

Linear Fractional p -Adic and Adelic Dynamical Systems

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Abstract

Using an adelic approach we simultaneously consider real and p -adic aspects of dynamical systems whose states are mapped by linear fractional transformations isomorphic to some subgroups of $GL(2, \mathbb{Q})$, $SL(2, \mathbb{Q})$ and $SL(2, \mathbb{Z})$ groups. In particular, we investigate behavior of these adelic systems when fixed points are rational. It is shown that any of these rational fixed points is p -adic indifferent for all but a finite set of primes. Thus only for finite number of p -adic cases a rational fixed point may be attractive or repelling. It is also shown that real and p -adic norms of any nonzero rational fixed point are connected by adelic product formula.

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1 Introduction

There are many dynamical systems whose states change in discrete time intervals. In such discrete time dynamical system its state changes by a mapping

$$f : X \rightarrow X, \tag{1}$$

where X is the state space and the map f describes how states evolve in time units. It is suitable to study evolution of such time discrete systems by iteration. If the state at the time $t = 0$ is $x_0 \in X$ and $f^n = f \circ \dots \circ f$ then after n iterations the state becomes

$$x_n = f^n(x_0). \tag{2}$$

The state space X has usually some natural additional structures such as hierarchies and distances between states. In particular, in physics and other related topics, the state space of very complex systems often displays a hierarchical structure. This implies that the classification of the states and their relationships may be based on an ultrametric, and in particular p -adic distance d_p . Recently much attention has been paid to some p -adic dynamical systems, since they have a lot of potential applications (for a review, see [1]).

Among the disordered systems, the mean field models for spin glasses whose ground states have ultrametric structure are of particular interest in mathematical physics [2]. Methods of p -adic analysis are applied to the investigation of replica symmetry breaking [3]. The ultrametricity arises from the basic properties of the field of p -adic numbers, the most important example of ultrametric spaces. This p -adic reformulation and further generalizations could be useful starting point in the study of the whole structure of spin glasses [4].

Also the cut and project method, commonly used in the study of quasicrystals and aperiodic order, has been recently extended [5]. Namely the key ingredients of this scheme, internal spaces, are no longer Euclidean but spaces with non-Euclidean topologies. Namely p -adic topologies or mixed Euclidean/ p -adic topologies are combined in the physical-internal space pair.

From the above examples it seems that ultrametricity is a common ingredient which cannot be avoided, and its p -adic treatment seems to be quite natural. Since 1987 there have been many constructions of p -adic physical models. In particular, p -adic numbers have been successfully employed in string theory, quantum mechanics and quantum cosmology (for a review, see [6], [7] and [8], respectively).

When each state of physical system is associated to several distinct hierarchical structures, the labeling of states by p -adics is no longer sufficient and appropriate index set becomes the ring of adeles. This was demonstrated for an asymmetric stochastic process on the adeles [9]. Ingredients of adeles are real and all p -adic numbers, which contain rationals in a dense way. According to [10] (see also motivations in [9]) rational numbers imply foundation of quantum theory on adelic spaces and some progress has been achieved in

string theory, quantum mechanics and quantum cosmology (for a review, see [6], [7] and [8], respectively).

Especially the unified aspect of adeles is a motivation to employ adelic approach to dynamical systems. In particular, we reconsider a class of the rational p -adic dynamical systems [11], [12] and construct the corresponding adelic dynamics with rational fixed points. We expect that this adelic approach will imply further developments of both p -adic and real dynamical systems.

In our case the state space X of the system is the adelic one and discrete dynamics is described by the mapping

$$f(x) = \frac{ax + b}{cx + d}, \quad (3)$$

where a, b, c, d are some rational numbers satisfying $ad - bc \neq 0$ and especially $ad - bc = 1$. It is worth noting that taking physical parameters to be rational numbers is not unnatural restriction but moreover gives a possibility to treat real and p -adic properties simultaneously and on an equal footing. In this way our approach differs from that in [11], where parameters $a, b, c, d \in \mathbb{C}_p$.

Linear fractional transformations (Möbius transformations) (3) and related $GL(2, \mathbb{C})$, $GL(2, \mathbb{C}_p)$ groups, and their subgroups, have very rich mathematical structure. They also have important applications in many parts of theoretical physics (see, e.g. [6] and [13]).

Let us also mention that if in this dynamical system we let parameters a, b, c, d to be sequences then we obtain recursion equation for magnetization arising in the study of the spin systems on the Bethe lattice. This implies that adelic linear fractional dynamical systems could be extended to the dynamics of spin systems on the random tree-like graphs over p -adic numbers, random rational p -adic systems and rationally perturbed monomial systems [14].

The paper is organized as follows. In Sec. 2 we briefly present some pertinent properties of p -adic numbers and adeles. Sec. 3 is devoted to an analysis in detail of real, p -adic and adelic properties of the above linear fractional dynamics (3). Obtained results are slightly extended, discussed and summarized in Sec. 4.

