Chapter 4

On a Class of Analytic Functions Related with Generalized Bazilevic Type Functions
The integral operator

\[ F(z) = \frac{a+1}{z^a} \int_0^z t^{a-1} f(t) \, dt, \quad \text{Re} \, a \geq 0, \]

known as Bernardi integral operator was defined by Bernardi [13] in 1969. This operator has many applications in geometric functions theory and is studied by many authors [1, 4, 54, 107, 126, 138]. The class \( P_k^\lambda(\rho) \) was introduced and studied by Moulis [73]. The class \( V_k^\lambda(\rho) \) was also introduced by Moulis [74]. Brief introduction of these classes along with some related classes and results is given in Section 2.2 and Section 2.4. Using the Bernardi integral operator and the above mentioned classes, Noor and Bukhari defined the class \( Q_k^\lambda(a, \gamma, \rho, b) \). In this chapter we shall use the class \( Q_k^\lambda(0, \gamma, \rho, b) \) and the class \( P_k \) to define a class of generalized Bazilevic functions of type \( \alpha \).

This chapter consists of four sections. In the first section a brief introduction along with definition of some related classes is given. We shall also introduce the class \( B_k^\lambda(\alpha, \gamma, \rho, b) \) in this section. Section two comprises on some preliminary results which are very important and useful for our further discussions, whereas in the next section, we shall investigate some results such as a necessary condition, coefficient bound, arc length and coefficient difference problem. All the contents of this chapter are already published in "Computer and Mathematics with Applications, Vol. 61(2011), 2456–2462", see [8].

4.1 Introduction

Let \( V_k^\lambda(\rho) \) denote the class of generalized bounded boundary rotation of order \( \rho \) related with Robertson functions and let \( P(b), \ b \neq 0 \) (complex), the class of caratheodory functions of complex order. Then using these classes, Noor and Bukhari [98] defined the following class of generalized close-to-convex functions.

**Definition 4.1.1** A function \( f \), analytic in \( E \) and of the form (2.1.1) belongs to the class \( T_k^\lambda(\rho, b) \), \( b \in \mathbb{C} - \{0\} \), \( k \geq 2 \), \( \lambda \) real, \( |\lambda| < \frac{\pi}{2} \) and \( 0 \leq \rho < 1 \), if there exists a
function \( g \in V_k^\lambda (\rho) \) such that

\[
\frac{f'}{g'} \in P(b), \ z \in E. \tag{4.1.1}
\]

Now using the Bernardi integral operator and the class \( T_k^\lambda (\rho, b) \), Noor and Bukhari [98] generalized the class of quasi-convex functions as follows.

**Definition 4.1.2** A function \( f \), analytic in \( E \) and of the form (2.1.1) belongs to the class \( Q_k^\lambda (a, \gamma, \rho, b) \) with \( \frac{f(z)(f(z))'}{z} \not= 0 \), \( \text{Re} \{a\} \geq 0 \), \( 0 < \gamma \leq 1 \), if there exists a function \( g \in T_k^\lambda (\rho, b) \) such that

\[
z f'(z) + \alpha f(z) = (a + 1) z (g'(z))^\gamma, \ z \in E. \tag{4.1.2}
\]

Using the above concepts we generalize the concept of Bazilevic functions of type \( \alpha \).

We define the class \( B_k^\lambda (\alpha, \gamma, \rho, b) \) as follows:

**Definition 4.1.3** A function \( f \in B_k^\lambda (\alpha, \gamma, \rho, b) \), if and only if, there exist \( g \in Q_k^\lambda (0, \gamma, \rho, b) \) such that

\[
\frac{zf''(z)}{f(z)} \left( \frac{f(z)}{zg'(z)} \right)^\alpha \in P_k, \ z \in E, \tag{4.1.3}
\]

where \( k \geq 2 \), \( \alpha > 0 \), \( 0 \leq \rho < 1 \), \( 0 < \gamma \leq 1 \), \( b \in \mathbb{C} - \{0\} \) and \( \lambda \) real, \( |\lambda| < \frac{\pi}{2} \).

### 4.2 Preliminary Lemmas

In this section we include some lemmas which are very useful for our investigations for the main results.

