minutes and that the assembly time is normally distributed.

12) A random sample of size 25 is drawn from a normal population. If $\bar{x} = 550$ and $s^2 = 49$, find

   a) a 95% two-sided confidence interval on μ,

   b) a 95% two-sided confidence interval on σ^2.

13) The diameter of steel rods manufactured on two different extrusion machines is being investigated. Two random samples of sizes $n_1 = 15$ and $n_2 = 18$ are selected, and the sample means and sample variances are $\bar{x}_1 = 8.73$, $s_1^2 = 0.30$, $\bar{x}_2 = 8.68$, and $s_2^2 = 0.34$, respectively. Assuming that $\sigma_1^2 = \sigma_2^2$, construct a 95% two-sided confidence interval on the difference in mean rod diameter.

14) Suppose that 30% of the items produced by a plant are of poor quality. Suppose also that a random sample of n items is to be taken, and let Q_n denote the proportion of the items in the sample that are of poor quality. Find a value of n such that

$$\mathrm{P}\left[0.2 \le Q_n \le 0.4\right] \ge 0.75$$

by using Chebyshev's Inequality.

15) A physicist makes 25 independent measurement of the specific gravity of a certain body. She knows that the limitations of her equipment are such that the standard deviation of each measurement is σ units.

   a) find a lower bound on the probability that the average of her measurements will differ from the actual specific gravity of the body by less than $\sigma/4$ units using Chebyshev's Inequality,

   b) approximate the same probability as in (a) using the central limit theorem.

16) Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution having the following probability density function,

$$f_X(x) = \begin{cases} \theta x^{\theta - 1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If the parameter θ is unknown, find the maximum likelihood estimator of θ.

17) Let $\bar{X}$ be the average of a random sample of size $n = 15$ from an unknown distribution with mean $\mu = 80$ and variance $\sigma^2 = 60$. Find a lower bound for $\mathrm{P}\left[75 < \bar{X} < 85\right]$.

18) The life in hours of a 75 watt light bulb is assumed to be approximately normally distributed with σ = 25 hours. A random sample of 20 light bulbs has an average life time of 1014 hours.

    a) Construct a 90% confidence interval on the mean life,

    b) Re-evaluate part (a) assuming that σ is unknown but its estimate is s = 25 hours.

    c) Determine the number of samples required to estimate the mean life to within 5 hours at 90% confidence.

19) The distance between A and C is measured in two stages: namely, AB and BC. Measurements on AB and BC are recorded as follows;
    AB : $100.5, 99.6, 100.1, 100.3, 99.5$
    BC : $50.2, 49.8, 50.0$
    Assuming that the distances are normally distributed,

    a) compute the sample mean and sample variance for the measured distances AB and BC,

    b) establish a 98% confidence interval on the actual distance AB,

    c) if the distance AC is the sum of the distances AB and BC, establish a 98% confidence interval on the actual length AC.

20) Suppose that 9 observations are selected at random from a normal distribution for which both the mean μ and the variance σ^2 are unknown, and suppose that for these 9 observations it is found that $\overline{X}$ = 22 and S^2 = 72.

    a) Carry out a test of the following hypotheses at a significance level of 0.05:

    $$H_o: \quad \mu \leq 20,$$
    $$H_a: \quad \mu > 20.$$

    b) Construct a 90% confidence interval on μ.

21) Let X be the thickness of a layer of silicon deposited on a substrate and assume that X follows a normal distribution. The target thickness is 7.5 microns. Suppose that 10 samples are tested and determined to have the following thicknesses

    7.50 7.55 7.55 7.40 7.45 7.35 7.45 7.40 7.45 7.50

    a) Determine point estimates of μ_X and σ_X^2.

    b) Find a 90% confidence interval for μ_X.

    c) Can it be stated with 90% confidence that the true mean thickness is not 7.5 microns?