2 p -Adic Numbers and Adeles

Rational numbers are significant in physics as well as in mathematics. Physical significance comes from the fact that result of any measurement is a rational number. According to the Ostrowski theorem, the set \mathbb{Q} of rational numbers is a dense subfield not only in the field \mathbb{R} of real numbers but also in the field \mathbb{Q}_p of p -adic numbers, for every prime number p . Consequently, a research on the basis of \mathbb{R} and \mathbb{Q}_p provides rather fine description of many properties related to rational numbers. The space of adeles \mathbb{A} is a suitable tool to consider \mathbb{R} and \mathbb{Q}_p simultaneously and on the equal footing. Below we

shall give a short introductory review of p -adic numbers and adeles relevant for this paper.

Let us recall that the first infinite set of numbers we encounter is the set \mathbb{N} of natural numbers. To have a solution of the simple linear equation $x + a = b$ for any $a, b \in \mathbb{N}$, one has to extend \mathbb{N} and to introduce the set \mathbb{Z} of integers. Requiring that there exists solution of the linear equation $nx = m$ for any $0 \neq n, m \in \mathbb{Z}$ one obtains the set \mathbb{Q} of rational numbers. Evidently these sets satisfy $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$. Algebraically \mathbb{N} is a semigroup, \mathbb{Z} is a ring, and \mathbb{Q} is a field.

To get \mathbb{Q} from \mathbb{N} only algebraic operations are used, but to obtain the field \mathbb{R} of real numbers from \mathbb{Q} one has to employ the absolute value which is an example of the norm (valuation) on \mathbb{Q} . Let us recall that a norm on \mathbb{Q} is a map $\|\cdot\| : \mathbb{Q} \rightarrow \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ with the following properties: (i) $\|x\| = 0 \leftrightarrow x = 0$, (ii) $\|x \cdot y\| = \|x\| \cdot \|y\|$, and $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{Q}$. In addition to the absolute value, for which we use usual arithmetic notation $|\cdot|_\infty$, one can introduce on \mathbb{Q} a norm with respect to each prime number p . Note that, due to the factorization of integers, any rational number can be uniquely written as $x = p^\nu \frac{m}{n}$, where p, m, n are mutually prime and $\nu \in \mathbb{Z}$. Then by definition p -adic norm (or, in other words, p -adic absolute value) is $|x|_p = p^{-\nu}$ if $x \neq 0$ and $|0|_p = 0$. One can verify that $|\cdot|_p$ satisfies all the above conditions and, moreover, strong triangle inequality, i.e. $|x + y|_p \leq \max(|x|_p, |y|_p)$. Thus p -adic norms belong to the class of non-Archimedean (ultrametric) norms. Up to the equivalence, there is only one p -adic norm for every prime number p . According to the Ostrowski theorem any nontrivial norm on \mathbb{Q} is equivalent either to the $|\cdot|_\infty$ or to one of the $|\cdot|_p$. One can easily show that $|m|_p \leq 1$ for any $m \in \mathbb{Z}$ and any prime p . The p -adic norm is a measure of divisibility of the integer m by prime p : the more divisible, the p -adic smaller. Using Cauchy sequences of rational numbers one can make completions of \mathbb{Q} to obtain $\mathbb{R} \equiv \mathbb{Q}_\infty$ and the fields \mathbb{Q}_p of p -adic numbers using norms $|\cdot|_\infty$ and $|\cdot|_p$, respectively. The cardinality of \mathbb{Q}_p is the continuum, like of \mathbb{Q}_∞ . p -Adic completion of \mathbb{N} gives the ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ of p -adic integers. Denote by $\mathbb{U}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$ multiplicative group of p -adic units.

Any p -adic number $x \in \mathbb{Q}_p$ can be presented in the unique way (unlike real numbers) as the sum of p -adic convergent series of the form

$$x = p^\nu (x_0 + x_1 p + \cdots + x_n p^n + \cdots), \quad \nu \in \mathbb{Z}, \quad x_n \in \{0, 1, \dots, p-1\}. \quad (4)$$

It resembles representation of a real number $y = \pm 10^\mu \sum_{k=0}^{-\infty} b_k 10^k$, $\mu \in \mathbb{Z}$, $b_k \in \{0, 1, \dots, 9\}$, but with the expansion in the opposite way. If $\nu \geq 0$ in (4), then $x \in \mathbb{Z}_p$. When $\nu = 0$ and $x_0 \neq 0$ one has $x \in \mathbb{U}_p$. Any negative integer can be easily presented starting from the representation of -1 :

$$-1 = p-1 + (p-1)p + (p-1)p^2 + \cdots + (p-1)p^n + \cdots. \quad (5)$$

Validity of (5) can be shown by elementary arithmetics, which is very similar to the real case, or treating it as the p -adic convergent geometric series.