**Lemma 4.2.1** Let \( p \) be analytic in \( E \) with \( p(0) = 1 \) belongs to the class \( P_k \). Then for \( z = re^{i\theta} \),

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |p(z)|^2 \ d\theta \leq \frac{1 + (k^2 - 1) r^2}{1 - r^2}.
\]

This result was proved by Noor [32].
Lemma 4.2.2 If \( h \) is analytic with \( h(0) = 1 \) and \( h \in P(b) \), then

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |h(z)|^2 \, d\theta \leq \frac{1 + \{4|b|^2 - 1\}r^2}{1-r^2}, \text{ for } z = re^{i\theta}.
\]

Lemma 4.2.3 Let \( f \in Q^\lambda_k(0, \gamma, \rho, b) \). Then

\[
\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ \frac{(zf'(z))'}{f'(z)} \right\} \, d\theta > -\left[ |b| + (1 - \rho) \left( \frac{k}{2} - 1 \right) \cos^2 \lambda \right] \gamma \pi,
\]

where \( z = re^{i\theta} \) and \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \).

Both the above results are proved in [98].

4.3 Main Results

This section comprises the results concerning the class \( B^\lambda_k(\alpha, \gamma, \rho, b) \). We derive the necessary conditions for the functions to be in the class \( B^\lambda_k(\alpha, \gamma, \rho, b) \), length of the boundary of the image domain (Length of the image domain), coefficient bound, coefficient difference and maximum value of the modulus for the functions in the class. Throughout this section we assume that \( k \geq 2, \alpha > 0, 0 \leq \rho < 1, 0 < \gamma \leq 1, b \in \mathbb{C} - \{0\} \) and \( \lambda \) real, \( |\lambda| < \frac{\pi}{2} \) unless otherwise stated.

4.3.1 Necessary Condition

In the following result we shall derive the necessary conditions for the functions \( f \) to be in the class \( B^\lambda_k(\alpha, \gamma, \rho, b) \).

Theorem 4.3.1 A function \( f \in B^\lambda_k(\alpha, \gamma, \rho, b) \), if and only if,

\[
\int_{\theta_1}^{\theta_2} \operatorname{Re} J(\alpha, f(z)) \, d\theta > \left[ \frac{k}{2} + \gamma \alpha \left( |b| + \left( \frac{k}{2} - 1 \right) (1 - \rho) \cos^2 \lambda \right) \right] \pi,
\]

(4.3.1)
where \( 0 \leq \theta_1 < \theta_2 \leq 2\pi, \ z = re^{i\theta}, \ r < 1, \) and

\[
J(\alpha, f(z)) = \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} + (\alpha - 1) \left\{ \frac{zf'(z)}{f(z)} \right\}.
\]

\( (4.3.2) \)

**Proof.** We can define, for \( z = re^{i\theta}, \ r \in (0, 1), \ \theta \) real, the following

\[
S(r, \theta) = \arg \left[ zf'(z) f^{\alpha-1}(z) \right], \tag{4.3.3}
\]

and

\[
V(r, \theta) = \arg \left[ (zg'(z))^{\alpha} \right]. \tag{4.3.4}
\]

The functions \( S(z), \ V(z) \) are periodic and continuous with period \( 2\pi \). Since \( f \in B_k^\lambda(\alpha, \gamma, \rho, b) \), therefore from [82], it follows that we can choose the branches of argument of \( S(z) \) and \( V(z) \) as

\[
|S(r, \theta) - V(r, \theta)| < \frac{k\pi}{4}. \tag{4.3.5}
\]

Now we have from (4.3.4) that

\[
\frac{d}{d\theta} V(r, \theta) = \alpha \Im \frac{d}{d\theta} \left\{ \log (re^{i\theta}g'(re^{i\theta})) \right\}
\]

\[
V(r, \theta_2) - V(r, \theta_1) = \alpha \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{(zg'(z))'}{g'(z)} \right\} d\theta.
\]

Since \( g \in Q_k^\lambda(0, \gamma, \rho, b) \), therefore by using Lemma 4.2.3, we obtain

\[
V(r, \theta_2) - V(r, \theta_1) = \alpha \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{(zg'(z))'}{g'(z)} \right\} d\theta > - \left[ |b| + (1 - \rho) \left( \frac{k}{2} - 1 \right) \cos^2 \lambda \right] \gamma \alpha \pi
\]