22) An official with the NCAA claims that only p = 0.01 = 1% of college football players use anabolic steroids. Others believe that $p > 0.01$. If 17 players from a random sample of 546 players tested positive to the drug, then what is your conclusion regarding the official claim? Use a significance level of α = 0.005.

23) Let p equal the proportion of women who agree that "men are basically selfish and self-centered." Suppose that in the past it was believed that p = 0.40. It is now claimed that p has increased. The *Detroit Free Press* (April 26, 1990) reported that 1260 out of a random sample of n = 3000 women agree with the statement. Test the claim that p has increased using a significance level of α = 0.01 and state your conclusion.

24) A manufacturer claims that the percentage of phosphorus in a fertilizer is at least 3 percent. Ten samples from a batch are taken and the percentage of phosphorus in each is measured. The ten measurements have a sample mean of 2.5 percent and a sample standard deviation of 0.5 percent. Is this sample mean *significantly* below the claimed value? You may assume that percent phosphorus is normally distributed.

25) An experiment was performed to compare the abrasive wear of two different laminated materials. Ten pieces of material 1 were tested by exposing each piece to a machine measuring wear. Eight pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The samples of material 1 gave an average wear of 85 units with a sample standard deviation of 4, while the samples of material 2 gave an average of 81 and a sample standard deviation of 5. Note that all data have been coded to simplify the calculations.

    a) Build a 90% confidence interval to compare the variability in the depth of wear of the two materials.

    b) Does material 2 have greater variability in depth of wear? **Using only the confidence interval you built in (a)**, draw a conclusion for a test at the 5% significance level. Explain your reasoning.

26) We are concerned with the performance of a process for the production of steel bolts. Upon inspection, 10 out of a random sample of 50 bolts do not pass specification requirements. It is desired to estimate the actual proportion p of bolts that meet specifications.

    a) Compute a point estimate for p.

    b) How large a sample is required if we want to be 97% confident that the error between our point estimate and the true proportion is less than 0.05?

    c) When testing at a significance level of 0.05, what is the approximate power of the test for $H_0 : p = 0.90$ vs $H_a : p < 0.90$ based on a sample of size 100 when the true proportion is 0.82?

27) Suspecting inadequate tightening on some units, engineers wish to collect data on the torque required to loosen bolts holding an assembly on a particular machine model. They wish to test the hypothesis $H_0 : \mu = 120$ against $H_a : \mu < 120$ using a random sample of size 16. The units of measurement are ft-lb and the standard deviation is assumed known and equal to 8 ft-lb. Suppose they define their critical region as $\bar{x} < 115$ ft-lb,

    a) what is the probability of type I error?

    b) what sample size should the engineers consider if they wish their test to have 98% power when the true mean torque is 112 ft-lb?

28) A structure is to rest on 100 piles. Nine test piles were driven at random locations into the supporting soil stratum and loaded until failure occurred. The resulting pile capacities (in tons) were 82, 76, 94, 90, 88, 92, 78, 85, and 80.

    a) What is a point estimate of the individual pile capacity to be used at the site?

    b) Establish a 98% confidence interval for the individual pile capacity to be used at the site.

    c) Standards are such that the standard deviation of the individual pile capacity cannot exceed 5 tons. Is it the case here? Test at the 5% significance level. Also report a p-value.

29) Two varieties, say 1 and 2, of girders are being considered for a bridge project. Specifically, type 1 is currently the standard, but type 2 might also be acceptable. Since type 2 is cheaper, it should be adopted, providing it does not have a decreased lifetime. To decide, fatigue testing is performed on both varieties and the number of cycles to failure are recorded.

    a) What hypotheses should be tested, and why?

    b) **In this context**, what is the type I error?

    c) **In this context**, what is the type II error?

