Cluster Synchronization of Coupled Systems with Nonidentical Linear Dynamics

Zhongchang Liu* and Wing Shing Wong

Abstract

This paper considers the cluster synchronization problem of generic linear dynamical systems whose system models are distinct in different clusters. These nonidentical linear models render control design and coupling conditions highly correlated if static couplings are used for all individual systems. In this paper, a dynamic coupling structure, which incorporates a global weighting factor and a vanishing auxiliary control variable, is proposed for each agent and is shown to be a feasible solution. Lower bounds on the global and local weighting factors are derived under the condition that every interaction subgraph associated with each cluster admits a directed spanning tree. The spanning tree requirement is further shown to be a necessary condition when the clusters connect acyclicly with each other. Simulations for two applications, cluster heading alignment of nonidentical ships and cluster phase synchronization of nonidentical harmonic oscillators, illustrate essential parts of the derived theoretical results.

Index Terms

Cluster synchronization; Coupled linear systems; Nonidentical systems; Graph topology

I. INTRODUCTION

Understanding the interaction of coupled individual systems continues to receive interest in the engineering research community [1]. Recently, more attention has been paid to cluster synchronization problems which have much wide applications, such as segregation into small subgroups for a robotic team [2] or physical particles [3], predicting opinion dynamics in social networks [4], and cluster phase synchronization of coupled oscillators [5], [6].

In the models reported in most of the literature, the clustering pattern is predefined and fixed; research focuses are on deriving conditions that can enforce cluster synchronization for

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*Corresponding author. E-mail: zcliu@ie.cuhk.edu.hk.
various system models [7]–[17]. Preliminary studies in [7]–[10] reported algebraic conditions on the interaction graph for coupled agents with simple integrator dynamics. Subsequently, a cluster-spanning tree condition is used to achieve intra-cluster synchronization for first-order integrators (discrete time [11] or continuous time [12]), while inter-cluster separations are realized by using nonidentical feed-forward input terms. For more complicated system models, e.g., nonlinear systems ([13]–[15]) and generic linear systems ([16], [17]), both control designs and inter-agent coupling conditions are responsible for the occurrence of cluster synchronization. For coupled nonlinear systems, e.g., chaotic oscillators, algebraic and graph topological clustering conditions are derived for either identical models ([13]) or nonidentical models ([14], [15]) under the key assumption that the input matrix of all systems is identical and it can stabilize the system dynamics of all individual agents via linear state feedback (i.e., the so-called QUAD condition). For identical generic linear systems which are partial-state coupled [16], [17], a stabilizing control gain matrix solved from a Ricatti inequality is utilized by all agents, and agents are pinned with some additional agents so that the interaction subgraph of each cluster contains a directed spanning tree.

The system models introduced above can describe a rich class of applications for multi-agent systems. A common characteristic is that the uncoupled system dynamics of all the agents can be stabilized by linear state feedback attenuated by a unique matrix (i.e., static state feedback). This simplification allows the derivation of coupling conditions to be independent of the control design of any agent, and thus offers scalability to a static coupling strategy. This kind of benefit still exists for nonidentical nonlinear systems which are full-state coupled, as all the system dynamics can be constrained by a common Lipchitz constant (Lipchitz can imply the QUAD condition [18]). However, for the class of partial-state coupled nonidentical linear systems, the stabilizing matrices for distinct linear system models are usually different. Then the coupling conditions under static couplings will be correlated with the control designs of all individual systems. This correlation not only harms the scalability of a coupling strategy but also increases the difficulty in specifying a graph topological condition on the interaction graph.

The goal of this paper is to achieve state cluster synchronization for partial-state coupled nonidentical linear systems, where agents with the same uncoupled dynamics are supposed to synchronize together. This is a problem of practical interest, for instance, maintaining different formation clusters for different types of interconnected vehicles, providing different synchronization frequencies for different groups of clocks using coupled nonidentical harmonic oscil-
lators, reaching different consensus values for people with different opinion dynamics, and so on. In order to relieve the difficulties in using the conventional static couplings, couplings with a dynamic structure is proposed by introducing a vanishing auxiliary variable which facilitates interactions among agents. With the proposed dynamic couplings, an algebraic necessary and sufficient condition, which is independent of the control design, is derived. This newly derived algebraic condition subsumes those published for integrator systems in [7]–[10] as special cases. Due to the entanglement between nonidentical system matrices and the parameters from the interaction graph, the algebraic condition is not straightforward to check. Thus, a graph topological interpretation of the algebraic condition is provided under the assumption that the interaction subgraph associated with each cluster contain a directed spanning tree. We also derive lower bounds for the local coupling strengths in different clusters, which are independent of the control design due to the dynamic coupling structure. This spanning tree condition is further shown to be a necessary condition when the clusters and the inter-cluster links form an acyclic structure. This conclusion reveals the indispensability of direct links among agents belonging to the same cluster, and further strengthens the sufficiency statement presented initially in [16]. Another contribution of the proposed dynamic couplings in comparison to those static couplings in [17] is that the lower bound of a global factor which weights the whole interaction graph is also independent of the control design. For this reason, the least exponential convergence rate of cluster synchronization is characterized more explicitly than that in [17]. The derived results in this paper are illustrated by simulation examples for two applications: cluster heading alignment of nonidentical ships and cluster phase synchronization of nonidentical harmonic oscillators.

The organization of this paper is as follows: Following this section, the problem formulation is presented in Section II. In Section III, both algebraic and graph topological conditions for cluster synchronization are developed. Simulation examples are provided in Section IV. Concluding remarks and discussions for potential future investigations follow in Section V.

II. Problem Statement

Consider a multi-agent system consisting of $L$ agents, indexed by $\mathcal{I} = \{1, \ldots, L\}$, and $N \leq L$ clusters. Let $\mathcal{C} = \{C_1, \ldots, C_N\}$ be a nontrivial partition of $\mathcal{I}$, that is, $\bigcup_{i=1}^{N} C_i = \mathcal{I}$, $C_i \neq \emptyset$, and $C_i \cap C_j = \emptyset$, $\forall i \neq j$. We call each $C_i$ a cluster. Two agents, $l$ and $k$ in $\mathcal{I}$, belong to the same cluster $C_i$ if $l \in C_i$ and $k \in C_i$. Agents in the same cluster are described by the
same linear dynamic equation:

$$\dot{x}_i(t) = A_ix_i(t) + B_iu_i(t), \quad l \in C_i, \quad i = 1, \ldots, N$$  \hspace{1cm} (1)$$

where $x_i(t) \in \mathbb{R}^n$ with initial value, $x_i(0)$, is the state of agent $l$ and $u_i(t) \in \mathbb{R}^{m_i}$ is the control input; $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m_i}$ are constant system matrices which are distinct for different clusters.