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and from (4.3.3), (4.3.4) and (4.3.5), we have

\[
S(r, \theta_2) - S(r, \theta_1) = S(r, \theta_2) + V(r, \theta_2) - V(r, \theta_2) + V(r, \theta_1) - V(r, \theta_1) - S(r, \theta_1)
\]

\[
= \{S(r, \theta_2) - V(r, \theta_2)\} - \{S(r, \theta_1) - V(r, \theta_1)\} + \{V(r, \theta_2) - V(r, \theta_1)\}
\]

\[
> -\frac{k\pi}{2} - \gamma\alpha \left\{ |b| + \left(\frac{k}{2} - 1\right) (1 - \rho) \cos^2 \lambda \right\} \pi
\]

\[
= -\left[\frac{k}{2} + \gamma\alpha \left\{ |b| + \left(\frac{k}{2} - 1\right) (1 - \rho) \cos^2 \lambda \right\} \right] \pi.
\]

Moreover, from (4.3.3), it follows that

\[
\frac{d}{d\theta} S(r, \theta) = \text{Re} \left\{ \frac{(zf'(z))'}{f'(z)} + (\alpha - 1) \frac{zf'(z)}{f(z)} \right\}.
\]

Therefore,

\[
\int_{\theta_1}^{\theta_2} \text{Re} J(a, \alpha, f(z)) d\theta > -\left[\frac{k}{2} + \gamma\alpha \left\{ |b| + \left(\frac{k}{2} - 1\right) (1 - \rho) \cos^2 \lambda \right\} \right] \pi.
\]

Which is the required proof. ■

By using the Lemma 2.5.20 we have the following result.

**Remark 4.3.1** The class \(B_k^\lambda(\alpha, \gamma, \rho, b)\) is univalent for \(k \leq \frac{2}{1+\gamma\alpha} \left[ \frac{1-\gamma\alpha|b|}{(1-\rho)\gamma\alpha \cos^2 \lambda} + \gamma\alpha \right].\)

### 4.3.2 Integral Representation

Now we derive the integral representation for the functions to be in the class \(B_k^\lambda(\alpha, \gamma, \rho, b).\)

**Theorem 4.3.2** A function \(f \in B_k^\lambda(\alpha, \gamma, \rho, b),\) if and only if,

\[
f(z) = \left\{ \alpha \int_0^z t^{\alpha-1} \left( \frac{G(t)}{t} \right)^{\alpha\gamma(1-\rho)e^{-i\lambda}} \cos \lambda (h(t))^{\alpha\gamma} p(t) dt \right\}^{\frac{1}{\gamma}}, \quad (4.3.6)
\]
where \( h \in P(b), \ p \in P_k \) and \( G \in R_k \), the class of bounded radius rotations.

**Proof.** From (4.1.3), we have

\[
 z^{1-\alpha} f'(z) (f)^{\alpha-1}(z) = (g'(z))^\alpha p(z) ,
\]

where \( g \in Q_k(0, \gamma, \rho, b) \) and \( p \in P_k \). Now using the Definition 4.1.2, \( g \in Q_k(0, \gamma, \rho, b) \), if and only if

\[
 g'(z) = (g'_1(z))^{\gamma} , \quad z \in E ,
\]

where \( g_1 \in T_k(\rho, b) \). This implies that

\[
 z^{1-\alpha} f'(z) (f(z))^{\alpha-1} = (g'_1(z))^{\gamma \alpha} p(z) .
\]

By Definition 4.1.1 along with (2.4.11), we have

\[
 z^{1-\alpha} f'(z) (f(z))^{\alpha-1} = \left[ (g'_2(z))^{(1-\rho)e^{-i\lambda} \cos \lambda} h(z) \right]^{\gamma \alpha} p(z) ,
\]

where \( g_2 \in V_k \) and \( h \in P(b) \). By Alexander type relation we have \( G(z) = zg'_2(z) \in R_k \). Therefore,

\[
 \alpha f'(z) (f(z))^{\alpha-1} = \alpha z^{\alpha-1} \left( \frac{G(z)}{z} \right)^{\gamma \alpha (1-\rho)e^{-i\lambda} \cos \lambda} (h(z))^{\gamma \alpha} p(z) .
\]

