A Practical Physical-Layer Network Coding Scheme for the Uplink of the Two-Way Relay Channel

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Abstract—In this paper, we consider the two-way relay channel with the two-phase protocol. In the first phase, two terminals simultaneously transmit their packets to the relay, creating a multiple-access channel (MAC), while in the second phase the relay sends a network-coded combination of the two packets to both terminals. We focus on the multiple-access phase and propose a practical decoding scheme based on practical binary block codes in order to obtain the combined packet at the relay. This scheme is found to perform well for a wide range of code rates and is superior to lattice coding at low rates.

I. INTRODUCTION AND MOTIVATION

In the two-way relay channel with network coding, the relay broadcasts the bit-wise modulo-2 sum (the “xor”) of the two packets. In the first phase, the two terminals A and B transmit their packets simultaneously such that the relay receives the superimposed signals, from which it tries to recover only the bit-wise sum of the two packets. It has been recognized by several authors that for the relay it is not necessary to decode the individual packets (see [1] for an excellent and up-to-date review). Apart from the uplink in the two-way relay (TWR) channel, the problem of decoding the binary sum of the transmitted packets appears also in collision-resolution mechanisms [2] and in network nodes which apply linear network coding and can make use of packet combinations.

Practical applications for two-way relaying include cellular networks which employ relays, wireless backhauls and satellite communications. In the following, we consider both the Gaussian and the fading symmetric multiple-access channel and discuss practical coding schemes which recover the modulo-2 sum of the packets. This work has been inspired by [3], who presented a non-binary sum-product algorithm for this task. In this paper, we compare the performance achieved with binary and non-binary decoders to several bounds on mutual information and find that applying linear codes with a corresponding extended decoder can provide a performance superior to the bounds for binary decoding, as well as to lattice coding.

Related work focusses primarily on lattice coding [1], [14], [4], [5], which has emerged as the prevailing including an interesting generalization of the basic network coding approach to more general functions of both packets [6], [7] and on symbol-wise processing [8]. In this paper, we focus on schemes which apply practical linear block codes.

MAC phase

Broadcast phase

Figure 1. Two-slot protocol for two-way relay channel

II. SYSTEM MODEL

In a TWR channel, the two terminals A and B exchange their packets $u_a$ and $u_b$ via a relay. As depicted in Fig. 1, this exchange can be performed in two phases:

1) During the MAC phase both terminals transmit their packet to the relay.

2) In the broadcast phase, the relay transmits the binary sum of both packets.

In the following, we only consider the MAC phase and assume that the relay requires the binary sum $u_{ab} = u_a \oplus u_b$.

We do not consider in this paper the cases in which the relay retransmits another function $f(u_a, u_b)$ of both messages or a symbol-wise function of the received symbols. In particular, we assume that the relay employs a soft-input decoder in order to recover the complete combined packet $u_{ab}$.

As drawn in Fig. 2, the two terminals A and B encode their messages $u_a$, $u_b \in \mathbb{F}_2^k$, each consisting of $k$ bits, into the codewords $c_a, c_b \in \mathbb{F}_2^k$ of length $n$ which are consequently mapped on BPSK or QPSK signals $x_a, x_b$ with transmit power $E_S = \mathbb{E} [|x_a|^2] = \mathbb{E} [|x_b|^2] = 1$. The received signal at the relay is given by

$$y = h_a x_a + h_b x_b + w$$

where $w$ is either real-valued or complex-valued Gaussian noise with variance $N_0/2$ per component and $h_a, h_b$ are fading coefficients or set to unity for the Gaussian channel.

Both terminals use the same binary linear channel code, such that we can write $c_a = u_a G$, $c_b = u_b G$ with the same generator matrix $G \in \mathbb{F}_2^{k \times n}$. This formulation holds for any binary linear block code, being LDPC and turbo codes the most prominent and practical examples. At the relay, we employ a soft decoder, e.g. the sum-product algorithm in the case of LDPC codes, in order to recover the sum of the two messages $u_{ab} = u_a \oplus u_b \in \mathbb{F}_2$. Note that all operations on $u_i$ and $c_i$ are carried out in the binary Galois field $\mathbb{F}_2$ while the signals $x, h, w$ and $y$ live in the real field $\mathbb{R}$, or the
Figure 2. System model for MAC phase of two-way relay channel

Table I

<table>
<thead>
<tr>
<th>c_a</th>
<th>c_b</th>
<th>c_ab</th>
<th></th>
<th>x_a</th>
<th>x_b</th>
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<td>2</td>
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</tbody>
</table>

complex field \( \mathbb{C} \). For BPSK modulation, the transmit signals are represented in Table I.