A. Interaction graph topology and graph partitions

A directed interaction graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is associated with system (1) such that each agent $l$ is regarded as a node, $v_l \in \mathcal{V}$, and a link from agent $k$ to agent $l$ corresponds to a directed edge $(v_k, v_l) \in \mathcal{E}$. An agent $k$ is said to be a neighbor of $l$ if and only if $(v_k, v_l) \in \mathcal{E}$. The adjacency matrix $\mathcal{A} = [a_{lk}] \in \mathbb{R}^{L \times L}$ has entries defined by: $a_{lk} \neq 0$ if $(v_k, v_l) \in \mathcal{E}$, and $a_{lk} = 0$ otherwise. In addition, $a_{ll} = 0$ to avoid self-links. Note that $a_{lk} < 0$ means that the influence from agent $k$ to agent $l$ is repulsive, while links with $a_{lk} > 0$ are cooperative. Define $\mathcal{L} = [b_{lk}] \in \mathbb{R}^{L \times L}$ as the Laplacian of $\mathcal{G}$, where $b_{ll} = \sum_{k=1}^L a_{lk}$ and $b_{lk} = -a_{lk}$ for any $k \neq l$.

Corresponding to the partition $C = \{C_1, \ldots, C_N\}$, a subgraph $\mathcal{G}_i$, $i = 1, \ldots, N$, of $\mathcal{G}$ contains all the nodes with indexes in $C_i$, and the edges connecting these nodes. See Fig. 1 for illustration. Without loss of generality, we assume that each cluster $C_i$, $i = 1, \ldots, N$, consists of $l_i \geq 1$ agents ($\sum_{i=1}^N l_i = L$), such that $C_1 = \{1, \ldots, l_1\}$, \ldots, $C_i = \{\sigma_i + 1, \ldots, \sigma_i + l_i\}$, \ldots, $C_N = \{\sigma_N + 1, \ldots, \sigma_N + l_N\}$ where $\sigma_1 = 0$ and $\sigma_i = \sum_{j=1}^{i-1} l_j$, $2 \leq i \leq N$. Then, the Laplacian $\mathcal{L}$ of the graph $\mathcal{G}$ can be partitioned into the following form:

$$\mathcal{L} = \begin{bmatrix}
L_{11} & L_{12} & \cdots & L_{1N} \\
L_{21} & L_{22} & \cdots & L_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
L_{N1} & L_{N2} & \cdots & L_{NN}
\end{bmatrix},$$  \hspace{1cm} (2)$$

where each $L_{ii} \in \mathbb{R}^{l_i \times l_i}$ specifies intra-cluster couplings and each $L_{ij} \in \mathbb{R}^{l_i \times l_j}$ with $i \neq j$, specifies inter-cluster influences from cluster $C_j$ to $C_i$, $i, j = 1, \cdots, N$. Note that $L_{ii}$ is not the Laplacian of $\mathcal{G}_i$ in general.

Construct a new graph by collapsing any subgraph of $\mathcal{G}$, $\mathcal{G}_i$, into a single node and define a directed edge from node $i$ to node $j$ if and only if there exists a directed edge in $\mathcal{G}$ from a node in $\mathcal{G}_i$ to a node in $\mathcal{G}_j$. We say $\mathcal{G}$ admits an acyclic partition with respect to $C$, if
the newly constructed graph does not contain any cyclic components. If the latter holds, by relabeling the clusters and the nodes in $G$, we can represent the Laplacian $L$ in a lower triangular form

$$L = \begin{bmatrix} L_{11} & 0 \\ \vdots & \ddots \\ L_{N1} & \cdots & L_{NN} \end{bmatrix},$$

so that each cluster $C_i$ receives no input from clusters $C_j$ if $j > i$. In Fig. 1, the two subgraphs $G_1$ and $G_2$ illustrate an acyclic partition of the whole graph.

**B. The cluster synchronization problem**

The main task in this paper is to achieve cluster synchronization for the states of systems in (1) via distributed couplings through the control inputs $u_i(t)$ which is defined as follows: for $l \in C_i$, $i = 1, \ldots, N$

$$u_i(t) = K_i \eta_l(t)$$

$$\dot{\eta}_l(t) = (A_i + B_i K_i) \eta_l + c \left[ c_i \sum_{k \in C_i} a_{ik}(\eta_k - \eta_l + x_l - x_k) + \sum_{k \notin C_i} a_{ik}(\eta_k - \eta_l + x_l - x_k) \right],$$

where $K_i$ is the control gain matrix to be specified; the vector $\eta_l(t) \in \mathbb{R}^n$, $l \in \mathcal{I}$ is an auxiliary control variable with initial value, $\eta_l(0)$; $c > 0$ is the global weighting factor for the whole interaction graph $\mathcal{G}$; each $c_i > 0$ is a local weighting factor used to adjust the intra-cluster coupling strength of cluster $C_i$. Note that the couplings in (4) takes a dynamic structure. The reasons why conventional static couplings (e.g., those in [13]–[17]) are not
The cluster synchronization problem is defined below.

**Definition 1**: A linear multi-agent system in (1) with couplings in (4) is said to achieve $N$-cluster synchronization with respect to the partition $C$ if the following holds: for any $x_l(0)$ and $\eta_l(0)$, $l \in I$, $\lim_{t \to \infty} \|x_l(t) - x_k(t)\| = 0 \ \forall k, l \in C_i$, $i = 1, \ldots, N$, $\lim_{t \to \infty} \eta_l(t) = 0 \ \forall l \in I$, and for any set of $x_l(0)$, $l \in I$ there exists a set of $\eta_l(0)$, $l \in I$ such that $\limsup_{t \to \infty} \|x_l(t) - x_k(t)\| > 0 \ \forall k, l \in C_i, \forall i = 1, \ldots, N$.

In the definition, all auxiliary variables, $\eta_l(t)$, $l \in I$ are required to decay to zero so as to guarantee that the control effort of every agent is essentially of finite duration. For state separations among distinct clusters, one should not expect them to happen for any set of $x_l(0)$’s and $\eta_l(0)$’s; an obvious counterexample is that all system states will stay at zero when $x_l(0) = \eta_l(0) = 0$ for all $l \in I$. Some assumptions throughout the paper are in order.

**Assumption 1**: Each of the pairs $(A_i, B_i)$, $i = 1, \ldots, N$ is stabilizable.

**Assumption 2**: Each $A_i$ has at least one eigenvalue on the closed right half plane. This assumption excludes trivial scenarios where all system states synchronize to zero.