This implies that

\[
 \frac{d}{dz} f^\alpha (z) = \alpha z^{\alpha-1} \left( \frac{G(z)}{z} \right)^{\gamma \alpha (1-\rho)e^{-i\lambda} \cos \lambda} (h(z))^{\gamma \alpha} p(z) .
\]

Integrating from 0 to \( z \), we obtain the required result. ■
4.3.3 Maximum Value of Modulus

**Theorem 4.3.3** Let \( f \in B_k^\lambda (\alpha, \gamma, \rho, b) \) and \( M (r) = \max_{|z|=r} |f (z)| \). Then

\[
M^\alpha (r) \leq \frac{2^{\alpha (1-\rho)} \left( \frac{1}{2} - 1 \right) \cos \lambda + \gamma \alpha - 1 |b|^{\gamma \alpha} (2 + k) r^\alpha}{_2F_1 \left( \alpha, \gamma \alpha \left\{ (1 - \rho) \left( \frac{1}{2} + 1 \right) \cos \lambda + 1 \right\} + 1, \alpha + 1; r \right)}, \tag{4.3.7}
\]

where \(_2F_1 \) is the hypergeometric function.

**Proof.** From the Theorem 4.3.2, we have

\[
f^\alpha (z) = \alpha \int_0^z t^{\alpha - 1} (G' (t) h (t))^{\gamma \alpha} p (t) \, dt, \quad z \in E,
\]

where \( G \in V_k^\lambda (\rho), \ h \in P (b) \) and \( p \in P_k \). It is well known from Lemma 2.4.19 that for \( \lambda \)-spirallike functions \( s_1 \) and \( s_2 \),

\[
G' (z) = \frac{(s_1(z)\left( \frac{1}{2} + \frac{1}{2} \right) (1 - \rho)}{(s_2(z)\left( \frac{1}{2} - \frac{1}{2} \right) (1 - \rho))}. \tag{4.3.8}
\]

Therefore,

\[
f^\alpha (z) = \alpha \int_0^z t^{\alpha - 1} \left[ \left( \frac{s_1(t)}{t} \left( \frac{1}{2} + \frac{1}{2} \right) (1 - \rho) \right)^{\gamma \alpha} h^{\gamma \alpha} (t) p (t) \, dt.
\]

Taking absolute value of both sides, we get

\[
|f^\alpha (z)| = \left| \alpha \int_0^z t^{\alpha - 1} \left[ \left( \frac{s_1(t)}{t} \left( \frac{1}{2} + \frac{1}{2} \right) (1 - \rho) \right)^{\gamma \alpha} h^{\gamma \alpha} (t) p (t) \, dt \right| \right| \leq \left| \alpha \int_0^r |t|^{\alpha - 1} \left[ \left( \frac{s_1(t)}{t} \left( \frac{1}{2} + \frac{1}{2} \right) (1 - \rho) \right)^{\gamma \alpha} |h (t)|^{\gamma \alpha} |p (t)| \, dt \right| \right|. \]

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Using the Lemma 2.2.4, Lemma 2.3.11 and the distortion results of the class, we obtain

\[ |f(z)|^\alpha \leq \alpha 2^{\gamma \alpha (1-\rho)(\frac{k}{2}-1)\cos \lambda -1} (2|b|)^{\gamma \alpha} (2 + k) \int_0^r \frac{t^{\alpha-1}dt}{(1 - t)^{\gamma \alpha \{ (1-\rho)(\frac{k}{2}+1) \cos \lambda + 1 \} + 1}}. \]

Since \( M(r) = \max_{|z|=r} |f(z)| \), therefore

\[ M^\alpha (r) \leq \alpha 2^{\gamma \alpha (1-\rho)(\frac{k}{2}-1)\cos \lambda -1} (2|b|)^{\gamma \alpha} (2 + k) \int_0^r \frac{t^{\alpha-1}dt}{(1 - t)^{\gamma \alpha \{ (1-\rho)(\frac{k}{2}+1) \cos \lambda + 1 \} + 1}}. \]