In the following, we will treat different approaches to recover the combined packet \( u_{ab} \) at the relay. Note that to this end it is in general not necessary to recover the individual messages \( u_a \) or \( u_b \).

III. Decoding Options

As a consequence of using the same linear code at both terminals, the binary sum of the two codewords \( c_{ab} \triangleq c_a \oplus c_b = u_{ab}G \) is also a codeword of this code, and it is precisely the codeword which corresponds to the message \( u_{ab} \). This motivates the following binary decoding scheme.

A. Binary Decoding

Since the relay is only interested in the combined packet \( u_{ab} \), the straightforward decoding approach is to derive the L-values with respect to the combined symbols \( c_{ab} = c_a \oplus c_b \),

\[
L^{(\text{bin})} \triangleq \ln \frac{P(c_{ab} = 1|y)}{P(c_{ab} = 0|y)} = \ln \frac{\exp \left( \frac{-(y-h_a+h_b)x_a}{N_0} \right) + \exp \left( \frac{-y+h_a+h_b}{N_0} \right)}{\exp \left( \frac{-(y-h_a+h_b)x_a}{N_0} \right) + \exp \left( \frac{-y+h_a+h_b}{N_0} \right)}
\]

(2)

\[
= \frac{4h_xh_y}{N_0} \ln \frac{\cosh \left( \frac{y-h_a+h_b}{N_0} \right)}{\cosh \left( \frac{y-h_a-h_b}{N_0} \right)}
\]

These L-values are fed to the soft-in decoder, which finds an estimate for \( u_{ab} \). This method seems optimum at first glance since the relay directly decodes for the desired message \( u_{ab} \), and we can directly derive the L-values for the corresponding codeword \( c_{ab} \). However, it will turn out that this is not the case.

B. Quaternary Decoding

A less obvious approach, pioneered by D. Wübben [3], considers the combination of both code bits into a two-dimensional vector \( [c_a, c_b]^T \in \mathbb{F}_2^2 \). We can consider the matrix codewords \( \begin{bmatrix} c_a \\ c_b \end{bmatrix} \in \mathbb{F}_2^{2 \times n} \) as forming a new code, which is defined by the parity-check equations

\[
\begin{bmatrix} c_a \\ c_b \end{bmatrix} \mathbf{H}^T = \mathbf{0}_{2 \times (n-k)}
\]

(3)

with the common parity-check matrix \( \mathbf{H} \), which is the parity-check matrix of the binary code, i.e. \( \mathbf{G} \mathbf{H}^T = \mathbf{0} \). This code is still a linear code with codewords of size \( 2 \times n \).

We can also express this code as a quaternary code by assigning to each bit vector the quaternary symbol \( \bar{c} \in \mathbb{F}_4 = \{0, 1, 2, 3\} \) according to Table I. Note, however, that this code is not a linear code in the Galois field \( \mathbb{F}_4 \). We can represent the quaternary symbols as elements of \( \mathbb{F}_4 \), but for the code to be linear, we require that any linear combination of two codewords yields again a codeword. In general, for a given linear code in \( \mathbb{F}_2^2 \), there is no representation in \( \mathbb{F}_4 \) such that this condition is fulfilled.

We nevertheless can employ a decoder in \( \mathbb{F}_4 \), e.g. Declercq’s non-binary belief propagation decoder based on FFT processing [9], with the binary parity-check matrix \( \mathbf{H} \). The decoder output is a quaternary message \( \bar{u} \in \mathbb{Z}_4^k \), which has to be mapped back to the binary values of \( u_{ab} \) according to Table I. From \( \bar{u} \) we can also extract the individual messages \( u_a \) and \( u_b \). Note that for correct decoding of \( u_{ab} \) it is not necessary that the individual messages be correctly decoding. Actually, for the Gaussian channel, the combined packet is typically recovered while the individual messages are not.