    Suppose forty girders of each variety are tested and the collected data are analyzed to produce the following confidence intervals;

$$90\%\text{ CI for } \quad \mu_1 - \mu_2 : \qquad (0.35, 2.97)$$

95% CI for	$\mu_1 - \mu_2$:	$(0.07, 3.25)$
98% CI for	$\mu_1 - \mu_2$:	$(-0.28, 3.59)$
99% CI for	$\mu_1 - \mu_2$:	$(-0.53, 3.84)$

Based on these confidence intervals, what would one conclude if you were performing the following **arbitrary** tests. Explain your reasoning.

d) Testing $H_0 : \mu_1 = \mu_2$ vs $H_a : \mu_1 \neq \mu_2$ at the .05 significance level.

e) Testing $H_0 : \mu_1 = \mu_2$ vs $H_a : \mu_1 \neq \mu_2$ at the .12 significance level.

f) Testing $H_0 : \mu_1 = \mu_2$ vs $H_a : \mu_1 \neq \mu_2$ at the .015 significance level.

g) Testing $H_0 : \mu_1 = \mu_2$ vs $H_a : \mu_1 > \mu_2$ at the .05 significance level.

30) Design engineers are working on a low-effort steering system that can be used in vans modified to fit the needs of disabled drivers. The old-type steering system required a force of 53 ounces to turn the van's 15-inch diameter steering wheel. It is hoped that the new design will reduce the average force required to turn the wheel. Engineers wish to test the hypothesis $H_0 : \mu = 53$ against $H_a : \mu < 53$, using a random sample of size 16. The standard deviation is assumed known and equal to 2 ounces. Suppose the engineers define the critical region as $\bar{x} < 52.5$,

a) what is the probability of type I error?

b) what is the power of this test when the true force is 51.5 ounces?

We know that the power can be increased with an increased sample size.

c) What sample size should the engineers consider if they wish their test to have 97% power when the true force is 52 ounces?

31) Engineers studied the effects of varying pressures on the density of cylindrical specimens made by dry pressing a ceramic compound. A mixture of Al_2O_3, polyvinyl alcohol, and water was prepared, dried overnight, crushed, and sieved to obtain 100 mesh size grains. These were pressed into cylinders at pressures from 2000 psi to 10000 psi, and cylinder densities were calculated. The table below gives the data that were obtained.

Pressure (psi)	**Density** (g/cc)
2000	2.486
2000	2.479
2000	2.472
4000	2.558
4000	2.570
4000	2.580
6000	2.646
6000	2.657
6000	2.653
8000	2.724
8000	2.774
8000	2.808
10000	2.861
10000	2.879
10000	2.858

a) Limiting yourself to those densities obtained under the 2000 and 4000 psi pressures, assess the strength of the evidence that increasing pressure increases the mean density of the resulting cylinders. Test at the 2%

significance and also report a p-value. If you are required to make assumptions to carry out the test, state these assumptions, indicate how each one may be checked, and if tests are available, test for the likely validity of these assumptions.

b) Assume that the 15 cylinder densities shown above represent a random sample from the population of interest. Give a 96% confidence interval for the proportion of densities lying between 2.5 and 2.7 g/cc.

32) Suppose that a sample of 9 steel reinforcing bars were tested for yield strength and the sample mean was found to be 20 kips.

a) What is the 90% confidence interval for the population mean if the population standard deviation is assumed to be equal to 3 kips?

b) How many additional bars must be tested to increase the confidence of the same interval found in (a) to 95%? (That is, what size sample yeilds the same interval as in (a) but at 95% confidence?)

33) The straight-line distance between two geodetic stations A and B is measured with the electronic ranging instrument called a *tellerometer*. The following are ten independent measurements of the distance:

45479.4 m	45479.6 m	45479.3 m	45479.5 m	45479.8 m
45479.2 m	45479.6 m	45479.5 m	45479.3 m	45479.1 m

a) Estimate the true distance.

b) Determine the 90% confidence interval for the true distance.

c) Suppose that an expert on the matter had previously claimed that the distance was 45479.6 m. What is the probability that a sample mean as short or shorter than the one obtained using the 10 observations above is observed assuming that the expert is correct?

