Optimization Model on Quadratic Programming Problem with Fuzzy

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Abstract: In this paper, the authors propose a computational procedure by using fuzzy approach to find the optimal solution of quadratic programming problems. The authors divide the calculation of the optimal solution into two stages. In the first stage the authors determine the unconstrained minimization and check its feasibility. The second stage, the authors explore the feasible region from initial point to another point until the authors get the optimal point by using Lagrange multiplier. A numerical example is included to support as illustration of the paper.

Key words: Fuzzy optimal solution, triangular fuzzy number, feasible set, quadratic programming, positive definite.

1. Introduction

The theory of quadratic programming problem is concerned on problems of constrained minimization where the constraint functions are linear and the objective is positive definite quadratic function [1-3]. Although it represents a natural transition from theory of linear programming to the theory of nonlinear programming problem, there are some important differences between their optimal solutions. If an optimum solution of quadratic programming problem exists then it is either an interior point or boundary point which is not necessarily an extreme point of the feasible region.

Quadratic programming problem represents real world situations involving a lot of parameters which value are assigned by experts. However, both experts and decision makers frequently do not precisely know the value of those parameters. Therefore it is useful to consider the knowledge of experts about the parameters as fuzzy data [4].


In this paper the authors try to solve quadratic programming problem by using active constraints method. Using the proposed method, the fuzzy optimal solution of QPP (quadratic programming problem) occurring in the real life situations can be easily obtained.

To fulfill the desired objective, the authors organize this paper as follows. In Section 2, some basic notations, definitions and arithmetic operations between two triangular fuzzy numbers are reviewed. In Section 3, formulation of QPP and the explorations of violated constraints are discussed. In Section 4, a
method is described to determine the optimal solution on the face and the intersection of violated constraints. In Section 5, the authors summarize the concept of active constraints in form of an algorithm for solving QPP. As an illustration, numerical examples are solved. In Section 6, it is conclusion as the end part of this paper.

2. Preliminary

Experimental details (sections titled Experimental Methods, Experimental Section, or Materials and Methods).

2.1 Basic Definition

Definition 2.1 [9] The characteristic function $\mu_A$ of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in $X$. This function can be generalized to a function $\mu_A$ such that the value assigned to the element of the universal set $X$ falls within a specified range, i.e., $\mu_A: X \rightarrow [0,1]$. The assigned value indicates the membership grade of the element in the set $A$. The function $\mu_A$ is called the membership function and the set $\tilde{A} = \{(x, \mu_A(x)) : x \in X\}$ defined by $\mu_A$ for each $x \in X$ is called a fuzzy set.

Definition 2.2 [10] A fuzzy number $\tilde{A} = (a, b, c)$ is said a triangular fuzzy number if its membership function is given by:

$$\tilde{A}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{x-c}{b-c}, & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

and $\alpha$-cuts corresponding to $\tilde{A} = (a, b, c)$ is given by Buckley, Esfandiar and Thomas [11].

$$\tilde{A}[\alpha] = \{(b-a)\alpha + a, -(c-b)\alpha + b\}, \alpha \in [0,1].$$

Definition 2.3 [3] A triangular fuzzy numbers $(a, b, c)$ are said to be non-negative fuzzy numbers if and only if $a > 0$.

Definition 2.4 Let $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers, then:

$\tilde{A} \leq \tilde{B}$ if and only if $a_1 \leq a_2$, $b_1 - a_1 \leq b_2 - a_2$, and $c_1 - b_1 \leq c_2 - b_2$;

$\tilde{A} \geq \tilde{B}$ if and only if $a_1 \geq a_2$, $b_1 - a_1 \geq b_2 - a_2$, and $c_1 - b_1 \geq c_2 - b_2$.

