DETERMINANTAL INEQUALITIES OF POSITIVE DEFINITE MATRICES

DAESHKI CHOI

(Communicated by I. Perić)

Abstract. Let $A_i$, $i = 1, \ldots, m$, be positive definite matrices with diagonal blocks $A_i^{(j)}$, $1 \leq j \leq k$, where $A_1^{(j)}, \ldots, A_m^{(j)}$ are of the same size for each $j$. We prove the inequality

$$\det \left( \sum_{i=1}^{m} A_i^{-1} \right) \geq \det \left( \sum_{i=1}^{m} (A_i^{(1)})^{-1} \right) \cdots \det \left( \sum_{i=1}^{m} (A_i^{(k)})^{-1} \right)$$

and more determinantal inequalities related to positive definite matrices.

1. Introduction

Notation. Throughout the paper, we will use the following notation:

- $I$ denotes the identity matrix of a proper size. We do not specify its order.
- $A \prec B$ ($A \preceq B$) is used to imply that $A$ and $B$ are Hermitian matrices such that $B - A$ is positive definite (semidefinite). In particular, a positive definite (positive semidefinite) matrix $A$ can be expressed as $A \succ 0$ ($A \succeq 0$).
- $\text{diag}(D_1, \ldots, D_k)$ denotes the block diagonal matrix whose diagonal blocks are $D_1, \ldots, D_k$.

Fischer’s inequality [1, Theorem 7.8.3] states that if $A$ is a positive definite matrix with diagonal blocks $A_1, \ldots, A_k$, then

$$\det A \leq \det A_1 \cdots \det A_k.$$ 

Let $A_i$, $i = 1, \ldots, m$, be positive definite matrices whose diagonal blocks are $n_j$-square matrices $A_i^{(j)}$ for $j = 1, \ldots, k$. Then the relation

$$\det \left( \sum_{i=1}^{m} A_i \right) \leq \det \left( \sum_{i=1}^{m} A_i^{(1)} \right) \cdots \det \left( \sum_{i=1}^{m} A_i^{(k)} \right)$$

follows directly from Fischer’s inequality. The main result of the paper is to show

$$\det \left( \sum_{i=1}^{m} A_i^{-1} \right) \geq \det \left( \sum_{i=1}^{m} (A_i^{(1)})^{-1} \right) \cdots \det \left( \sum_{i=1}^{m} (A_i^{(k)})^{-1} \right).$$


Keywords and phrases: Determinantal inequalities, Fischer’s inequality, determinants of block matrices, positive definite matrices.
2. Proof of the Main inequality

The following is a well-known result [1, Corollary 7.7.4].

**Lemma 1.** If $0 \prec A \preceq B$, then $B^{-1} \preceq A^{-1}$ and $\det(A) \preceq \det(B)$.

We expect the following is known, but we include a proof as we do not know a reference.

**Lemma 2.** Let $P = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ be a positive definite matrix. Then $P$ can be factorized as $P = T^*T$ with $T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$ being conformally partitioned as $P$.

**Proof.** Since $A$ is positive definite, it can be factorized as $A = X^*X$ for an invertible matrix $X$. Since $P$ is positive definite, the Schur complement $C - B^*A^{-1}B$ is also positive definite. Thus there exists a matrix $Z$ such that $C - B^*A^{-1}B = Z^*Z$. If $T$ is defined by $T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$, where $Y = (X^*)^{-1}B$, then a direct computation shows $P = T^*T$. $\square$

The following is in [2, Corollary 1].

**Lemma 3.** Let $T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$, where $X$ and $Z$ are square matrices. Then

$$\det(I + T^*T) \geq \det(I + X^*X)\det(I + Z^*Z).$$

The following theorem is equivalent to Theorem 1.1 in [3]. Here we give a simple proof using Lemma 3.