Using the norm $|\cdot|_p$ one can introduce p -adic metric $d_p(x, y) = |x - y|_p$, which satisfies all necessary properties of metric with strong triangle inequality, i.e. $d_p(x, y) \leq \max(d_p(x, z), d_p(z, y))$ which is of the non-Archimedean (ultrametric) form. Consequently $d_p(x, y)$ is a distance between p -adic numbers x and y . Using this metric, \mathbb{Q}_p becomes an ultrametric space with p -adic topology. Because of ultrametricity, the p -adic spaces have some exotic (from the real point of view) properties and usual illustrative examples are: a) any point of the ball $B_\mu(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^\mu\}$ can be taken as its center instead of a ; b) any ball can be regarded as a closed as well as an open set; c) two balls may not have partial intersection, i.e. they are disjoint sets or one of them is a subset of the other; and c) all triangles are isosceles. \mathbb{Q}_p is a zerodimensional and a totally disconnected topological space. \mathbb{Z}_p is a compact and \mathbb{Q}_p is a locally compact space.

In analogy with real case one can introduce p -adic algebraic extensions. Here the situation is much richer. The quadratic equation

$$z^2 - \tau = 0, \quad \tau = p, \varepsilon, \varepsilon p, \quad \text{where } \varepsilon = \sqrt[p-1]{1} \text{ and } p \neq 2, \quad (6)$$

has not a solution in \mathbb{Q}_p and one must introduce quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$ with elements $z = x + \sqrt{\tau}y$, $x, y \in \mathbb{Q}_p$. When $p = 2$ there are seven quadratic extensions $\mathbb{Q}_2(\sqrt{\tau})$, where $\tau = -1, \pm 2, \pm 3, \pm 6$. However $\mathbb{Q}_p(\sqrt{\tau})$ are not algebraically closed and one has to make higher extensions. Algebraically closed and topologically complete extension is \mathbb{C}_p , which is an infinite dimensional vector space. For a more details about p -adic numbers and their algebraic extensions, see, e.g. [15].

Real and p -adic numbers are continual extrapolations of rational numbers along all possible nontrivial and inequivalent metrics. To consider real and p -adic numbers simultaneously and on equal footing one uses concept of adèles. An adèle x (see, e.g. [16]) is an infinite sequence

$$x = (x_\infty, x_2, x_3, \dots, x_p, \dots), \quad (7)$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set \mathcal{P} of primes p one has $x_p \in \mathbb{Z}_p$. Componentwise addition and multiplication make the ring structure of the set \mathbb{A} of all adèles, which is the union of restricted direct products in the following form:

$$\mathbb{A} = \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P}), \quad \mathbb{A}(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p. \quad (8)$$

A multiplicative group of ideles \mathbb{A}^* is a subset of \mathbb{A} with elements $x = (x_\infty, x_2, x_3, \dots, x_p, \dots)$, where $x_\infty \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $x_p \in \mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$

with the restriction that for all but a finite set \mathcal{P} one has that $x_p \in \mathbb{U}_p$. Thus the whole set of ideles is

$$\mathbb{A}^* = \bigcup_{\mathcal{P}} \mathbb{A}^*(\mathcal{P}), \quad \mathbb{A}^*(\mathcal{P}) = \mathbb{R}^* \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p^* \times \prod_{p \notin \mathcal{P}} \mathbb{U}_p. \quad (9)$$

A principal adele (idele) is a sequence $(x, x, \dots, x, \dots) \in \mathbb{A}$, where $x \in \mathbb{Q}$ ($x \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$). \mathbb{Q} and \mathbb{Q}^* are naturally embedded in \mathbb{A} and \mathbb{A}^* , respectively.

Let \mathbb{P} be set of all primes p . Denote by \mathcal{P}_i , $i \in \mathbb{N}$, subsets of \mathbb{P} . Then $\mathcal{P}_i \prec \mathcal{P}_j$ if $\mathcal{P}_i \subset \mathcal{P}_j$. It is evident that $\mathbb{A}(\mathcal{P}_i) \subset \mathbb{A}(\mathcal{P}_j)$ when $\mathcal{P}_i \prec \mathcal{P}_j$. Spaces $\mathbb{A}(\mathcal{P})$ have natural Tikhonov topology and adelic topology in \mathbb{A} is introduced by inductive limit: $\mathbb{A} = \lim \text{ind}_{\mathcal{P} \in \mathbb{P}} \mathbb{A}(\mathcal{P})$. A basis of adelic topology is a collection of open sets of the form $W(\mathcal{P}) = \mathbb{V}_\infty \times \prod_{p \in \mathcal{P}} \mathbb{V}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p$, where \mathbb{V}_∞ and \mathbb{V}_p are open sets in \mathbb{R} and \mathbb{Q}_p , respectively. Note that adelic topology is finer than the corresponding Tikhonov topology. A sequence of adeles $a^{(n)} \in \mathbb{A}$ converges to an adele $a \in \mathbb{A}$ if *i*) it converges to a componentwise and *ii*) if there exist a positive integer N and a set \mathcal{P} such that $a^{(n)}, a \in \mathbb{A}(\mathcal{P})$ when $n \geq N$. In the analogous way, these assertions hold also for idelic spaces $\mathbb{A}^*(\mathcal{P})$ and \mathbb{A}^* . \mathbb{A} and \mathbb{A}^* are locally compact topological spaces.