2.2 Fuzzy Arithmetic

The following concepts and results are introduced from [9, 11]. Let $\tilde{A}[\alpha] = [a_\alpha, a_\alpha^+]$ and $\tilde{B}[\alpha] = [b_\alpha, b_\alpha^+]$ be two closed, bounded, intervals of real numbers. If * denotes addition, subtraction, multiplication, or division, then $[a_\alpha, a_\alpha^+] * [b_\alpha, b_\alpha^+] = [\beta, \sigma]$ where:

$[\beta, \sigma] = \{a \ast b : a \leq a_\alpha, \quad b_\alpha \leq b \leq b_\alpha^+\}$

If * is division, the authors must assume that zero does not belong to $[b_\alpha, b_\alpha^+]$. the authors may simplify the above equation as follows:

Addition

$[a_\alpha, a_\alpha^+] \oplus [b_\alpha, b_\alpha^+] = [a_\alpha + b_\alpha, a_\alpha^+ + b_\alpha^+]$

Subtraction

$[a_\alpha, a_\alpha^+] \ominus [b_\alpha, b_\alpha^+] = [a_\alpha - b_\alpha^+, a_\alpha^+ - b_\alpha]$

Division

$[a_\alpha, a_\alpha^+] \odot [b_\alpha, b_\alpha^+] = [a_\alpha + a^+_\alpha] \ominus \frac{1}{b^+_\alpha} \odot \frac{1}{b^-_\alpha}$

Multiplication

$[a_\alpha, a_\alpha^+] \otimes [b_\alpha, b_\alpha^+] = [\beta, \sigma]$ where

$$\beta = \min \{a_\alpha \cdot b_\alpha, a_\alpha^+ \cdot b_\alpha^+, a_\alpha^+ \cdot b_\alpha, a_\alpha \cdot b_\alpha^+\}$$

$$\sigma = \max \{a_\alpha \cdot b_\alpha, a_\alpha^+ \cdot b_\alpha^+, a_\alpha^+ \cdot b_\alpha, a_\alpha \cdot b_\alpha^+\}$$

Remark 2.5 Multiplication may be further simplified as follows. For $\tilde{A} = (a, b, c)$ and $\tilde{B} = (x, y, z)$ be a non-negative triangular fuzzy numbers, then:

$$\tilde{A} \otimes \tilde{B} = \left\{ \begin{array}{c} (ax, by, cz), \quad a \geq 0 \\ (ax, by, cz), a < 0, c \geq 0 \\ (az, by, cx), \quad c < 0 \end{array} \right\}$$

3. Problem Formulation

A quadratic function on $E^n$ is to be considered in this paper which is defined by

$$f(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_j^T d_{ij} x_j + \sum_{j=1}^{n} c_j^T x_j + q$$

(1)

where $q, c_j$ and $D = [d_{ij}]_{n \times n}$ are constant scalar
quantities that can be written in vector-matrix notation as
\[ f(x) = \frac{1}{2}x^TDx + c^Tx + q \]  
(2)
in which \( D = [d_{ij}]_{nxn} \), \( c = (c_1,...,c_n)^T \), and \( x = (x_1,...,x_n)^T \).

Without loss of generality, the authors consider \( D \) to be a symmetric matrix and if \( D \) is a positive definite, then \( f(x) \) is given by Eq. (2), it can be called as a positive definite quadratic function.

The set of all feasible solution is so-called the feasible region which will be considered in this paper, is a closed set defined by
\[ F = \{ x | Ax \leq b, x \geq 0 \} \]  
(3)
where \( A \) is \((m \times n)\) matrix and \( b \) is a vector in \( E^n \).

Since \( f(x) \) given by Eq. (2) is positive definite quadratic function, then \( f(x) \) is strictly convex in \( x \), therefore \( f(x) \) attains a unique minimum at
\[ x^{(0)} = -D^{-1}c \]  
(4)
which is called unconstrained minimum of \( f(x) \). As mention in Section 1, \( x^{(0)} \) can be an interior point or boundary point of feasible region. However, there is one more possibility that \( x^{(0)} \) can be an exterior point. Therefore, if \( x^{(0)} \in F \), then \( x^{(0)} \) becomes the optimal solution of the QPP [2].

Another advantage of strictly convexity properties of \( f(x) \) is that if \( x^{(0)} \) is an exterior point, then definitely, \( x^* \) the optimal solution of the considered problem on the boundary of the feasible region. Therefore, \( x^* \) must be located on one of the active or equality constraints or on the intersection of several active (equality) constraints [1-3].