To deal with stable $A_i$’s, one may introduce distinct feed-forward terms in $u_l(t)$ as studied in [11], [12]. In order to segregate the system states according to the uncoupled system dynamics in (1), an additional mild assumption is made on the system matrices $A_i$’s, namely, they can produce distinct trajectories. Rigorously, for any $i \neq j$, the solutions $x_i(t)$ and $x_j(t)$ to the linear differential equations $\dot{x}_i(t) = A_i x_i(t)$ and $\dot{x}_j(t) = A_j x_j(t)$, respectively satisfy $\limsup_{t \to \infty} \|x_i(t) - x_j(t)\| > 0$ for almost all initial states $x_i(0)$ and $x_j(0)$ in the Euclidean space $\mathbb{R}^n$.

**Assumption 3**: Every block $L_{ij}$ of $L$ defined in (2) has zero row sums, i.e., $L_{ij} 1_{l_j} = 0$.

This assumption guarantees the invariance of the clustering manifold

$$\{x(t) = [x_1^T(t), \ldots, x_L^T(t)]^T : x_1(t) = \cdots = x_{\sigma_1}(t), \ldots, x_{\sigma_{N+1}}(t) = \cdots = x_L(t)\}.$$

It is imposed frequently in the literature to result in cluster synchronization for various multi-agent systems (see [7]–[10], [13], [14], [16], [17]). To fulfill it, one can let positive and negative weights be balanced for all of the links directing from one cluster to any agent in another cluster. The negative weights for inter-cluster links is supposed to provide desynchronizing influences. Note also that with Assumption 3 each $L_{ii}$ is the Laplacian of a subgraph $G_i$, $i = 1, \ldots, N$.

**Notation**: $1_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$. The identity matrix of dimension $n$ is $I_n \in \mathbb{R}^{n \times n}$.preferred will be explained in details in the main part.
The symbol $\text{blockdiag}\{M_1, \ldots, M_N\}$ represents the block diagonal matrix constructed from the $N$ matrices $M_1, \ldots, M_N$. "$\otimes$" stands for the Kronecker product. A symmetric positive (semi-) definite matrix $S$ is represented by $S > 0 (S \geq 0)$. $\text{Re}\lambda(A)$ is the real part of the eigenvalue of a square matrix $A$, and $\sigma(A)$ is the spectrum of $A$.

III. CONDITIONS FOR ACHIEVING CLUSTER SYNCHRONIZATION

In this section, we first present a necessary and sufficient algebraic clustering condition that entangles parameters from the Laplacian $L$ and the system matrices $A_i$’s. Then, we present some graph topological conditions which offer more intuitive interpretations.

The following discussion makes use of the weighted graph Laplacian

$$L_c = \begin{bmatrix} c_1 L_{11} & \cdots & L_{1N} \\ \vdots & \ddots & \vdots \\ L_{N1} & \cdots & c_N L_{NN} \end{bmatrix} \in \mathbb{R}^{L \times L}, \quad (5)$$

and the following matrix:

$$\hat{L}_c = \begin{bmatrix} c_1 \hat{L}_{11} & \cdots & \hat{L}_{1N} \\ \vdots & \ddots & \vdots \\ \hat{L}_{N1} & \cdots & c_N \hat{L}_{NN} \end{bmatrix} \in \mathbb{R}^{(L-N) \times (L-N)}, \quad (6)$$

where each $\hat{L}_{ij}$, $i, j = 1, \ldots, N$ is a block matrix defined as

$$\hat{L}_{ij} = \tilde{L}_{ij} - 1_i \gamma_{ij}^T,$$  \quad (7)

with

$$\gamma_{ij} = [b_{\sigma_i+1,\sigma_j+2}, \ldots, b_{\sigma_i+1,\sigma_j+l_j}]^T \in \mathbb{R}^{l_j-1},$$

$$\tilde{L}_{ij} = \begin{bmatrix} b_{\sigma_i+2,\sigma_j+2} & \cdots & b_{\sigma_i+2,\sigma_j+l_j} \\ \vdots & \ddots & \vdots \\ b_{\sigma_i+l_i,\sigma_j+2} & \cdots & b_{\sigma_i+l_i,\sigma_j+l_j} \end{bmatrix} \in \mathbb{R}^{(l_i-1) \times (l_j-1)}.$$

The two matrices $L_c$ and $\hat{L}_c$ have the following relation, whose proof is shown in Appendix I.

**Lemma 1:** Under Assumption 3, each diagonal block $L_{ii}$ in $L_c$ has exactly one zero eigenvalue if and only if the corresponding matrix $\hat{L}_{ii}$ defined in (7) is nonsingular. Moreover, $L_c$ defined in (5) has exactly $N$ zero eigenvalues if and only if the matrix $\hat{L}_c$ defined in (6)
is nonsingular.

A. Algebraic clustering conditions

Under Assumption 1, for each \( i = 1, \ldots, N \) there exists a matrix \( P_i > 0 \) satisfying the Riccati equation
\[
P_i A_i + A_i^T P_i - P_i B_i B_i^T P_i = -I.
\]
(8)

Choose the control gain matrices as \( K_i = -B_i^T P_i \), and denote \( \hat{A} = \text{blockdiag}\{I_{l_1-1} \otimes A_1, \ldots, I_{l_N-1} \otimes A_N\} \). Then, we have the following algebraic condition to check the cluster synchronizability.

**Theorem 1:** Under Assumptions 1 to 3, the multi-agent system in (1) with couplings in (4) achieves \( N \)-cluster synchronization if and only if the matrix \( \hat{A} - c \hat{L}_c \otimes I_n \) is Hurwitz, where \( \hat{L}_c \) is defined in (6).

The proof is given in Appendix II. The matrix \( \hat{A} - c \hat{L}_c \otimes I_n \) contains parameters from the interaction graph that entangle intimately with those from the system dynamics. In general, it is not possible to verify the above synchronization condition by simply comparing the eigenvalues of \( \hat{L}_c \) with those of \( A_i \)'s. However, one can do so for a homogeneous multi-agent system as stated in the following corollary.

**Corollary 1:** Under Assumptions 1 to 3, and with identical system parameters: \( A_i = A, \)
\( B_i = B, K_i = K, \) for all \( i = 1, \ldots, N, \) a multi-agent system in (1) with couplings in (4) achieves \( N \)-cluster synchronization if and only if the following holds:
\[
\min_{\sigma(\hat{L}_c)} \text{Re} \lambda(c \hat{L}_c) > \max_{\sigma(A)} \text{Re} \lambda(A).
\]
(9)

A sketch of the proof for this corollary is given in Appendix III.