3 Linear Fractional Dynamical Systems

It is worth to recall some basic notions from the theory of dynamical systems valid for mapping (1) and its iterations (2) at real and p -adic spaces. Let us introduce an index v to denote real ($v = \infty$) and p -adic ($v = p$) cases simultaneously. A *fixed point* ξ is a solution of the equation $f(\xi) = \xi$. If there exists a neighborhood $V(\xi)$ of the fixed point ξ such that for any point $x_n \in V(\xi)$, $x_n \neq \xi$ holds: *(i)* $|x_n - \xi|_v < |x_{n-1} - \xi|_v$, i.e. $\lim_{n \rightarrow \infty} x_n = \xi$, then ξ is called an *attractor*; *(ii)* $|x_n - \xi|_v > |x_{n-1} - \xi|_v$, then ξ is a *repeller*; and *(iii)* $|x_n - \xi|_v = |x_{n-1} - \xi|_v$, then ξ is an *indifferent point*. Basin of attraction $A(\xi)$ of an attractor ξ is the set

$$A(\xi) = \{x_0 \in \mathbb{Q}_v : \lim_{n \rightarrow \infty} x_n \rightarrow \xi\}. \quad (10)$$

A Siegel disk is called a ball $V_r(\xi)$ if every sphere $S_\rho(\xi)$, $\rho < r$ is an invariant sphere of the mapping $f(x)$, i.e. if an initial point $x_0 \in S_\rho(\xi)$ then all iterations x_n also belong to $S_\rho(\xi)$. The union of all Siegel disks $V_r(\xi)$ with the same center ξ is called a maximum Siegel disk and denoted by $SI(\xi)$.

If the mapping (1) has the first derivative in the fixed point ξ then it is useful to employ the following properties: $|f'(\xi)|_v > 1$ - attractor, $|f'(\xi)|_v < 1$ - repeller and $|f'(\xi)|_v = 1$ - indifferent point.

It is worth noting that the general form of a linear fractional transformation is given by

$$f(z) = \frac{az + b}{cz + d}, \quad (11)$$

where $a, b, c, d, z \in \mathbb{C}$ or \mathbb{C}_p with conditions $z \neq -\frac{d}{c}$, $c \neq 0$, $ad - bc \neq 0$, and complex plane may be extended by the point at infinity. There is an isomorphism between map (11) and 2×2 matrices

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det F \neq 0, \quad (12)$$

which are elements of $GL(2, \mathbb{C})$ or $GL(2, \mathbb{C}_p)$ groups. Since map (11) remains the same under change $a \rightarrow \lambda a, b \rightarrow \lambda b, c \rightarrow \lambda c, d \rightarrow \lambda d$, one can choose a suitable λ and redefine a, b, c, d so that $\det F = ab - cd = 1$. Thus one obtains $SL(2, \mathbb{C})$ and $SL(2, \mathbb{C}_p)$. If $ad - bc = 1$ the above map (11) is still invariant under transformation $a \rightarrow -a, b \rightarrow -b, c \rightarrow -c, d \rightarrow -d$, and the corresponding projective linear groups are $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm E\}$ and $PSL(2, \mathbb{C}_p) = SL(2, \mathbb{C}_p)/\{\pm E\}$, where E is the unit 2×2 matrix.

We shall below mainly consider rational dynamical systems given by the following mapping

$$f(x) = \frac{ax + b}{cx + d}, \quad x \in \mathbb{A}, \quad (13)$$

where $a, b, c, d \in \mathbb{Q}$ with conditions $x \neq -\frac{d}{c}$, $c \neq 0$ and $ad - bc = 1$. The corresponding group of matrices F , with $\det F = 1$, is $SL(2, \mathbb{Q})$, which is isomorphic to $Sp(2, \mathbb{Q})$ - the group of symplectic 2×2 matrices with rational entries.

It is worth mentioning that the map (13) preserves the cross-ratio

$$\frac{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)} = \frac{(f(\alpha_1) - f(\alpha_3))(f(\alpha_2) - f(\alpha_4))}{(f(\alpha_1) - f(\alpha_4))(f(\alpha_2) - f(\alpha_3))} \quad (14)$$

between any different points $x = \alpha_1, \alpha_2, \alpha_3, \alpha_4$.