Now for \( t = ru \), we have

\[ M^\alpha (r) \leq \alpha 2^{\gamma \alpha (1-\rho)(\frac{k}{2}-1)\cos \lambda -1} (2|b|)^{\gamma \alpha} (2 + k) \int_0^1 \frac{u^{\alpha-1}du}{(1 - ru)^{\gamma \alpha \{ (1-\rho)(\frac{k}{2}+1) \cos \lambda + 1 \} + 1}}. \]

Using (2.8.7) for \( a = \gamma \alpha \{ (1-\rho)(\frac{k}{2}+1) \cos \lambda + 1 \} + 1 \), \( b = \alpha \) and \( c = b + 1 \), we obtain

\[ M^\alpha (r) \leq \frac{2^{\gamma \alpha (1-\rho)(\frac{k}{2}-1)\cos \lambda + \gamma \alpha -1} |b|^{\gamma \alpha} (2 + k) \int_0^1 \frac{u^{\alpha-1}du}{(1 - ru)^{\gamma \alpha \{ (1-\rho)(\frac{k}{2}+1) \cos \lambda + 1 \} + 1}}}{\Gamma(\alpha \{ (1-\rho)(\frac{k}{2}+1) \cos \lambda + 1 \} + 1; 1 + \alpha ; r)}. \]

Hence the proof is complete. \( \blacksquare \)

### 4.3.4 Arc Length Problem

Let \( C_r \) denote the closed curve which is the image of the circle \(|z| = r < 1\) under the mapping \( w(z) = f(z) \) and, let \( L_r (f(z)) \) denote the length of \( C_r \). Also let \( M(r) = \max_{|z|=r} |f(z)| \). We now prove the following.

**Theorem 4.3.4** Let \( f \in B_k^\alpha (\alpha, \gamma, \rho, b) \), \( 0 < \alpha \leq 1 \), \( 0 < \gamma < 1 \) and

\[ \gamma \alpha \{ 4(1-\rho)(\frac{k}{2}-1)\cos^2 \lambda + 1 \} > 1. \]

Then, for \( M(r) = \max_{|z|=r} |f(z)| \) and \( r \to 1 \),

\[ L_r F(z) \leq C_k (\alpha, \gamma, \rho, \lambda, b) M^{1-\alpha} (r) \left( \frac{1}{1-r} \right)^{\gamma \alpha \{ (1-\rho)\cos \lambda (\frac{k}{2}-1) \cos \lambda + 2 \} + 1}, \quad (4.3.9) \]

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where $C_k(\alpha, \gamma, \rho, \lambda, b)$ is a constant depending upon $\lambda, \alpha, \gamma, \rho, b$, and $k$ only.

**Proof.** We know that, for $z = re^{i\theta}$, $0 < r < 1$, $0 \leq \theta \leq 2\pi$,

$$L_r F(z) = \int_{0}^{2\pi} |zf'(z)| d\theta. \quad (4.3.10)$$

Now from Theorem 4.3.2, we have

$$zf'(z) = z^\alpha f^{1-\alpha}(z) (g'(z))^{\alpha} p(z),$$

where $g \in Q_k^\lambda(0, \gamma, \rho, b)$ and $p \in P_k$. Since $g \in Q_k^\lambda(0, \gamma, \rho, b)$, therefore using (4.1.1) and (4.1.2), we get

$$g'(z) = (g_1^1(z) h(z))^\gamma,$$

where $g_1 \in V_k^\lambda(\rho)$ and $h \in P(b)$. This implies that from (2.4.11) and (4.3.10) that

$$L_r F(z) = \int_{0}^{2\pi} \left| z^\alpha f^{1-\alpha}(z) \left( (g_2^2(z))^{(1-\rho)} e^{-i\lambda \cos \lambda h(z)} \right)^{\gamma \alpha} p(z) \right| d\theta, \quad (4.3.11)$$

where $h \in P(b)$, $p \in P_k$ and $g_2 \in V_k$. Now, using Lemma 2.4.16 for $g_2 \in V_k$, there exists a starlike function $s$ and $h_1 \in P$ such that

$$zg_2'(z) = s(z) \left( h_1(z) \right)^{\frac{1}{2}-1}, \quad z \in E. \quad (4.3.12)$$