The L-values for this quaternary code are given as log-likelihood vectors (one vector for each code symbol) as

\[
\mathbf{L} = \ln \begin{bmatrix} P(\bar{c} = 0|y) \\ P(\bar{c} = 1|y) \\ P(\bar{c} = 2|y) \\ P(\bar{c} = 3|y) \end{bmatrix} = -\frac{1}{N_0} \begin{bmatrix} (y + h_a + h_b)^2 \\ (y + h_a - h_b)^2 \\ (y - h_a + h_b)^2 \\ (y - h_a - h_b)^2 \end{bmatrix} + \ell_1
\]

(4)

where \( \ell_1 \) is an irrelevant additive constant. It is interesting to observe that the expression for the quaternary L-value in (4) is much simpler than the one for its binary counterpart (2).

C. Comparison of Binary and Quaternary Decoding

We have performed simulations for both decoding methods using the LDPC code family of DVB-S2 of length \( n = 64800 \) bits [10]. This code family comprises a set of irregular LDPC codes with rates in \( R = \{ \frac{1}{3}, \frac{1}{2}, 1, 2 \} \). We have chosen these codes for several reasons: they are standardized and therefore facilitate reproducibility, they offer a wide range of rates, and they are sufficiently long to perform close (within 1 dB) to the Shannon limit. Fig. 3 shows the BER of the combined message \( u_{ab} \) for both decoding approaches over a Gaussian channel and with BPSK modulation. We can observe that for low code rates (operated at low SNR), quaternary decoding is significantly better than binary decoding, while for medium to high code rates there is no appreciable difference.

This raises the question on the fundamental limits for both approaches. In the next section, we will give some...
partial answers to this question. An intuitive explanation for the performance loss of binary decoding lies in the loss of information in the calculation of the binary L-values: this step discards part of the information contained in the receive signal \( y \).

IV. MUTUAL INFORMATIONS

In this section, we compute several mutual informations between the received signal \( y \) and different transmit signals. In all cases, we assume that the signals \( c_a \) and \( c_b \) are i.i.d. uniform Bernoulli variables, and hence both \( c_{ab} \) and \( \bar{c} \) are also uniformly distributed.

A. Network-Coded Signal

We can define and compute the mutual information between the bit-wise binary sum \( c_{ab} \) and the received signal \( y \) as

\[
C_{ab} = I(c_{ab}; y) = \sum_{c_{ab}=0}^{1} \int_{-\infty}^{\infty} p(c_{ab}, y) \log \frac{p(c_{ab}, y)}{p(c_{ab})p(y)} \, dy \leq 1
\]

This capacity constitutes the upper bound for binary decoding.

There is no closed-form expression for this mutual information, but for numerical computation we can express \( C_{ab} \) as an expectation of easily-generated random variables. With

\[
p(c_{ab} = 0, y) = \frac{1}{4} \left( p(y|\bar{c} = 0) + p(y|\bar{c} = 3) \right)
p(c_{ab} = 1, y) = \frac{1}{4} \left( p(y|\bar{c} = 1) + p(y|\bar{c} = 2) \right)
\]

we obtain after some calculations

\[
C_{ab} = 1 - \frac{1}{2} \mathbb{E}_{w, h_a, h_b} \left[ \log \left( 1 + \frac{b_{1,0} + b_{0,1}}{1 + b_{1,1}} \right) \right.
\]

\[
+ \left. \log \left( 1 + \frac{b_{1,0} + b_{0,-1}}{1 + b_{1,-1}} \right) \right]
\]

\[= \text{ld} \left( 1 + \frac{b_{1,0} + b_{0,1}}{1 + b_{1,1}} \right)
\]

\[+ \text{ld} \left( 1 + \frac{b_{1,0} + b_{0,-1}}{1 + b_{1,-1}} \right)
\]

\[= \text{ld} \left( 1 + b_{1,0} + b_{0,1} + b_{1,1} \right)
\]

\[+ \text{ld} \left( 1 + b_{-1,0} + b_{0,1} + b_{-1,1} \right)
\]

\[C_{c} = 2 - \frac{1}{2} \mathbb{E}_{w, h_a, h_b} \left[ \log \left( 1 + \frac{b_{1,0} + b_{0,1}}{1 + b_{1,1}} \right) \right.
\]

\[+ \left. \log \left( 1 + \frac{b_{1,0} + b_{0,-1}}{1 + b_{1,-1}} \right) \right]
\]

This capacity corresponds to the sum-rate constraint of the multiple-access channel with binary input.