To have an adelic system, it must be satisfied $|f_p(x_p)|_p \leq 1$ in

$$f_{\mathbb{A}}(x) = \left(f_{\infty}(x_{\infty}), f_2(x_2), f_3(x_3), \dots, f_p(x_p), \dots \right), \quad x \in \mathbb{A}, \quad (15)$$

for all but a finite set \mathcal{P} of prime numbers p . In other words, there has to be a prime number q such that $|f_p(x_p)|_p \leq 1$ for all $p > q$. It is obvious that q depends not only on parameters $a, b, c, d \in \mathbb{Q}$, which characterize system,

but also depends on $x \in \mathbb{Q}_p$. For this reason, let us consider existence of a prime number q such that

$$\left| \frac{ax+b}{cx+d} \right|_p \leq 1, \quad c \neq 0, \quad d \neq 0, \quad x \in \mathbb{Z}_p, \quad (16)$$

for all $p > q$. Requiring that c and d are nonzero rational numbers then there exists an enough large prime number q such that $|c|_p = |d|_p = 1$, $|cx+d|_p = 1$ and $|ax+b|_p \leq 1$ when $|x|_p \leq 1$ for all $p > q$. Then it follows the existence of prime q such that (16) is satisfied for all but a finite set of primes, i.e. this function has necessary adelic properties.

For the function (11) we find the following two fixed points:

$$\xi_{1,2} = \frac{a-d \pm \sqrt{(a-d)^2 + 4bc}}{2c} = \frac{a-d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2c} \quad (17)$$

with properties

$$f(\xi_1) \cdot f(\xi_2) = \xi_1 \cdot \xi_2 = -\frac{b}{c}, \quad f'(\xi_1) \cdot f'(\xi_2) = 1. \quad (18)$$

Let us note that (17) can be rewritten in the form

$$\xi_{1,2} = \frac{a-d \pm \sqrt{(\text{Tr } F)^2 - 4 \det F}}{2c}, \quad (19)$$

where F is the corresponding matrix (12).

For the fixed points it is important to notice that if the point ξ_1 is attractive ($|f'(\xi_1)|_v < 1$) then the point ξ_2 is repelling ($|f'(\xi_2)|_v > 1$) and vice versa. The indifferent fixed points always emerge in the pair. These facts immediately follow from the relation (18) that holds for the mapping associated with dynamical system, we consider.

Generally, these points belong to \mathbb{C} in real case and \mathbb{C}_p in p -adic case, and their analysis we postpone for later consideration. Now we are mainly interested in cases $ad-bc = 1$ and when fixed points are rational, because they then belong simultaneously to real and p -adic numbers. To this end we are going to analyze the following six possibilities: **(A)** $b = 0$, **(B)** $b = c$, $d = a$, **(C)** $b = -c$, $d = a + 2c$, **(D)** $b = -c$, $d = a - 2c$, **(E)** $d = -a + 2$ and **(F)** $d = -a - 2$ which give rational values of $\sqrt{(a-d)^2 + 4bc} = \sqrt{(a+d)^2 - 4(ad-bc)}$ in (17).

3.1 Case A: $b = 0$, $ad = 1$.

We have the following rational function with fixed points:

$$f(x) = \frac{x}{d(cx+d)}, \quad \xi_1 = \frac{1-d^2}{cd}, \quad \xi_2 = 0, \quad d \neq 0. \quad (20)$$

For further analysis we need

$$f'(x) = \frac{1}{(cx+d)^2}, \quad f'(\xi_1) = d^2, \quad f'(\xi_2) = \frac{1}{d^2}. \quad (21)$$

We shall see now that dynamics $f(x) = \frac{x}{d(cx+d)}$ has attractive, repelling and indifferent fixed points which do not depend on parameter $c \neq 0$, and that there exists such finite set \mathcal{P} of primes that x_1 and x_2 are indifferent points for $p \notin \mathcal{P}$.

3.1.1 Fixed points in real and p -adic cases

Here we have three distinct possibilities for fixed points $\xi_1 = \frac{1-d^2}{cd}$ and $\xi_2 = 0$. Let $\mathcal{P}' = \{p \in \mathbb{P} : |d|_p < 1\}$ and $\mathcal{P}'' = \{p \in \mathbb{P} : |d|_p > 1\}$, where \mathbb{P} is the set of all prime numbers.

(i) Fixed point ξ_1 is *attractive* ($|f'(\xi_1)|_v < 1$) and ξ_2 is *repelling* ($|f'(\xi_2)|_v > 1$) iff $|d|_\infty < 1$ in the real case and $p \in \mathcal{P}'$ in the p -adic case.

(ii) Fixed point ξ_1 is *repelling* ($|f'(\xi_1)|_v > 1$) and ξ_2 is *attractive* ($|f'(\xi_2)|_v < 1$) iff $|d|_\infty > 1$ in the real case and $p \in \mathcal{P}''$ in the p -adic case.

(iii) Fixed points ξ_1 and ξ_2 are *indifferent* ($|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$) iff $|d|_\infty = 1$ in the real case and $p \notin \mathcal{P}' \cup \mathcal{P}''$ in the p -adic case.