Also from Lemma 2.3.5 it is given that $s_1$ is $\lambda$-spirallike function, if and only if, there is a starlike function $s$ such that

$$s_1(z) = z \left[ \frac{s(z)}{z} \right]^{e^{-i\lambda \cos \lambda z}}, \quad z \in E. \quad (4.3.13)$$
From (4.3.11), (4.3.12) and (4.3.13), we obtain

\[ L_r F (z) = \int_0^{2\pi} \left| z^\alpha f^{1-\alpha} (z) \left( \frac{s_1(z)}{z} \right)^{\gamma \alpha (1-\rho)} (h_1 (z))^{\gamma \alpha (1-\rho) (\frac{t}{2} - 1)} e^{-i\lambda \cos \lambda} h^\alpha (z) p (z) \right| d\theta, \]

(4.3.14)

where \( h_1 \in P, \ h \in P (b), \ p \in P_k \) and \( s_1 \) is \( \lambda \)-spiral-like function. Since \( z = re^{i\theta} \) and \( M (r) = \max_{|z|=r} |f (z)| \), therefore

\[ L_r F (z) \leq 2\pi r^{\alpha + \gamma \alpha (1-\rho)} M^{1-\alpha} (r) e^{\frac{\alpha \ln 2\lambda}{4}} \]

\[ \frac{1}{2\pi} \int_0^{2\pi} \left| (s_1(z))^{\gamma \alpha (1-\rho)} |h(z)|^{\gamma \alpha} |p(z)| |(h_1(z))^{\gamma \alpha (1-\rho) (\frac{t}{2} - 1)} \cos^2 \lambda \right| d\theta. \]

We now using the generalized Holder inequality for \( m_1 = 2, \ m_2 = \frac{2}{\gamma \alpha}, \ m_3 = \frac{4}{1-\gamma \alpha}, \ m_4 = \frac{4}{1-\gamma \alpha} \) are all positive for \( 0 < \alpha \leq 1, \ 0 < \gamma < 1 \) such that \( \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} = 1 \). Therefore, we have

\[ L_r F (z) \leq 2\pi r^{\alpha + \gamma \alpha (1-\rho)} M^{1-\alpha} (r) e^{\frac{\alpha \ln 2\lambda}{4}} \left( \frac{1}{2\pi} \int_0^{2\pi} |(s_1(z))|^{\frac{4 \gamma \alpha (1-\rho)}{1-\gamma \alpha}} d\theta \right)^{\frac{1-\gamma \alpha}{4}} \]

\[ \left( \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \right)^{\frac{1}{2}} \left( \frac{1}{2\pi} \int_0^{2\pi} |(h_1(z))|^{\frac{4 \gamma \alpha (1-\rho) (\frac{t}{2} - 1) \cos^2 \lambda}{1-\gamma \alpha}} d\theta \right)^{\frac{1-\gamma \alpha}{4}} \]

\[ \left( \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \right)^{\frac{\gamma \alpha}{4}} . \]
By Lemma 2.2.2, Lemma 4.2.1, Lemma 4.2.2 and a subordination result, we have

\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |s_1(z)| \frac{4\gamma\alpha(1-\rho)}{1-\gamma\alpha} \, d\theta \right)^{\frac{1-\gamma\alpha}{4}} \leq \left( \frac{1}{2\pi} \right)^{\frac{1-\gamma\alpha}{4}} \left( \frac{1}{1-r} \right)^{2\gamma\alpha(1-\rho)\cos\lambda - \frac{1-\gamma\alpha}{4}},
\]

\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 \, d\theta \right)^{\frac{\gamma\alpha}{2}} \leq \left( 1 + \frac{4 |b|^2 - 1}{1 - r^2} \right) \frac{2^{\gamma\alpha}}{T},
\]

\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 \, d\theta \right)^{\frac{1}{2}} \leq \left( \frac{1 + (k^2 - 1) r^2}{1 - r^2} \right)^{\frac{1}{2}},
\]