For the Gaussian channel without noise, this channel reduces to the discrete memoryless channel depicted in Fig.4, which has a capacity of \( \frac{3}{2} \) bits per channel use. This means that for code rates above \( \frac{3}{4} \) it is impossible to recover the individual messages.

B. Quaternary Signal

The mutual information between the quaternary signal \( \bar{c} \) and the received signal is given by

\[
C_{\bar{c}} = I(\bar{c}; y) = \sum_{\bar{c} = 0}^{3} \int_{-\infty}^{\infty} p(\bar{c}, y) \log \frac{p(\bar{c}, y)}{p(\bar{c})p(y)} \, dy \leq 2
\]

\[C_{\bar{c}} = 2 - \frac{1}{2} \mathbb{E}_{w, h_a, h_b} \left[ \log \left( 1 + b_{1,0} + b_{0,1} + b_{1,1} \right) \right.
\]

\[+ \left. \log \left( 1 + b_{-1,0} + b_{0,1} + b_{-1,1} \right) \right]
\]

This can be expressed as

\[
C_{\bar{c}} = 1 - \frac{1}{2} \mathbb{E}_{w, h_a, h_b} \left[ \log \left( 1 + \frac{b_{1,0} + b_{0,1}}{1 + b_{1,1}} \right) \right.
\]

\[+ \left. \log \left( 1 + \frac{b_{1,0} + b_{0,-1}}{1 + b_{1,-1}} \right) \right]
\]

This can be expressed as

\[
c_{\bar{c}} = 1 - \frac{1}{2} \mathbb{E}_{w, h_a, h_b} \left[ \log \left( 1 + \frac{b_{1,0} + b_{1,1}}{1 + b_{0,1}} \right) \right.
\]

\[+ \left. \log \left( 1 + \frac{b_{1,0} + b_{1,-1}}{1 + b_{0,-1}} \right) \right]
\]
D. Single-User Bound

An upper bound for the rates of all binary signals is given by the single-user bound, for which we set \( x_b = 0 \) and obtain the capacity of the binary-input channel

\[
C_{\text{BPSK}} = I(c_a; y | x_b = 0) = 1 - \mathbb{E}_{w,a}\left[ \log(1 + b_1,0) \right]
\]  

(12)

For the AWGN channel, this capacity can be expressed with the J-function of \([13]\) as

\[
C_{\text{BPSK}}^{(\text{AWGN})} = J \left( \frac{2E_b}{N_0} \right)
\]

(13)

, while for Rayleigh fading, this capacity is given with the \( \beta \)-function, \( \beta(x) \triangleq \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k} \), as

\[
C_{\text{BPSK}}^{(\text{Rayleigh})} = \frac{1}{\ln 2} \cdot \frac{2}{1 + \frac{2E_b}{N_0}} \beta \left( \sqrt{\frac{2E_b}{N_0}} \right)
\]

(14)

Note that for the real-valued channels, the SNR is actually \( \frac{2E_b}{N_0} \) since the noise variance is by convention \( \sigma_w^2 = \frac{N_0}{2} \).

E. Relations between Mutual Informations

It can be shown by evaluating (9), (11) and (12) that

\[
C_{\text{BPSK}} + C_a = C_b
\]

For binary decoding, we can consider the rate \( R_{ab}^{(\text{bin})} = R_{b}^{(\text{bin})} \) corresponding to the transmission and correct decoding of the message \( u_b \) or \( u_a \). Both rates are equal due to the symmetry of the channel. We denote the rate of the combined packet with binary decoding by \( R_{ab}^{(\text{bin})} \). For BPSK modulation, these rates are simply the code rates. For binary decoding, these rates are limited by

\[
\begin{align*}
R_{a}^{(\text{bin})} & \leq C_a \quad (15) \\
R_{ab}^{(\text{bin})} & \leq C_{ab} \quad (16)
\end{align*}
\]

In general, i.e. also for quaternary decoding, it holds

\[
\begin{align*}
R_{a} & \leq \min \left\{ \frac{1}{2} C_b, C_{\text{BPSK}} \right\} \quad (17) \\
R_{a} + R_{ab} & \leq C_b \quad (18)
\end{align*}
\]