**Remark 1:** In words, the algebraic condition (9) states that the weighted graph Laplacian \( \hat{L}_c \) has exactly \( N \) zero eigenvalues, and all the nonzero eigenvalues have large enough positive real parts to dominate the unstable system dynamics described by \( A \). This condition implies that related results in [7]–[9] are special cases with \( A = 0, B = 1 \) and \( K = 1 \). It also includes part of the results in [10], which are obtained for identical double integrators. Note that with identical system parameters, one can use static controllers without involving the auxiliary variables \( \eta_i \)'s. However, in that case the synchronized state in each cluster depends linearly on the initials states \( x_i(0) \)'s only. For certain initial state sets, state separations in the limit cannot be guaranteed.
B. Graph topological conditions

The matrix  \( \hat{A} - c \hat{L}_c \otimes I_n \) in Theorem 1 can be proven to be Hurwitz for certain graph topologies in conjunction with some lower bounds on the weighting factors. To do so, the following well-known result for subgraphs will be useful.

**Lemma 2 ([19]):** Let \( G_i \) be a non-negatively weighted subgraph. Then, the Laplacian \( L_{ii} \) of \( G_i \) has a simple zero eigenvalue and all the nonzero eigenvalues have positive real parts if and only if \( G_i \) contains a directed spanning tree.

By Lemma 1, there exists a positive definite matrix \( \hat{W}_i \in \mathbb{R}^{(l_i-1) \times (l_i-1)} \) such that
\[
\hat{W}_i \hat{L}_{ii} + \hat{L}_{ii}^T \hat{W}_i > 0, \quad i = 1, \ldots, N,
\]
if the corresponding subgraph \( G_i \) satisfies the conditions in Lemma 2. Denote
\[
\hat{\mathcal{W}} = \text{blockdiag}\{\hat{W}_1, \ldots, \hat{W}_N\},
\]
and let
\[
\hat{\mathcal{L}}_o = \hat{\mathcal{L}}_c - \hat{\mathcal{L}}_d,
\]
with \( \hat{\mathcal{L}}_d = \text{blockdiag}\{c_1 \hat{L}_{11}, \ldots, c_N \hat{L}_{NN}\} \). The following theorem states the main result of this subsection.

**Theorem 2:** Under Assumptions 1 to 3, a multi-agent system in (1) with couplings in (4) achieves \( N \)-cluster synchronization exponentially fast with the least rate of \( \frac{1}{2}[c - \lambda_{\text{max}}(\hat{A} + \hat{A}^T)] \), if each subgraph, \( G_i \), contains only cooperative edges and has a directed spanning tree, and the weighting factors satisfy
\[
c > \max_{i \in \{1, \ldots, N\}} \lambda_{\text{max}}(A_i + A_i^T),
\]
and for each \( i = 1, \ldots, N \)
\[
c_i \geq \frac{\lambda_{\text{max}}(\hat{\mathcal{W}}) - \lambda_{\text{min}}(\hat{\mathcal{W}} \hat{\mathcal{L}}_o + \hat{\mathcal{L}}_o^T \hat{\mathcal{W}})}{\lambda_{\text{min}}(\hat{W}_i \hat{L}_{ii} + \hat{L}_{ii}^T \hat{W}_i)},
\]
where each \( \hat{W}_i \) satisfies (10).

**Proof:** Following the proof of the sufficiency part of Theorem 1, we need to show that the system
\[
\hat{\zeta}(t) = (\hat{\mathcal{A}} - c \hat{\mathcal{L}}_c \otimes I_n)\zeta(t)
\]
is exponentially stable under the conditions in Theorem 2. First, these conditions guarantee the existence of positive definite matrices, \( \hat{W}_i \)'s, satisfying (10). Hence, (13) can be written as

\[
c_i \lambda_{\min}(\hat{W}_i \hat{L}_{ii} + \hat{L}_i^T \hat{W}_i) + \lambda_{\min}(\hat{W} \hat{L}_o + \hat{L}_o^T \hat{W}) \geq \lambda_{\max}(\hat{W})
\]

for \( i = 1, \ldots, N \). These inequalities together with Weyl’s eigenvalue theorem ([20]) yield the following:

\[
\lambda_{\min}(\hat{W} \hat{L}_c + \hat{L}_c^T \hat{W}) = \lambda_{\min}(\hat{W} \hat{L}_d + \hat{L}_d^T \hat{W}) + \lambda_{\min}(\hat{W} \hat{L}_o + \hat{L}_o^T \hat{W}) \geq \lambda_{\max}(\hat{W}),
\]

which further implies that

\[
\hat{W} \hat{L}_c + \hat{L}_c^T \hat{W} \geq \hat{W}.
\] (15)

Now, consider the Lyapunov function candidate \( V(t) = \zeta(t)^T (\hat{W} \otimes I_n) \zeta(t) \) for the system (14). Taking time derivative on both sides of \( V(t) \), one gets

\[
\dot{V}(t) = \zeta^T(t) (\hat{W} \otimes I_n) (\hat{A} - c\hat{L}_c \otimes I_n)
\]
\[
+ (\hat{A} - c\hat{L}_c \otimes I_n)^T (\hat{W} \otimes I_n) \zeta(t)
\]
\[
= \zeta^T(t) [(\hat{W} \otimes I_n) (\hat{A} + \hat{A}^T) - c(\hat{W} \hat{L}_c + \hat{L}_c^T \hat{W}) \otimes I_n] \zeta(t)
\]
\[
\leq \zeta^T(t) [(\hat{W} \otimes I_n) (\hat{A} + \hat{A}^T) - c\hat{W} \otimes I_n] \zeta(t)
\]
\[
\leq -[c - \lambda_{\max}(\hat{A} + \hat{A}^T)] V(t),
\]

where the last inequality follows from (12). This confirms the exponential stability of system (14), and therefore cluster synchronization can be achieved exponentially fast with the least rate of \( \frac{1}{2} [c - \lambda_{\max}(\hat{A} + \hat{A}^T)] \).

We have the following comments on the condition in (12):

1) From the above proof, one can find another lower bound for \( c \) as follows:

\[
c > \frac{\lambda_{\max}((\hat{W} \otimes I_n)(\hat{A} + \hat{A}^T))}{\lambda_{\min}(\hat{W} \hat{L}_c + \hat{L}_c^T \hat{W})},
\] (16)
This bound is tighter than that in (12) since the inequality in (15) and $\lambda_{\max}(\hat{W}) > 0$, $\lambda_{\max}(A_i + A_i^T) \geq 0$ for any $i$ imply that the right-hand side (RHS) of (16) $\leq \frac{\lambda_{\max}(\hat{W}) \lambda_{\max}(\hat{A} + \hat{A}^T)}{\lambda_{\max}(\hat{W})}$ = RHS of (12). However, this tighter bound only guarantees that $\dot{V}(t) < 0$, and does not provide a lower bound on the convergence rate, which could be quite slow. Moreover, the RHS of (16) involves all the $c_i$’s in $\hat{L}_c$, and no known distributed algorithm is available for the computation.