\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |h_1(z)| \frac{4\gamma\alpha(1-\rho)}{2^{\gamma\alpha}} \, d\theta \right)^{\frac{1-\gamma\alpha}{4}} \leq \left( c(\lambda_1) \right)^{\frac{1-\gamma\alpha}{4}} \left( \frac{1}{1-r} \right)^{2\gamma\alpha(1-\rho)(\frac{k}{2} - 1)\cos^2\lambda - \frac{1-\gamma\alpha}{4}},
\]

with \( \lambda_1 = \frac{4\gamma\alpha(1-\rho)(\frac{k}{2} - 1)}{1-\gamma\alpha} \cos^2\lambda > 1 \). Therefore, we have

\[
L_\rho F(z) \leq 2^{\frac{1+3\gamma\alpha}{4}} \pi^{\frac{3+\gamma\alpha}{2}} r^{\alpha + \gamma\alpha(\rho-1)} M^{1-\alpha}(r) e^{\frac{\sin 2\lambda}{4} k |b|^{\gamma\alpha} \left( c(\lambda_1) \right)^{\frac{1-\gamma\alpha}{4}}} \times \left( \frac{1}{1-r} \right)^{2\gamma\alpha(1-\rho)\cos\lambda - \frac{1-\gamma\alpha}{4} + \gamma\alpha(1-\rho)(\frac{k}{2} - 1)\cos^2\lambda - \frac{1-\gamma\alpha}{4} + \frac{\gamma\alpha}{4} + \frac{1}{2}}.
\]

This implies that

\[
L_\rho F(z) \leq C_k(\alpha, \gamma, \rho, \lambda, b) M^{1-\alpha}(r) \left( \frac{1}{1-r} \right)^{\gamma\alpha(1-\rho)\cos\lambda(\frac{k}{2} - 1)\cos^2\lambda + 1}, r \to 1.
\]
4.3.5 Rate of Coefficient Growth

The problem of growth and asymptotic behavior of coefficients is well-known. In the upcoming results, we investigate this problem.

**Theorem 4.3.5**  Let $f \in B_k^\lambda (\alpha, \gamma, \rho, b), \ 0 < \alpha \leq 1, \ 0 < \gamma < 1$

and $\gamma \alpha \{4 (1 - \rho) \left(\frac{k}{2} - 1\right) \cos^2 \lambda + 1\} > 1$. Then for $n \geq 2$

$$|a_n| \leq C_{k_1} (\alpha, \gamma, \rho, \lambda, b) M^{1-\alpha} \left(\frac{1}{r}\right) \left[\gamma \alpha \left(\frac{(1-\rho) \cos \lambda (\frac{k}{2} - 1) \cos \lambda + 2) + 1\right)ight], \quad (4.3.15)$$

where $C_{k_1} (\alpha, \gamma, \rho, \lambda, b)$ is constant depending upon $\lambda, \alpha, \gamma, \rho, b$ and $k$ only.

**Proof.** Since with $z = re^{i\theta}$, Cauchy theorem gives

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} z f'(z) e^{-i\theta} d\theta$$

$$n |a_n| \leq \int_0^{2\pi} |z f'(z) e^{-i\theta}| d\theta = \frac{1}{2\pi r^n} L_r F(z).$$

Using Theorem [4.3.4] we obtain

$$n |a_n| \leq \frac{1}{2\pi r^n} C_k (\alpha, \gamma, \rho, \lambda, b) M^{1-\alpha} \left(\frac{1}{r}\right) \left[\gamma \alpha \left(\frac{(1-\rho) \cos \lambda (\frac{k}{2} - 1) \cos \lambda + 2) + 1\right)\right].$$

Taking $r = 1 - \frac{1}{n}$, we have

$$|a_n| \leq C_{k_1} (\alpha, \gamma, \rho, \lambda, b) M^{1-\alpha} \left(n\right) \left(n\right) \gamma \alpha \left[\frac{(1-\rho) \cos \lambda (\frac{k}{2} - 1) \cos \lambda + 2) + 1\right]^{-1}, \quad n \to \infty,$$

which is the required result. ■

4.3.6 Coefficient Difference

In the theorem below, we discuss the coefficient difference problem for the class $B_k^\lambda (\alpha, \gamma, \rho, b)$. 83
Theorem 4.3.6 Let $f \in \mathcal{B}_k^* (\alpha, \gamma, \rho, b)$, $0 < \alpha \leq 1$, $0 < \gamma < 1$ and
\[ \gamma \alpha (1 - \rho) \left( \frac{b}{2} - 1 \right) \cos^2 \lambda > 1. \] Then for $n \geq 2$,
\[ ||a_{n+1}|| - |a_n| \leq C_{k_2} (\alpha, \gamma, \rho, \lambda, b) M^{1-\alpha} (n)^{\gamma \alpha \beta (1-\rho) \cos \lambda \left( \frac{b}{2} - 1 \right) \cos^2 \lambda} \frac{1}{2 \pi r^n} \int_0^{2\pi} \left| z - \xi \right| |z f' (z)| \, d\theta, \]
where $C_{k_2} (\alpha, \gamma, \rho, \lambda, b)$ is a constant depending upon $\lambda$, $\alpha$, $\gamma$, $\rho$, $b$ and $k$ only.