3.1.2 Adelic aspects of fixed points

According to the above results one has the following two adelic fixed points $\xi^{(i)}$:

$$\xi^{(i)} = \left(\xi_\infty^{(i)}, \xi_2^{(i)}, \xi_3^{(i)}, \xi_5^{(i)}, \dots, \xi_p^{(i)}, \dots \right), \quad \xi^{(i)} \in \mathbb{A}, \quad i = 1, 2, \quad (22)$$

where $\xi_\infty^{(1)} = \xi_p^{(1)} = \frac{1-d^2}{cd} \in \mathbb{Q}$ and $\xi_\infty^{(2)} = \xi_p^{(2)} = 0$ for any $p \in \mathbb{P}$. Structure of adelic fixed points $\xi^{(i)}$ depends only on the values of rational parameter d . When $d = \pm 1$ both adeles $\xi^{(1)}$ and $\xi^{(2)}$ have indifferent points in the real

and all p -adic positions, i.e. they are adelicly indifferent. However, when $d \neq \pm 1$ there emerge also attractive and repelling points in real as well as in p -adic cases but only for finite sets of primes. Namely if parameter $d \in \mathbb{Q}$ is such that $|d|_\infty < 1$ (or $|d|_\infty > 1$), $|d|_p < 1$ for $p \in \mathcal{P}'$ and $|d|_p > 1$ for $p \in \mathcal{P}''$ then adelic fixed point $\xi^{(1)}$ has one real attractive (or repelling) point and finite p -adic attractive and repelling points which correspond to \mathcal{P}' and \mathcal{P}'' , respectively. The vice versa situation is for the second adelic point $\xi^{(2)}$.

The above results can be summarized as follows:

$$\xi_v^{(1)} = \frac{1-d^2}{cd} = \begin{cases} \text{attractive,} & |d|_v < 1 \quad (p \in \mathcal{P}') \\ \text{repelling,} & |d|_v > 1 \quad (p \in \mathcal{P}'') \\ \text{indifferent,} & |d|_v = 1 \quad (p \notin \mathcal{P} = \mathcal{P}' \cup \mathcal{P}''), \end{cases}$$

$$\xi_v^{(2)} = 0 = \begin{cases} \text{attractive,} & |d|_v > 1 \quad (p \in \mathcal{P}') \\ \text{repelling,} & |d|_v < 1 \quad (p \in \mathcal{P}'') \\ \text{indifferent,} & |d|_v = 1 \quad (p \notin \mathcal{P} = \mathcal{P}' \cup \mathcal{P}''). \end{cases}$$

3.2 Case B: $c = b$, $d = a$, $a^2 - b^2 = 1$.

In this case we have the following function with its fixed points:

$$f(x) = \frac{ax+b}{bx+a}, \quad \xi_1 = 1, \xi_2 = -1. \quad (23)$$

For the sequel we need

$$f'(x) = \frac{1}{(a+bx)^2}, \quad f'(\xi_1) = \frac{1}{(a+b)^2}, \quad f'(\xi_2) = \frac{1}{(a-b)^2}. \quad (24)$$

3.2.1 Fixed points in real and p -adic cases

There are again three distinct possibilities. Let now $\mathcal{P}' = \{p \in \mathbb{P} : |a-b|_p < 1\}$ and $\mathcal{P}'' = \{p \in \mathbb{P} : |a-b|_p > 1\}$.

(i) Fixed point ξ_1 is *attractive* ($|f'(\xi_1)|_v < 1$) and ξ_2 is *repelling* ($|f'(\xi_2)|_v > 1$) iff $|a-b|_\infty < 1$ in the real case and $p \in \mathcal{P}'$ in the p -adic case.

(ii) Fixed point ξ_1 is *repelling* ($|f'(\xi_1)|_v > 1$) and ξ_2 is *attractive* ($|f'(\xi_2)|_v < 1$) iff $|a-b|_\infty > 1$ in the real case and $p \in \mathcal{P}''$ in the p -adic case.

(iii) Fixed points ξ_1 and ξ_2 are *indifferent* ($|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$) iff $|a-b|_\infty = 1$ in the real case and $p \notin \mathcal{P}' \cup \mathcal{P}''$ in the p -adic one.

3.2.2 Adelic aspects of fixed points

According to the above results one has the following two adelic fixed points $\xi^{(i)}$:

$$\xi^{(i)} = \left(x_\infty^{(i)}, \xi_2^{(i)}, \xi_3^{(i)}, \xi_5^{(i)}, \dots, \xi_p^{(i)}, \dots \right), \quad \xi^{(i)} \in \mathbb{A}, \quad i = 1, 2, \quad (25)$$

where $\xi_\infty^{(1)} = \xi_p^{(1)} = +1$ and $\xi_\infty^{(2)} = \xi_p^{(2)} = -1$ for any $p \in \mathbb{P}$. Structure of adelic fixed points $\xi^{(i)}$ depends only on the rational values of $a - b$. When $a - b = \pm 1$ both adeles $\xi^{(1)}$ and $\xi^{(2)}$ have indifferent points in the real and all p -adic positions, i.e. they are adelicly indifferent. However, when $a - b \neq \pm 1$ there emerge also attractive and repelling points in real as well as in p -adic cases but only for finite sets of primes. Namely, if $a - b \in \mathbb{Q}$ is such that $|a - b|_\infty < 1$ (or $|a - b|_\infty > 1$), $|a - b|_p < 1$ for $p \in \mathcal{P}'$ and $|a - b|_p > 1$ for $p \in \mathcal{P}''$ then adelic fixed point $\xi^{(1)}$ has one real attractive (or repelling) point and finite p -adic attractive and repelling points which correspond to \mathcal{P}' and \mathcal{P}'' , respectively. The vice versa situation is for the second adelic point $\xi^{(2)}$.