V. SIMULATION RESULTS AND DISCUSSION

In Fig. 5, the required \( E_b/N_0 \) for each code rate to obtain a BER of \( 10^{-4} \) is plotted, along with the mutual informations \( C_{ab}, C_{\text{BPSK}} \) according to (5)(12) and the achievable rate for lattice coding \( \frac{1}{2} \log \left( 1 + \frac{2E_b}{N_0} \right) \) \([14],[1]\). The points for \( R_{ab}^{(\text{bin})} \) and \( R_{ab}^{(\text{quat})} \) correspond to the points in Fig. 3 in which the BER curves intersect with the horizontal line for \( \text{BER} = 10^{-4} \).

From Fig. 5, we can make several interesting observations:

- For binary decoding, the points \( R_{ab}^{(\text{bin})} \) are within 1 dB of \( C_{ab} = I(c_{ab}; y) \), which is consistent with the performance of this code family over the AWGN channel.

- For low code rates, \( R \in \{ \frac{1}{2}, \frac{1}{4} \} \), quaternary decoding performs better than the achievable rate of lattice coding and for \( R = \frac{1}{4} \) it is even superior to \( C_{ab} \). This means that with the rate 1/4 code from the DVB-S2 standard and a non-binary decoding, we achieve better performance than any code with a binary decoder.

This last observation confirms that quaternary decoding is not limited by (5). The natural question, which remains open in this paper, is on the fundamental limit for recovering the combined message \( u_{ab} \).

Fig. 6 depicts the required SNRs for the Rayleigh fading channel with BPSK. For this channel, the gap between binary and quaternary decoding increases to several dBs. For the fading channel, it is also possible to recover the individual messages for the entire range of code rates, and it is interesting
to observe that for low and medium code rates, at the same SNR, we can recover both the combined as well as the individual messages. In this case, the points for $R_a$ and $R_{ab}$ coincide, which means that according to (18), the curve $\frac{1}{2}C_e$ constitutes an upper bound. For high rates, it requires more SNR to recover the individual message, while for gap between binary vs. quaternary decoding for the combined message shrinks. (For the problem at hand, decoding of the individual messages can be considered as a by-product of the quaternary decoder.)

We can also observe that for most code rates, the rate $R_a > C_a$, which means that quaternary decoding for the individual message with the given finite-length codes is superior to binary decoding with any code. $C_a$ also is the capacity for the individual message if we apply independent random codes for each message. This implies that for the TWR channel, linear (and implementable) codes can outperform the information-theoretic limit for random coding. This is in line with the findings in [14], [1] for lattice codes.

The required SNRs for given rates for the complex-valued Rayleigh fading channel with QPSK modulation are shown in Fig. (7). While the complex-valued Gaussian TWR and the complex-valued Rayleigh fading channel can be separated into two real-valued channels, this is not possible for the fading TWR with a complex fading coefficient. For this reason, the results in Fig. (7) are not a scaled version of Fig. (6). The computation of the L-values for QPSK is a relatively straightforward, albeit not trivial, extension of the BPSK case, involving a marginalization. For the complex-valued fading TWR channel, there is a considerable gap between binary and quaternary decoding at all rates, with an advantage of about 5 dB for the quaternary scheme at the lowest rate. In contrast to the real-valued TWR fading channel, the achievable rates for the individual and the combined message coincide now for all cases and are therefore limited by $\frac{1}{2}C_e$.

VI. CONCLUSION

We have considered the MAC phase of the two-way relay channel with the two-phase protocol, in which the relay retransmits the xorred message of both terminals. In order to obtain the combined message $u_k \oplus u_b$ at the relay, we apply practical binary linear block codes (in particular the LDPC codes from the DVB-S2 standard) and consider two decoding options: (1) direct binary decoding for $u_k \oplus u_b$ based on the linearity of the code, (2) quaternary decoding for both messages simultaneously and mapping back to the combined message. We find that quaternary decoding is always superior or equal to binary decoding and in some cases the gap amounts to several dBs. For low code rates on the Gaussian two-way relay channel, quaternary decoding with standard linear block codes is found to perform better than the upper bound for lattice coding.

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