In the conventional approach, the values of the parameter of QPP models must be well defined and precise. However, in a real life this is not a realistic assumption. In the real problems there may exists uncertainty about the parameters. In such a situation, the parameters of QPP with \( m \) fuzzy constraints and \( n \) fuzzy variables may be formulated as follows:

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\[ \text{Minimize } \tilde{Z}(x) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{d}_{ij} \tilde{x}_i \tilde{x}_j + \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j + \tilde{q} \]  
(5)
subject to
\[ \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \tilde{x}_j \leq \sum_{i=1}^{m} \tilde{b}_i \]  
(6)
where \( \tilde{c} = (c_j)_{m \times 1}, \tilde{A} = (a_{ij})_{m \times n}, \tilde{D} = (d_{ij})_{n \times n} \) are positive definite and symmetric of fuzzy numbers and all variable, \( \tilde{x} = (\tilde{x}_1,...,\tilde{x}_n) \) are non-negative fuzzy numbers.

The fuzzy unconstrained minimum of \( Z \) in Eq. (5) is given by:
\[ \tilde{x}^{(0)} = -\tilde{D}^{-1} \cdot \tilde{c} \]  
(7)

**Definition 3.1** Any set of \( x_j \) which satisfies the set of the constraints in Eq. (6) is called feasible solution for Eqs. (5)-(6). Let \( F \) be the set of all feasible solution of Eq. (6). The authors shall say that \( x^* \in F \) is an optimal feasible solution for Eqs. (5)-(6) if \( \tilde{Z}(x^*) \leq \tilde{Z}(x) \) for all \( x \in F \).

**Remark 3.2** The fuzzy optimal solution of QPP problem Eqs. (5)-(6) will be a triangle fuzzy numbers \( \tilde{x}^*[\alpha] = [x_1^*(\alpha),x_2^*(\alpha)] \) if it is satisfied the following conditions:
- \( \tilde{x}^* \) are a non-negative fuzzy numbers
- \( \tilde{A} \otimes \tilde{x}^* \leq b \);
- \( x_1^*(\alpha) \) monotonically increasing, \( \alpha \in [0,1] \);
- \( x_2^*(\alpha) \) monotonically decreasing, \( \alpha \in [0,1] \);
- \( x_1^*(\alpha) \leq x_2^*(\alpha) \).

**4. Constraints and Lagrange Multipliers**

In this section, the authors describe how to search the critical point on the boundary and the vertex of feasible region if the authors find the unconstrained minimum which does not feasible.

Let the authors consider the QPP given by
\[ \min_x \frac{1}{2} x^T D x + c^T x + q \]  
(8)
subject to
\[ A x = b \]  
(9)
where \( D \) is an \((n \times n)\) positive definite matrix and \( A \) is an \((m \times n)\) matrix, \( q \in E^n \).

The Lagrange function of the above problem is define by
\[ L(x,\lambda) = \frac{1}{2} x^T D x + c^T x + q + \lambda^T (A x - b) \]  
(10)
and the stationary point \((x^*,\lambda^*)\) will occurs if
\[ \nabla_x L(x^*,\lambda^*) = Dx^* + c + A^T \lambda^* = 0 \]  
(11)
\[ \nabla_2 L(x^*, \lambda^*) = Ax^* - b = 0 \]  
(12)
The solution is
\[ (x^*) = M^{-1} \left( \begin{array}{c} -c \\ b \end{array} \right) \]
where \( M = \begin{pmatrix} D & A^T \\ A & 0 \end{pmatrix} \) is a square matrix of dimension \((m \times n) \times (m \times n)\). If \( M \) is nonsingular, then the solution \((x^*, \lambda^*)\) exists and unique.