The above results can be summarized as follows:

$$\xi_v^{(1)} = 1 = \begin{cases} \text{attractive,} & |(a - b)|_v < 1 \quad (p \in \mathcal{P}') \\ \text{repelling,} & |(a - b)|_v > 1 \quad (p \in \mathcal{P}'') \\ \text{indifferent,} & |(a - b)|_v = 1 \quad (p \notin \mathcal{P}' \cup \mathcal{P}''), \end{cases}$$

$$\xi_v^{(2)} = -1 = \begin{cases} \text{attractive,} & |(a - b)|_v > 1 \quad (p \in \mathcal{P}') \\ \text{repelling,} & |(a - b)|_v < 1 \quad (p \in \mathcal{P}'') \\ \text{indifferent,} & |(a - b)|_v = 1 \quad (p \notin \mathcal{P}' \cup \mathcal{P}''). \end{cases}$$

3.3 Case C: $b = -c$, $d = a + 2c$, $(a + c)^2 = 1$.

This is the case with fused fixed points

$$f(x) = \frac{ax - c}{cx + a + 2c}, \quad \xi_1 = \xi_2 = -1. \quad (26)$$

For further investigation we need

$$f'(x) = \frac{1}{(cx + a + 2c)^2}, \quad f'(\xi_1) = f'(\xi_2) = 1. \quad (27)$$

3.3.1 Fixed points in real and p -adic cases

In this special case we have the only one possibility. Namely, due to $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused fixed point $\xi_1 = \xi_2 = -1$ is indifferent one in real as well as in all p -adic cases (i.e. for all primes $p \in \mathbb{P}$).

3.3.2 Adelic aspects of fixed points

According to the above results one has only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = (\xi_\infty, \xi_2, \xi_3, \xi_5, \dots, \xi_p, \dots), \quad \xi \in \mathbb{A}, \quad (28)$$

where $\xi_\infty = \xi_p = -1$ for any $p \in \mathbb{P}$. This is one pure adelic indifferent point for any rational values of parameters a and c constrained by relation $(a + c)^2 = 1$ and $c \neq 0$.

3.4 Case D: $b = -c$, $d = a - 2c$, $(a - c)^2 = 1$.

In this case one has again mapping with fused fixed points, i.e.

$$f(x) = \frac{ax - c}{cx + a - 2c}, \quad \xi_1 = \xi_2 = 1. \quad (29)$$

In the following we need

$$f'(x) = \frac{1}{(cx + a - 2c)^2}, \quad f'(\xi_1) = f'(\xi_2) = 1. \quad (30)$$

3.4.1 Fixed points in real and p -adic cases

In this special case we have the only one possibility. Namely, due to $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused fixed point $\xi_1 = \xi_2 = 1$ is indifferent one in real as well as in all p -adic cases.

3.4.2 Adelic aspects of fixed points

According to the above results one has only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = (\xi_\infty, \xi_2, \xi_3, \xi_5, \dots, \xi_p, \dots), \quad \xi \in \mathbb{A}, \quad (31)$$

where $\xi_\infty = \xi_p = 1$ for any $p \in \mathbb{P}$. This is one pure adelic indifferent point for any rational values of parameters a and c constrained by relation $(a - c)^2 = 1$ and $c \neq 0$.

3.5 Case E: $d = -a + 2$, $(a - 1)^2 + bc = 0$.

This is another case with double fixed point:

$$f(x) = \frac{ax + b}{cx - a + 2}, \quad \xi_1 = \xi_2 = \frac{a - 1}{c}. \quad (32)$$

For further investigation we need

$$f'(x) = \frac{1}{(cx - a + 2)^2}, \quad f'(\xi_1) = f'(\xi_2) = 1. \quad (33)$$

3.5.1 Fixed points in real and p -adic cases

Due to $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused rational fixed point $\xi_1 = \xi_2 = \frac{a-d}{2c}$ is indifferent one in real as well as in all p -adic cases.

3.5.2 Adelic aspects of fixed points

According to the above results we have only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = (\xi_\infty, \xi_2, \xi_3, \xi_5, \dots, \xi_p, \dots), \quad \xi \in \mathbb{A}, \quad (34)$$

where $\xi_\infty = \xi_p = \frac{a-1}{c}$ for any $p \in \mathbb{P}$. This is one pure adelic indifferent point for any rational values of parameters a, b and c constrained by relation $(a - 1)^2 + bc = 0$ and $c \neq 0$.