2) Note that the role of $c$ is more essential in stabilizing the unstable modes of the system matrices, $A_i$’s, than in strengthening the connective ability of the interaction graph. A global weighting factor similar to $c$ is utilized in a related paper [17] where the clustering problem for identical linear systems are solved. In that paper, the global factor serves as a parameter in a Ricatti inequality so as to result in a control gain matrix. However, using a larger value for that factor does not necessarily increase the convergence rate. In contrast, the selection of $c$ in this paper is independent of the control design in (8). And the rate of convergence can be improved definitely by using a larger value for $c$.

The following two remarks explain why we prefer the dynamic couplings in (4) than static couplings when dealing with nonidentical linear systems.

**Remark 2:** To achieve state cluster synchronization for a group of generic linear systems, a natural choice of static couplings is the following (slightly modified from couplings of homogeneous linear systems in [16], [17]): for each $l \in C_i, i = 1, \ldots, N$

$$u_l(t) = K_i \left[ c_i \sum_{k \in C_i} b_{lk} x_k(t) + \sum_{k \notin C_i} b_{lk} x_k(t) \right] \quad (17)$$

However, following a similar procedure as in [17], one will need the following condition

$$c_i \lambda_{\min}( (\hat{W}_i \hat{L}_{ii} + \hat{L}_{ii}^T \hat{W}_i) \otimes P_i B_i B_i^T P_i ) \geq \rho, \quad (18)$$

for every $i = 1, \ldots, N$, where $\rho = \lambda_{\max}( (\hat{W} \otimes I_n) PBB^T P ) - \lambda_{\min}( PBB^T P (\hat{W} L_o \otimes I_n) + (L_o^T \hat{W} \otimes I_n) PBB^T P )$. To compute $\rho$, one needs information on the control design of all agents, i.e., $B^T P = \text{blockdiag}\{ I_{l-1} \otimes B_1 P_1, \ldots, I_{N-1} \otimes B_N P_N \}$. This fact renders the selection of local weighting factors, $c_i$’s, a centralized decision. Moreover, (18) cannot be satisfied by any $c_i$ in the nontrivial case that $\rho > 0$, and $P_i B_i B_i^T P_i$ is singular for some $i$.

In contrast to (18), the condition (13) specifies explicitly the requirements for $c_i$’s, and it is
independent of the design of control gain matrices. In this sense, the dynamic couplings in (4) are preferable to the static ones in (17).

**Remark 3:** For nonidentical nonlinear systems of the form, \( \dot{x}_l(t) = f_i(x_l, t), \ l \in C_i \), static couplings are used to result in closed-loop systems as follows ([14], [15]):

\[
\dot{x}_l(t) = f_i(x_l, t) - \Gamma \left[ c_i \sum_{k \in C_i} b_{lk} x_k(t) + \sum_{k \notin C_i} b_{lk} x_k(t) \right],
\]

where \( \Gamma \) is a constant (usually nonnegative-definite) matrix. It was shown that clustering conditions involve the graph Laplacian only (see [15]) if all individual self-dynamics are constrained by the so-called QUAD condition: for any \( x, y \in \mathbb{R}^n \),

\[
(x - y)^T [f_i(x) - f_i(y) - \Gamma(x - y)] \leq -\omega(x - y)^T (x - y), \quad \omega > 0 \text{ is a prescribed positive scalar.}
\]

For generic linear systems with static couplings in (17), this QUAD condition requires that for any \( x \in \mathbb{R}^n \),

\[
x^T (A_i - \Gamma) x \leq -\omega x^T x \quad \text{with} \ \Gamma = B_i K_i \text{ for all } i = 1, \ldots, N.
\]

Given a \( \Gamma \), for the existence of control gains \( K_i \)'s, one needs all \( B_i \)'s to satisfy \( \text{Rank}(B_i) = \text{Rank}([B_i \ \Gamma]) \). However, this rank condition is too restrictive. For example, for the models in (21), an applicable choice of \( \Gamma \) is \( I_2 \), but then \( \text{Rank}(B_i) < \text{Rank}([B_i \ \Gamma]) \) and thus no \( K_i \) can be solved from \( \Gamma = B_i K_i \). In contrast, the dynamic couplings in (4) do not impose such constraints on the system models.

Generally, it is not always necessary to let every subgraph contain a directed spanning tree. In fact, agents belonging to a common cluster may not need to have direct connections at all as long as the algebraic condition in Theorem 1 is satisfied. This point is illustrated by a simulation example in the next section. Nevertheless, the spanning tree condition turns out to be necessary under some particular graph topologies as stated by the corollary below.

**Corollary 2:** Let \( G \) be an interaction graph with an acyclic partition as in (3), and let the edge weights of every subgraph \( G_i \) be nonnegative. Under Assumptions 1 to 3, a multi-agent system (1) with couplings in (4) achieves \( N \)-cluster synchronization if and only if every \( G_i \) contains a directed spanning tree, and the weighting factors satisfy

\[
c \cdot c_i > \frac{\max_{\sigma(A_i)} \text{Re}\lambda(A_i)}{\min_{\sigma(L_{ii})} \text{Re}\lambda(L_{ii})}, \quad \forall i = 1, \ldots, N, \quad (19)
\]

where each \( \hat{L}_{ii} \) is defined in (7).

**Proof:** By Theorem 1, we can examine the stability of \( \hat{A} - c \hat{L}_c \otimes I_n \). Let \( T_i \in \mathbb{R}^{(l_i - 1) \times (l_i - 1)}, \ i = 1, \ldots, N \), be a set of nonsingular matrices such that \( T_i^{-1} \hat{L}_{ii} T_i = J_i \), where \( J_i \) is the Jordan form of \( \hat{L}_{ii} \). Denote \( T = \text{blockdiag}\{T_1 \otimes I_n, \ldots, T_N \otimes I_n\} \). Then, the
block triangular matrix $T^{-1}(\hat{A} - c\hat{L}c \otimes I_n)T$ has diagonal blocks $A_i - \tilde{c}_i\lambda_k(\hat{L}_{ii})I_n$, where $\tilde{c}_i = c \cdot c_i$, $k = 1, \ldots, l_i - 1$, $i = 1, \ldots, N$. Hence, the matrix $\hat{A} - c\hat{L}c \otimes I_n$ is Hurwitz if and only if $\tilde{c}_i \min_k \text{Re}\lambda_k(\hat{L}_{ii}) > \max_m \text{Re}\lambda_m(A_i)$ for any $i$. This claim is equivalent to the conclusion of this corollary due to Lemma 2, the first claim of Lemma 1, and Assumption 2 that requires $\max_m \text{Re}\lambda_m(A_i) \geq 0$.