34) The effective life of a component used in a jet-turbine aircraft engine is a random variable with mean 5000 hr and standard deviation 40 hr. The distribution of effective life is fairly close to a normal distribution. The engine manufacturer introduces an improvement into the manufacturing process for this component that increases the mean life to 5050 hr and decreases the standard deviation to 30 hr. Suppose that a random sample of n_1 = 16 components is selected from the "old" process and a random sample of n_2 = 25 components is selected from the "improved" process. What is the probability that the average of the sample from the "improved" process exceeds the average of the sample from the "old" process by at least 25 hr?

35) An article in the journal *Hazardous Waste and Hazardous Materials* (Vol. 6, 1989) reported the results of an analysis of the weight of calcium in standard cement and cement doped with lead. Reduced levels of calcium would indicate that the hydration mechanism in the cement is blocked, allowing water to wash away various parts of the cement structure. Ten sample of standard cement had an average weight percent calcium of 90.0, with a sample standard deviation of 5.0, while 15 samples of the lead-doped cement had an average weight percent calcium of 87.0, with a sample standard deviation of 4.0. Assume that the weight percent calcium is normally distributed and find a 95% confidence interval on the difference in means.

36) The journal *Human Factors* (pp. 375-380, 1962) reports a study in which 14 subjects were asked to park two cars having substantially different wheel bases and turning radii. The time in seconds was recorded for each car and subject, and the resulting data are shown below.

Subject	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Auto 1	37.0	25.8	16.2	24.2	22.0	33.4	23.8	58.2	33.6	24.4	23.4	21.2	36.2	29.8
Auto 2	17.8	20.2	16.8	41.4	21.4	38.4	16.8	32.2	27.8	23.2	29.6	20.6	32.2	53.8

Make and state any required assumptions and find a 90% confidence interval on the difference in the two means. In what way should this data be compared in order to draw conclusions about the cars?

37) In an article by Ophir, El-Gad, and Snyder entitled "Inference for the Fraction of Dry Cells with Internal Shorts," (*Journal of Quality Technology*, 1988) a series of experiments were conducted to reduce the proportion of cells being scrapped by a battery plant because of internal shorts. At the beginning of the study about 6% of the cells were being scrapped because of internal shorts. At a point in the work described in the paper, in a sample of 235 cells produced under a particular trial set of plant operating conditions, there were 9 cells with shorts.

   a) Construct a 95% confidence interval for the true proportion, p, of batteries being scrapped at this point.

   b) Using the confidence interval, do you think the proportion of batteries being scrapped has been reduced under these trial plant operating conditions?

   c) Formally test the hypothesis that the batteries being scrapped has been reduced at the 5% significance level and report a p-value.

38) In a paper entitled "Comparing Fractions Conforming for Two Methods of Operating a Pelletizing Process", Greiner, Grim, Larson, and Lukomski studied several different methods of running a pelletizing process. Two of these involved using a mix with 20% reground powder with respectively small (condition 1) and long (condition 2) shot sizes. Of 100 pellets produced under condition 1, 38 proved to conform to specifications, while of 100 pellets produced under condition 2, 29 proved to conform to specifications.

   a) Build a 90% confidence interval for comparing the two methods of process operations.

   b) Do the two methods produce the same proportions of pellets conforming to specifications? Test at $\alpha = 0.05$ and report a p-value.

39) It is said that "doctors bury their mistakes, architects cover them with ivy, engineers write long reports which never see the light of day" ("Foundation Failures", D.W. Joyce, *Civil Engineering*, April 1983). One area in which engineering mistakes are critical is dam-building. How large a sample is necessary to estimate the percentage of non-federal earth dams in the United States that are in need of immediate repair to within 1% with at least 90% confidence?