3.6 Case F: $d = -a - 2$, $(a + 1)^2 + bc = 0$.

As in the previous three cases one has here fusion of fixed points. Namely,

$$f(x) = \frac{ax + b}{cx - a - 2}, \quad \xi_1 = \xi_2 = \frac{a + 1}{c}. \quad (35)$$

We also employ

$$f'(x) = \frac{1}{(cx - a - 2)^2}, \quad f'(\xi_1) = f'(\xi_2) = 1. \quad (36)$$

3.6.1 Fixed points in real and p -adic cases

Since $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused fixed point $\xi_1 = \xi_2 = \frac{a-d}{2c}$ is indifferent one in real as well as in all p -adic cases.

3.6.2 Adelic aspects of fixed points

From the above results one has only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = (\xi_\infty, \xi_2, \xi_3, \xi_5, \dots, \xi_p, \dots), \quad \xi \in \mathbb{A}, \quad (37)$$

where $\xi_\infty = \xi_p = \frac{a+1}{c}$ for any $p \in \mathbb{P}$. This is one pure adelic indifferent point for any rational values of parameters a and c constrained by relation $(a+1)^2 + bc = 0$ and $c \neq 0$.

4 Concluding Remarks

Let us recall that in linear fractional function (13) we have considered parameters a, b, c and d as some rational numbers. This is a natural requirement, since measured values of physical quantities are rational, and it gives us also a possibility to investigate the corresponding dynamical systems as the adelic ones. To have in (13) fixed points ξ_1 and ξ_2 , which depend on these parameters, rational we have made restriction on the discriminant $(a-d)^2 + 4bc$ so that it is a square and then $\sqrt{(a-d)^2 + 4bc}$ is a rational, too. We have found six such cases and investigated their real, p -adic and adelic properties in detail. However in a more general setting $\sqrt{(a-d)^2 + 4bc}$ may belong to the field $\mathbb{C} = \mathbb{R}(\sqrt{-1})$ of usual complex numbers and the field $\mathbb{Q}_p(\sqrt{\tau})$ of p -adic quadratic extensions or even \mathbb{C}_p (see the Sec. 2). The most general case would be investigation of mapping (11). This is mathematically also interesting (cf. Ref. [11]) but its potential physical content should be justified.

We have analyzed mapping (13) with condition $ad-bc = 1$ which is related to $SL(2, \mathbb{Q})$. Since function (13) and its derivative $f'(x) = (ad-bc)/(cx+d)^2$ are invariant under scale transformation $a \rightarrow \lambda a, b \rightarrow \lambda b, c \rightarrow \lambda c, d \rightarrow \lambda d$ it follows that any $ad-bc = r^2 \neq 0$ case can be transformed to $ad-bc = 1$ rescaling by $\lambda = 1/r \in \mathbb{Q}$. As a consequence of this scale invariance our evaluation can be easily extended to $ad-bc = r^2 \in \mathbb{Q}^*$ case.

It is worth noting that our analysis in Sec. 3 contains also some transformations related to the modular group, i.e. transformations of the form (13) with $a, b, c, d \in \mathbb{Z}$ satisfying $ad-bc = 1$. Modular group identifies with quotient group $SL(2, \mathbb{Z})/\{\pm E\}$, where E is the unit 2×2 matrix. We found the following five different such transformations:

$$f_1(x) = \frac{\pm x}{cx \pm 1}, \quad f_2(x) = \frac{ax + a \mp 1}{(-a \pm 1)x - a \pm 2}, \quad f_3(x) = \frac{(-c \pm 1)x - c}{cx + c \pm 1}, \quad (38)$$

$$f_4(x) = \frac{(ax - a \pm 1)}{(a \mp 1)x - a \pm 2}, \quad f_5(x) = \frac{(c \pm 1)x - c}{cx - c \pm 1}. \quad (39)$$

One can introduce product of norms on the multiplicative group of ideles \mathbb{A}^* as

$$|\alpha| = \prod_v |\alpha_v|_v, \quad \alpha = \{\alpha_v\}_v \in \mathbb{A}^*, \quad (40)$$

where v runs through primes p and ∞ . Here the product in the right hand side makes sense because $|\alpha_p|_p = 1$ for all but a finite set of $p \in \mathbb{P}$. This yields to the remarkable adelic product formula

$$|r| = |r|_\infty \prod_{p \in \mathbb{P}} |r|_p = 1, \quad r \in \mathbb{Q}^*. \quad (41)$$

This product formula connects real and all p -adic norms of the same nonzero rational number. It presents the simplest example of adelic product formulas which connect real and p -adic counterparts of a rational quantity. Note that $|r|^{s+it} = 1$, where $s, t \in \mathbb{R}$, is a slight extension of (41) and is an example of adelic multiplicative character. It is obvious that r in (41) can be replaced by nonzero parameters a, b, c, d as well as by fixed points ξ_1, ξ_2 . It can be also applied to the invariant relation (14) and to the function $f(x)$ defined in (13), when $x \in \mathbb{Q}^*$. One of the important consequences of adelic product formula (41) is that a real quantity can be expressed as product of inverse p -adic counterparts.

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