This corollary reveals the indispensability of direct links among agents in the same cluster under an acyclicly partitioned interaction graph. Note that such direct communication requirements for intra-cluster agents is not necessary under a nonnegatively weighted interaction graph (see [11], [12], [15] for references).

Remark 4: It is worth mentioning for the condition in (19) that one can set $c_i = 1$ for all $i$, and adjust the global factor $c$ only to result in cluster synchronization. In contrast, without the acyclic partitioning structure, the local weighting factors $c_i$’s need to satisfy the lower bound conditions in (13). Note that (19) specifies the tightest lower bound for $c$, while a lower bound reported in [16] for identical linear systems via Lyapunov stability analysis can be quite loose.

IV. Simulation Examples

In this section, we provide application examples for cluster synchronization of nonidentical linear systems. We also conduct numerical simulations using these models to illustrate the derived theoretical results.

A. Example 1: Heading alignment of nonidentical ships

Consider a group of four ships with the interaction graph described by Fig. 2(a), where ship 1 and 2 (respectively, ship 3 and 4) are of the same type. The purpose is to synchronize...
the heading angles for ships of the same type. The steering dynamics of a ship is described by the well-known Nomoto model [21]:

\[ \dot{\psi}_l(t) = v_l(t) \]

\[ \dot{v}_l(t) = -\frac{1}{\tau_i} v_l(t) + \frac{\kappa_i}{\tau_i} u_l(t) \] (20)

where \( \psi_l \) is the heading angle (in degree) of a ship \( l \in I \), \( v_l \) (deg/s) is the yaw rate, and \( u_l \) is the output of the actuator (e.g., the rudder angle). The parameter \( \tau_i \) is a time constant, and \( \kappa_i \) is the actuator gain, both of which are related to the type of a ship. Define for \( i = 1, 2 \) the system matrices

\[
A_i = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau_i} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \frac{\kappa_i}{\tau_i} \end{bmatrix},
\]

(21)

and assume that \( \tau_1 = 42.21, \tau_2 = 107.3, \kappa_1 = 0.181, \kappa_2 = 0.185 \). The solutions to the Riccati equations in (8) are given by \( P_1 = \begin{bmatrix} 22.3 & 233.2 \\ 233.2 & 3915.4 \end{bmatrix} \) and \( P_2 = \begin{bmatrix} 34 & 580 \\ 580 & 16875 \end{bmatrix} \), which lead to the control gain matrices \( K_1 = -[1 \quad 16.79] \) and \( K_2 = -[1 \quad 29.09] \). Since \( \max_{i=1,2} \lambda_{\text{max}}(A_i + A_i^T) = 0.99 \), we set \( c = 1 \) according to (12).

The weighted graph Laplacian is given by

\[
L_c = \begin{bmatrix} 0 & 0 & 5 & -5 \\ -c_1 & c_1 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -c_2 & c_2 \end{bmatrix},
\]

which yields \( \hat{L}_c = \begin{bmatrix} c_1 & 4 \\ -1 & c_2 \end{bmatrix} \) using the definition in (6). So, \( \hat{L}_{11} = 1, \hat{L}_{22} = 1 \), and for any \( \tilde{W}_1 > 0 \) and \( \tilde{W}_2 > 0 \), the inequalities in (10) hold. We choose \( \tilde{W}_1 = \tilde{W}_2 = 1 \). It follows that \( \lambda_{\text{max}}(\tilde{W}) = 1, \lambda_{\text{min}}(\tilde{W}\tilde{L}_o + \tilde{L}_o^T\tilde{W}) = -3 \), and \( \lambda_{\text{min}}(\tilde{W}_i\tilde{L}_{ii} + \tilde{L}_{ii}^T\tilde{W}_i) = 2 \) for \( i = 1, 2 \). Then, we can choose \( c_1 = c_2 = 2 \) so that the inequalities in (13) are satisfied. Simulation result in Fig. 3(a) shows that cluster synchronization is achieved for the heading angles (the velocity \( v_l(t) \) of every agent will converge to zero as shown in Fig. 3(b)).

Now, let \( c_1 = 0 \) so that agents 1 and 2 in cluster \( C_1 \) have no direct connection. Cluster synchronization is still achieved as shown in Fig. 3(c). This example illustrates that intra-cluster connections are not necessary for cluster synchronization under a cyclicly partitioned
(a) Under the graph in Fig. 2(a), cluster synchronization is achieved with $c_1 = c_2 = 2$.

(b) Under the graph in Fig. 2(a), the velocity $v(t)$ of every ship converges to zero.

(c) Under the graph in Fig. 2(a), cluster synchronization is achieved with $c_1 = 0$, $c_2 = 2$.

(d) Under the acyclic partitioning graph in Fig. 2(b), $\psi$'s in the first cluster cannot synchronize together.

Fig. 3. Evolutions of $\psi(t)$ and $v(t)$ for the ship heading steering dynamics in (20) connected with graphs in Fig. 2.

interaction graph. However, under an acyclic partition as in Fig. 2(b), the first cluster of agents, having no direct connections, cannot achieve state synchronization as shown in Fig. 3(d).

B. Example 2: Cluster synchronization of oscillators

The studied cluster synchronization problem for nonidentical linear systems may find applications in the coexistence of oscillators with different frequencies. To see this, let us consider two clusters of coupled harmonic oscillators with graph topology in Fig. 4(a), where the first cluster contains a sender $s_1$ and two receivers $r_1$ and $r_2$, the second cluster contains a sender $s_2$ and two receivers $r_3$ and $r_4$, and the four receivers are coupled by some directed links. Assume the angular frequencies of the two clusters of oscillators are $w_1 = 2$ rad/s and
Fig. 4. (a) Interaction graph partitioned into two clusters $C_1 = \{s_1, r_1, r_2\}$ and $C_2 = \{s_2, r_3, r_4\}$. (b) Evolutions of the first components of the nonidentical harmonic oscillators under the graph in Fig. 4(a).

\[ w_2 = 2.5 \text{ rad/s}, \text{ respectively. So, the dynamic equation of each oscillator is} \]

\[
\begin{align*}
\dot{x}_{1i}(t) &= x_{2i}(t), \\
\dot{x}_{2i}(t) &= -w_i^2 x_{1i}(t) + u_i(t), \quad t \in C_i, \quad i = 1, 2
\end{align*}
\]  

which corresponds to the following system matrices:

\[
A_i = \begin{bmatrix} 0 & 1 \\ -w_i^2 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad i = 1, 2.
\]

The objective is to let the receivers of each cluster follow the state of the sender.