40) The "supergopher" is a device invented to drill through Arctic pack ice. It is a cone-shaped apparatus 5 feet high, 4 feet wide, and wound with a copper coil. Water heated to 180^{o}F is pumped through the coil. This allows the gopher to melt a vertical round shaft through the ice. Let X denote the distance (depth) that the gopher can drill per hour. These data are obtained on 10 test holes (in feet/hour):

2.0	1.7	2.6	1.5	1.4
2.1	3.0	2.5	1.8	1.4

Find a 95% confidence interval for the variance of the distance that can be drilled in an hour.

41) In 1969, in the Unite States, on average 8% of household waste was metal. Because of the increase in recycling efforts, it is hoped that this figure has been reduced. An experiment is run to verify this contention.

   a) Set up the appropriate null and alternative hypotheses for the experiment.

   b) Explain in a practical sense what has occurred if a Type I error is committed.

   c) Explain in a practical sense what has occurred if a Type II error is committed.

42) A physically stable jar filling process is known to have an associated standard deviation of $\sigma = 1.6$g net weight. Suppose that with a declared (label) weight of 135g, process engineers have set a target mean net will weight at $135 + 3\sigma = 139.8$g. Suppose further that in a routine check of filling process performance, intended to detect any changes of the process mean from its target value, a sample of $n = 25$ jars produced a sample mean fill weight of 139.0g.

a) What does this value of $\bar{x}$ have to say about the plausibility of the current process mean actually being at the target mean of 139.8g? Use a formal testing procedure.

b) What is the probability of a type II error when the true mean is 138.5g?

c) How large a sample is required if we want the power of our test to be 0.90 when the true mean is 139.0g?

d) How large a sample is required if we want the power of our test to be 0.95 when the true mean is 139.5g?

43) Part of a data set of W. Armstrong (*Analysis of Survivial Data* by Cox and Oakes) gives number of cycles to failure of ten springs of a particular type under a stress of 950 N/mm^2. They are (in units of 1000 cycles): 225, 171, 198, 189, 189, 135, 162, 135, 117, 162. Test the manufacturer's claim of a mean number of cycles to failure of 185.

44) The data set collected by W. Armstrong also includes spring longevity at 900 N/mm^2. They are (in units of 1000 cycles): 216, 162, 153, 216, 225, 216, 306, 225, 243, 189. Test the hypothesis of equal mean lifetimes with an alternative of increased lifetime accompanying a reduction in stress level.

45) D. Kim did some crude tensile strength testing on pieces of some nominally 0.012in diameter wire of various lengths. Below are Kim's measured strengths (in kg) for pieces of wire of length 25cm:

4.00	4.65	4.70	4.50	4.40	4.50	4.50	4.20

Is this data sufficient evidence to question the manufacturer's claim of a tensile strength variability f $\sigma = 0.15$? Test at the $\alpha = 0.05$ significance level. Also report a p-value.

46) A study of two types of materials used in electrical conduits (tubes used to house electrical wires) is to be conducted. The purpose of the study is to compare the strength of one to the other. Strength is to be assessed by measuring the load in pounds required to crush a 6 inch piece of the conduit to 40% of its original diameter. The primary question to be answered is, "Does material I on average withstand a heavier load than material II?". However, before this question can be answered, we must consider the question, "Is $\sigma_1^2 = \sigma_2^2$" as reasonable results on comparing the mean can be obtained only if equality of variance can be assumed. Test for equality of variance at the $\alpha = 0.2$ significance level the following data;

Material I	Material II
$n_1 = 25$	$n_2 = 16$
$\bar{x}_1 = 380$ lb	$\bar{x}_2 = 370$ lb
$s_1^2 = 100$	$s_2^2 = 400$

47) Two different formulations of an oxygenated motor fuel are being tested to study their road octane numbers. The variance of road octane number of formulation 1 is $\sigma_1^2 = 1.5$, and for formulation 2 it is $\sigma_2^2 = 1.2$. We wish to test at $\alpha = 0.05$.

a) How large a sample is required to achieve a power of 0.90 when the two formulations differ by 1?

b) What is the power of the test based on a sample size of 20 when the formulations differ by 1?