**Theorem 1.** Let $C_i \succ 0$ and $D_i \succ 0$ be $n_i$-square matrices for $i = 1, \ldots, k$ and $D = \text{diag}(D_1, \ldots, D_k)$. Then

$$\det(I + C^{-1}D) \geq \det(I + C_1^{-1}D_1) \cdots \det(I + C_k^{-1}D_k). \quad (2.1)$$

**Proof.** By a standard continuity argument, we may assume that $D_i$ are positive definite. In this case, it is also enough to show the inequality

$$\det(I + C^{-1}) \geq \det(I + C_1^{-1}) \cdots \det(I + C_k^{-1}) \quad (2.2)$$

by the following argument:

$$\det(I + C^{-1}D) = \det(I + (D^{-\frac{1}{2}}CD^{-\frac{1}{2}})^{-1})$$

$$\geq \det(I + (D_1^{-\frac{1}{2}}C_1D_1^{-\frac{1}{2}})^{-1}) \cdots \det(I + (D_k^{-\frac{1}{2}}C_kD_k^{-\frac{1}{2}})^{-1})$$

$$= \det(I + C_1^{-1}D_1) \cdots \det(I + C_k^{-1}D_k).$$
Moreover, mathematical induction allows us to prove (2.2) for \( k = 2 \). By Lemma 2, there exists a matrix \( T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix} \) being conformally partitioned as \( C^{-1} \) such that \( C^{-1} = T^*T \). Then we have

\[
\det(I + C^{-1}) = \det(I + T^*T) \geq \det(I + X^*X) \det(I + Z^*Z)
\]

by Lemma 3. Now it is enough to show \((X^*X)^{-1} \preceq C_1\) and \((Z^*Z)^{-1} \preceq C_2\), since the relations and the above inequality imply

\[
\det(I + C^{-1}) \geq \det(I + C_1^{-1}) \det(I + C_2^{-1})
\]

by Lemma 1. From

\[
C = (T^*T)^{-1} = \begin{bmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y + Z^*Z \end{bmatrix}^{-1},
\]

we have

\[
C_1 = (X^*X - X^*Y(Y^*Y + Z^*Z)^{-1}Y^*X)^{-1}
\]

by the block inverse theorem [1]. Thus \( C_1 \geq (X^*X)^{-1} \). Similarly, we have

\[
C_2 = (Y^*Y + Z^*Z - Y^*X(X^*X)^{-1}X^*Y)^{-1}
\]

\[
= (Y^*(I - X(X^*X)^{-1}X^*)Y + Z^*Z)^{-1}
\]

\[
= (Z^*Z)^{-1}. \quad \square
\]

**Corollary 1.** Let \( A \) be positive definite. If \( A_i \) and \( B_i, \ i = 1, \ldots, k, \) are the \( n_i \)-square diagonal blocks of \( A \) and \( A^{-1} \), respectively, then

\[
\det(I + (A_iB_i)^{-1}) \leq 2^{n_i} \leq \det(I + A_iB_i), \ i = 1, \ldots, k.
\]

**Proof.** Fix \( i \). If \( C = A, D_i = A_i, \) and \( D_j \) is the zero matrix for all \( j \neq i \) in (2.1), then we have \( \det(I + A_iB_i) \geq 2^{n_i} \). Similarly, if \( C = A, D_i = B_i^{-1}, \) and \( D_j \) is the zero matrix for all \( j \neq i \) in (2.1), we have \( 2^{n_i} \geq \det(I + A_i^{-1}B_i^{-1}) \). \( \square \)

We can generalize (2.1) using the following result [2, Theorem 1]:

**Lemma 4.** Let \( T_i = \begin{bmatrix} X_i & Y_i \\ O & Z_i \end{bmatrix}, \ i = 1, \ldots, m, \) be \( n_i \)-square conformally partitioned matrices. Then

\[
\det(\sum_{i=1}^{m} T_i^*T_i) \geq \det(\sum_{i=1}^{m} X_i^*X_i) \det(\sum_{i=1}^{m} Z_i^*Z_i).
\]

The following is the main theorem of the paper.

**Theorem 2.** (Main) Let \( A_i, \ i = 1, \ldots, m, \) be positive definite matrices whose diagonal blocks are \( n_j \)-square matrices \( A_i^{(j)} \) for \( j = 1, \ldots, k \). Then

\[
\det(\sum_{i=1}^{m} A_i^{-1}) \geq \det(\sum_{i=1}^{m} (A_i^{(1)})^{-1}) \cdots \det(\sum_{i=1}^{m} (A_i^{(k)})^{-1}).
\]
Proof. We use the same argument as we did in Theorem 1. Using mathematical induction on \( k \), we may assume \( k = 2 \). By Lemma 2, for each \( i = 1, \ldots, m \) there exists a matrix \( T_i = \begin{bmatrix} X_i & Y_i \\ O & Z_i \end{bmatrix} \) being conformally partitioned as \( A_i^{-1} \) such that \( A_i^{-1} = T_i^*T_i \). Then