By a similar design procedure as in the previous example, we can set $K_1 = -[0.1231 \ 1.1163]$, $K_2 = -[0.0554 \ 1.0539]$, $c = 6$, and $c_1 = c_2 = 13$. Simulation result in Fig. 4(b) shows the synchronous oscillations of the harmonic oscillators with two distinct angular frequencies.

V. CONCLUSIONS

This paper investigates the state cluster synchronization problem for multi-agent systems with nonidentical generic linear dynamics. By using a dynamic structure for coupling strategies, this paper derives both algebraic and graph topological clustering conditions which are independent of the control designs. For future studies, cluster synchronization which can only be achieved for the system outputs is a promising topic, especially for linear systems with parameter uncertainties or for heterogeneous nonlinear systems. For completely heterogenous linear systems, research works following this line are conducted by the authors in [22] and others in [23]. For nonlinear heterogeneous systems, the new theory being established
for complete output synchronization problems [24], [25] may be further extended. Another interesting challenge existing in cluster synchronization problems is to discover other graph topologies that meet the algebraic conditions.

**APPENDIX I**

**PROOF OF LEMMA 1**

**Proof:** Let $S = \text{blockdiag}\{S_1, \ldots, S_N\}$, where $S_i = \begin{bmatrix} 1 & 0 \\ 1_{l_i-1} & I_{l_i-1} \end{bmatrix} \in \mathbb{R}^{l_i \times l_i}$ for $i = 1, \ldots, N$. Clearly, $S_i$ has the inverse matrix $S_i^{-1} = \begin{bmatrix} 1 & 0 \\ -1_{l_i-1} & I_{l_i-1} \end{bmatrix}$. By direct computation one can show that

$$S_i^{-1}L_{ij}S_j = \begin{bmatrix} 0 & \gamma_{ij} \\ 0 & \hat{L}_{ij} \end{bmatrix}.$$  

This implies the first claim when $i = j$.

For the second claim, consider that

$$S^{-1}L_cS = \begin{bmatrix} 0 & \gamma_{11} & \cdots & 0 & \gamma_{1N} \\ 0 & c_1\hat{L}_{11} & \cdots & 0 & \hat{L}_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \gamma_{N1} & \cdots & 0 & \gamma_{NN} \\ 0 & \hat{L}_{N1} & \cdots & 0 & c_N\hat{L}_{NN} \end{bmatrix}.$$  

Rearrange the columns and rows of $S^{-1}L_cS$ by permutation and similarity transformations to get the following block upper-triangular matrix

$$\begin{bmatrix} 0_{1 \times N} & \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{1 \times N} & \gamma_{N1} & \cdots & \gamma_{NN} \\ 0_{(L-N) \times N} & \hat{L}_c \end{bmatrix},$$  

where $\hat{L}_c$ is defined in (6). Then, the second claim of this lemma follows immediately.
APPENDIX II

PROOF OF THEOREM 1

Proof: The closed-loop system equations for (1) using couplings (4) are given as

\[ \dot{z}_l = A_{ci}z_l - c \left[ c_i \sum_{k \in C_i} b_{lk} E z_k + \sum_{k \notin C_i} b_{lk} E z_k \right], \quad (23) \]

for all \( l \in C_i, \ i = 1, \ldots, N \), where \( z_l = [x_l^T, \eta_l^T]^T \) and

\[ A_{ci} = \begin{bmatrix} A_i & B_i K_i \\ 0 & A_i + B_i K_i \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ -I_n & I_n \end{bmatrix}. \quad (24) \]

Let \( e_l(t) := z_l(t) - z_{\sigma_i+1}(t) \) for \( l \in C_i \) and \( l \neq \sigma_i + 1, \ i = 1, \ldots, N \). It follows from (23) and Assumption 3 that

\[ \dot{e}_l(t) = A_{ci}e_l(t) - c \left[ c_i \sum_{k \in C_i} (b_{lk} - b_{\sigma_i+1,k}) E \xi_k(t) \right] + \sum_{k \notin C_i} (b_{lk} - b_{\sigma_i+1,k}) E \xi_k(t), \quad (25) \]

Define a nonsingular transformation matrix \( Q \) as follows:

\[ Q = \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} I_n & 0 \\ -I_n & I_n \end{bmatrix}, \quad (26) \]

and let \( \varepsilon_l(t) := [\xi_{l1}^T(t), \xi_{l2}^T(t)]^T = Q^{-1}e_l(t) \). Clearly, \( \xi_l = x_l - x_{\sigma_i+1} \) and \( \zeta_l = \eta_l - \eta_{\sigma_i+1} - x_l + x_{\sigma_i+1} \). By (25), one can obtain the following dynamic equations:

\[ \dot{\xi}_l(t) = (A_i + B_i K_i)\xi_l(t) + B_i K_i \zeta_l(t), \]

\[ \dot{\zeta}_l(t) = A_i \zeta_l(t) - c \left[ c_i \sum_{k \in C_i} (b_{lk} - b_{\sigma_i+1,k}) \zeta_k(t) \right] + \sum_{k \notin C_i} (b_{lk} - b_{\sigma_i+1,k}) \zeta_k(t), \]

for \( l \in C_i \) and \( l \neq \sigma_i + 1, \ i = 1, \ldots, N \). Since \( K_i \) stabilizes \( (A_i, B_i) \), the variable \( \varepsilon_l(t) \) tends to zero as \( t \to \infty \) if and only if \( \zeta_l(t) \) tends to zero. Denote

\[ \zeta(t) = [\xi_{\sigma_i+2}^T(t), \ldots, \xi_{\sigma_i+1+l_i}^T(t), \ldots, \xi_{\sigma_N+2}^T(t), \ldots, \xi_{\sigma_N+l_N}^T(t)]^T, \]
which evolves with the following differential equation

$$
\dot{\zeta}(t) = \left( \hat{A} - c\hat{L}_c \otimes I_n \right) \zeta(t).
$$

(27)

Clearly, $\zeta(t)$ and every $\varepsilon_i(t)$ (hence every $e_i(t)$) all converge to zero if and only if $\hat{A} - c\hat{L}_c \otimes I_n$ is Hurwitz. That is, we have shown that $\lim_{t \to \infty} \| x_i(t) - x_k(t) \| = 0$ and $\lim_{t \to \infty} \| \eta_l(t) - \eta_k(t) \| = 0$, $\forall l, k \in \mathcal{C}_i$, $i = 1, \ldots, N$.