5. Algorithm

The results shown in previous section can be used to obtain an algorithm to find the fuzzy optimal solution of QPP. The brief algorithm as follows:

Step 1 : Compute \( \tilde{x}^{(0)} \), the unconstrained minimum of \( f(x) \) by using equation given by Eq. (7);

Step 2 : If \( \tilde{x}^{(0)} \) satisfies all the constraints provided by the problem, then stop, and \( \tilde{x}^{(0)} \) becomes the fuzzy optimal solution of the QPP. But if \( \tilde{x}^{(0)} \) is outside of feasible region, then push all the indexes of the constraints violated by \( \tilde{x}^{(0)} \) onto the set \( S \), where \( S = \{ j \mid |A_j^T x| > b_j, j \in \{1, \ldots, m\} \} \) for further investigation;

Step 3 : Compute \( \tilde{x}^*_j \) the constrained minimum of \( f(x) \) subject to equality constraint \( j \) where \( j \in S \). If \( \tilde{x}^*_j \) for one \( j \in S \) is feasible then \( \tilde{x}^*_j \) is the fuzzy optimal solution of QPP and stop. Otherwise, search the fuzzy optimal solution of QPP which might be located on the equality or intersection of two and more equality violated constraints by \( \tilde{x}^{(0)} \) according to the method explained in Secsion 4.

6. Numerical Example

In this section, the authors illustrate an example to show the capability of the constraints exploration method. The example is taken from Ref. [12]. The objective function is to minimize the quadratic function and the constraint function consists of two linear functions where the first constraint is a sign of equality and the second constraint is a sign inequality.

\[ \text{Min } z = (x_1 - 1)^2 + (x_2 - 2)^2 \]  
(13)
subject to
\[ -x_1 + x_2 = 1 \]  
(14)
and
\[ x_1 + x_2 \leq 2 \]  
(15)
and
\[ (x_1, x_2) \geq (0, 0). \]

The fuzzy QPP of the problem Eqs. (13)-(15) with \( \alpha \in [0,1] \) can be written as

\[ \text{Minimize } Z_\alpha(x) = (\tilde{x}_1 - \tilde{1})^2 + (\tilde{x}_2 - \tilde{2})^2 \]
subject to
\[ \tilde{x}_1 + \tilde{x}_2 = \tilde{1} \]
\[ \tilde{x}_1 \leq \tilde{2} \]
and \((\tilde{x}_1, \tilde{x}_2)\) are nonnegative fuzzy numbers.

According to the algorithm in Section 5, the optimal solution of the fuzzy QPP is summarized as follows:

Step 1 : The unconstrained minimum by using Eq. (7) is:
\[ \tilde{x}^{(0)} = \left( \begin{array}{c} \frac{a+1}{2}, -\alpha + 2 \\ \frac{3a+1}{2}, -\alpha + 3 \end{array} \right); \]

Step 2 : Obviously, \( \tilde{x}^{(0)} \) only satisfied the constraint Eq. (14) and violated of constraints Eq. (15).

Therefore, the authors are only focusing on finding the optimal point on the constraint Eq. (15);

Step 3 : By using Lagrangian function which is discussed in previous section, the authors get:
\[ \tilde{x}^*_{15} = \left( \begin{array}{c} 8a^2 - 37a^2 + 106a - 73 \\ 2(a^2 + 2a + 1) \end{array} \right) \cdot \left( \begin{array}{c} 4a^2 - 21a^2 + 34a - 21 \\ 2(a^2 + 2a + 1) \end{array} \right) \]
\[ \left( \begin{array}{c} (a - 2)(22a^2 - 13a + 5) \\ (a + 1)^2 \end{array} \right) \cdot \left( \begin{array}{c} (a - 2)(3a - 5)(2a - 5) \\ (a + 1)^2 \end{array} \right) \]
By using Definition 2.4, the authors obtain that the constrained minimum, \( \tilde{x}^*_{15} \) is satisfied all the constraints, then the optimal solution for the example is \( \tilde{x}^*_{15} = \tilde{x}^* \).

At \( \alpha = 1 \),
\[ \tilde{x}^* = (1/2, 3/2) \]
which is obtained in Ref. [12].

7. Conclusions

In this paper the authors solve the fuzzy quadratic programming problem by using constraints exploration method. The fuzzy solution are characterized by fuzzy numbers through the use of the concept of violation constraints by the fuzzy unconstrained and the optimal solution. By using this approach, the fuzzy optimal solution of quadratic programming problem which is occurring in the real life situation can be easily obtained.
References