Proof. We know that
\[ ||(n + 1) \xi a_{n+1} - na_n|| \leq \frac{1}{2 \pi r^n} \int_0^{2\pi} \left| z - \xi \right| |z f' (z)| \, d\theta. \]

Since
\[ z f' (z) = z^{\alpha + 1 - \alpha} (z) \left( \frac{s_1 (z)}{z} \right)^{\gamma \alpha (1 - \rho)} (h_1 (z))^{\gamma \alpha (1 - \rho)} \left( \frac{b}{2} - 1 \right) e^{-i \lambda \cos \lambda} h^{\gamma \alpha} (z) p (z), \]
where $h_1 \in P$, $h \in P (b)$, $p \in P_k$ and $s_1$ is $\lambda$-spirallike function, therefore
\[ ||(n + 1) \xi a_{n+1} - na_n|| \leq \frac{1}{2 \pi r^n} \int_0^{2\pi} \left| z - \xi \right| \left| z^{\alpha + 1 - \alpha} (z) \left( \frac{s_1 (z)}{z} \right)^{\gamma \alpha (1 - \rho)} \left( h_1 (z) \right)^{\gamma \alpha (1 - \rho)} \left( \frac{b}{2} - 1 \right) e^{-i \lambda \cos \lambda} h^{\gamma \alpha} (z) p (z) \right| \, d\theta, \]
\[ \leq e^{\frac{\pi \sin 2\lambda}{4} M^{1-\alpha} (r)^{\rho^2 - \gamma \alpha (1 - \rho)}} \frac{1}{2 \pi r^n} \int_0^{2\pi} \left| z - \xi \right| |s_1 (z)| \left| s_1 (z) \right|^{\gamma \alpha (1 - \rho) - 1} \left| h_1 (z) \right|^{\gamma \alpha (1 - \rho)} \left( \frac{b}{2} - 1 \right) \cos^2 \lambda \left| h (z) \right|^{\gamma \alpha} |p (z)| \, d\theta \right\}. \]
Using the Lemma 2.1.1, we get
\[ ||(n + 1) \xi a_{n+1} - na_n|| \leq \frac{1}{2 \pi r^n} M^{1-\alpha} (r)^{\rho^2 - \gamma \alpha (1 - \rho)} e^{\frac{\pi \sin 2\lambda}{4}} \frac{2 r^2}{1 - r^2} \]
\[ \int_0^{2\pi} \left| s_1 (z) \right|^{\gamma \alpha (1 - \rho) - 1} \left| h_1 (z) \right|^{\gamma \alpha (1 - \rho)} \left( \frac{b}{2} - 1 \right) e^{-i \lambda \cos \lambda} \left| h (z) \right|^{\gamma \alpha} |p (z)| \, d\theta. \]
Now by using similar procedure as given in Theorem 4.3.4 we obtain

\[ |(n+1) \xi a_{n+1} - na_n| \leq C_{k_2} (\alpha, \gamma, \rho, \lambda, b) M^{1-\alpha} (r) \]

\[ r^{\alpha - \gamma \alpha (1-\rho) - n} \left( \frac{1}{1 - r} \right)^{\gamma \alpha \left[ (1-\rho) \cos \lambda \left( \left( \frac{k}{2} - 1 \right) \cos \lambda + 2 \right) + 1 \right] - 2 \cos \lambda - 1}. \]

Putting \(|\xi| = r = \frac{n}{1+n}\), we have the required result. ■

### 4.4 Conclusion

In this chapter we used the class \( Q_k^\lambda (0, \gamma, \rho, b) \) and the class \( P_k \) to define a class of generalized Bazilevic functions of type \( \alpha \). We have investigated some results such as a necessary condition, coefficient bound, arc length and coefficient difference problem.