Next, we prove that $\eta_l(t)$ for any $l \in \mathcal{I}$ vanishes as $t \to \infty$. To this end, for each $i = 1, \ldots, N$, let $\eta_l(t)$ be the solution of $\dot{\eta}_l(t) = (A_i + B_iK_i)\eta_l(t)$ with an arbitrary initial value $\eta_l(0)$. Since $\sum_{k \in \mathcal{C}_l} b_{lk} = 0$ $\forall l \in \mathcal{I}$ by Assumption 3, we have that

$$
\dot{\eta}_l(t) = (A_i + B_iK_i)\eta_l(t)
$$

$$
= (A_i + B_iK_i)\eta_l(t) - c \left[ c_i \left( \sum_{k \in \mathcal{C}_i} b_{lk}(\eta_{\sigma_i+1} - x_{\sigma_i+1}) \right) + \sum_{j=1, j \neq i}^{N} \left( \sum_{k \in \mathcal{C}_j} b_{lk}(\eta_{\sigma_j+1} - x_{\sigma_j+1}) \right) \right],
$$

for any $l \in \mathcal{C}_i$. Subtracting the above from (4b) yields

$$
\dot{\eta}_l(t) - \dot{\eta}_l(t) = (A_i + B_iK_i)(\eta_l(t) - \eta_l(t))
$$

$$
- c \left( c_i \sum_{k \in \mathcal{C}_i} b_{lk}\dot{\zeta}_k + \sum_{j=1, j \neq i}^{N} \sum_{k \in \mathcal{C}_j} b_{lk}\dot{\zeta}_k \right).
$$

The above system is exponentially stable and driven by inputs which all converge to zero exponentially fast. Therefore, for any $\eta_l(0), l \in \mathcal{I}$, we have $\eta_l(t) \to \eta_l(t) \to 0$ $\forall l \in \mathcal{C}_i$, as $t \to \infty$.

Lastly, we show that inter-cluster state separations can be achieved for any initial states $x_l(0)$’s by selecting $\eta_l(0)$’s properly. Given any set of $x_l(0)$, $l \in \mathcal{I}$, choose $\eta_l(0)$, $l \in \mathcal{I}$ such that $x_l(0) - \eta_l(0) = x_{\sigma_i+1}(0) - \eta_{\sigma_i+1}(0)$ for all $l \in \mathcal{C}_i$, $i = 1, \ldots, N$, and $\limsup_{t \to \infty} \| e^{A_l t}[x_l(0) - \eta_l(0)] - e^{A_l t}[x_l(0) - \eta_l(0)] \| \neq 0$ for any $i \neq j$. Considering the definition of $\zeta_l$ and the linear differential equation (27), one has $x_l(t) - \eta_l(t) = x_{\sigma_i+1}(t) - \eta_{\sigma_i+1}(t)$ for all $t > 0$. This together with (4) lead to the following dynamics

$$
\dot{x}_l(t) - \dot{\eta}_l(t) = A_i(x_l(t) - \eta_l(t)), \forall l \in \mathcal{C}_i.
$$
It follows that

\[ x_l(t) = e^{A_l t}[x_l(0) - \eta_l(0)] + \eta_l(t) \]
\[ \rightarrow e^{A_l t}[x_l(0) - \eta_l(0)], \ \forall l \in \mathcal{C}_i \text{ as } t \rightarrow \infty. \]

Therefore, \( \limsup_{t \to \infty} \|x_l(t) - x_k(t)\| \neq 0 \ \forall l \in \mathcal{C}_i, \ \forall k \in \mathcal{C}_j, \ \forall i \neq j. \) This completes the proof. \hfill \blacksquare

**APPENDIX III**

**PROOF OF COROLLARY 1**

**Proof:** The proof for the necessity and sufficiency of (9) is straightforward using the results in Lemma 1 and Theorem 1, and thus is omitted for simplicity. We only show that state separations are possible for any initial states \( x_l(0), l \in \mathcal{I} \) by using the dynamic couplings even for systems with identical parameters.

Constellate the states \( z_l(t) = [x_l^T(t), \eta_l^T(t)]^T \) of all \( L \) agents to form

\[ z(t) := [z_1^T(t), z_2^T(t), \ldots, z_L^T(t)]^T. \]

It follows that

\[ \dot{z}(t) = (I_L \otimes A_c - cL_c \otimes E)z(t), \quad (28) \]

with \( A_c = \begin{bmatrix} A & BK \\ 0 & A + BK \end{bmatrix} \) and \( E = \begin{bmatrix} 0 & 0 \\ -I_n & I_n \end{bmatrix} \). One can derive, after a series of manipulations, that

\[ z(t) \rightarrow \left( \sum_{i=1}^{N} \mu_i \nu_i^T \right) \otimes e^{A_c t} z(0), \quad \text{as } t \rightarrow \infty, \]

where each \( \nu_i = [\nu_{i1}, \ldots, \nu_{iL}]^T \in \mathbb{R}^L \) is a left eigenvector of \( L_c \) such that \( \nu_i^T L_c = 0, \nu_i^T \mu_i = 1, \) and \( \nu_i^T \mu_j = 0, \forall i \neq j, \) with \( \mu_1 = [1_{l_1}, 0_{L-l_1}]^T, \mu_2 = [0_{l_1}, 1_{l_2}, 0_{L-l_1-l_2}]^T, \ldots, \mu_N = [0_{L-l_N}, 1_{l_N}]^T. \) It then follows from the definitions of \( z_l(t) \) and \( z(t) \) that for all \( l \in \mathcal{C}_i, \)

\[ x_l(t) \rightarrow \sum_{k=1}^{L} \nu_{ik} [e^{A_k t} x_k(0) + (e^{(A+BK)t} - e^{A_l t})\eta_k(0)] \]
\[ \rightarrow e^{A_l t} \sum_{k=1}^{L} \nu_{ik} [x_k(0) - \eta_k(0)], \quad \text{as } t \rightarrow \infty. \]

Since \( A \) is non-Hurwitz, \( e^{A_l t} \) is nonzero as \( t \rightarrow \infty. \) Then, for any set of initial states \( x_l(0), \)
\( l \in \mathcal{I} \), one can always find a set of \( \eta_l(0) \), \( l \in \mathcal{I} \) such that \( \limsup_{t \to \infty} \|x_l(t) - x_k(t)\| \neq 0 \) for any two agents \( l \in \mathcal{C}_i \) and \( k \in \mathcal{C}_j \), \( i \neq j \). This completes the proof. 

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