\[
\det(\sum_{i=1}^{m} A_i^{-1}) \geq \det(\sum_{i=1}^{m} X_i^*X_i) \det(\sum_{i=1}^{m} Z_i^*Z_i)
\]

by Lemma 4. Now it is enough to show \( (X_i^*X_i)^{-1} \preceq A_i^{(1)} \) and \( (Z_i^*Z_i)^{-1} \preceq A_i^{(2)} \) for each \( i \), since the relations and the inequality above imply

\[
\det(\sum_{i=1}^{m} A_i^{-1}) \geq \det(\sum_{i=1}^{m} A_i^{(1)})^{-1}) \det(\sum_{i=1}^{m} A_i^{(2)}))^{-1}
\]

by Lemma 1. From

\[
A_i = (T_i^*T_i)^{-1} = \begin{bmatrix} X_i^*X_i & X_i^*Y_i \\ Y_i^*X_i & Y_i^*Y_i + Z_i^*Z_i \end{bmatrix}^{-1},
\]

we have

\[
A_i^{(1)} = (X_i^*X_i - X_i^*Y_i(Y_i^*Y_i + Z_i^*Z_i)^{-1}Y_i^*X_i)^{-1}
\]

and thus \( A_i^{(1)} \succ (X_i^*X_i)^{-1} \). Similarly,

\[
A_i^{(2)} = (Y_i^*Y_i + Z_i^*Z_i - Y_i^*X_i(X_i^*X_i)^{-1}X_i^*Y_i)^{-1}
\]

\[
= (Y_i^*(I - X_i(X_i^*X_i)^{-1}X_i^*)Y_i + Z_i^*Z_i)^{-1}
\]

\[
= (Z_i^*Z_i)^{-1}. \quad \square
\]

3. More inequalities

Here we show more inequalities related to Theorem 1. The following will be used without proof (See [1, Theorem 7.7.8]).

**Lemma 5.** If \( S \subseteq \{1, 2, \ldots, n\} \) is an index set, then \( A(S)^{-1} \preceq A^{-1}(S) \), where \( B(T) \) denotes the principle submatrix of \( B \) determined by deletion of the rows and columns indicated by \( T \).

The following presents additional inequalities of determinants. One of them is the inequality in Theorem 1. We contains it here since it is proved in a different way.

**Theorem 3.** Let \( C_i \succ 0 \) and \( D_i \succeq 0 \) be \( n_i \)-square matrices for \( i = 1, \ldots, k \) and \( D = \text{diag}(D_1, \ldots, D_k) \). Then we have the following results:

(a) \( \det(I + CD) \leq \det(I + C_1D_1) \cdots \det(I + C_kD_k). \)

\(^1\)The inequality is equivalent to the inequality in [3, Theorem 1.2].
(b) If \( D \preceq C^{-1} \), then \( D_i \preceq C_i^{-1} \) and
\[
\det(I - CD) \leq \det(I - C_1D_1) \cdots \det(I - C_kD_k).
\]

(c) \( \det(I + C^{-1}D) \geq \det(I + C_1^{-1}D_1) \cdots \det(I + C_k^{-1}D_k) \). (Theorem 1)

(d) If \( D \preceq C \), then \( D_i \preceq C_i \) and
\[
\det(I - C^{-1}D) \leq \det(I - C_1^{-1}D_1) \cdots \det(I - C_k^{-1}D_k).
\]

**Proof.** (a) follows directly from Fischer’s inequality:
\[
\det(I + CD) = \det(I + \sqrt{DC} \sqrt{D})
\leq \prod_{i=1}^{k} \det(I + \sqrt{D_i}C_i \sqrt{D_i})
= \prod_{i=1}^{k} \det(I + C_iD_i).
\]

Assume \( D \preceq C^{-1} \). Since \( I - \sqrt{D_i}C_i \sqrt{D_i}, \ i = 1, \ldots, k, \) are the diagonal blocks of the positive semidefinite matrix \( I - \sqrt{DC} \sqrt{D} \), the relation \( D_i \preceq C_i^{-1} \) holds for all \( i \) and (b) also follows from Fischer’s inequality.

Now we prove (c). Let \( B = (C + D)^{-1} \). Then \( B \) is a positive definite matrix such that \( D \preceq B^{-1} \). By (b), \( D_i \preceq B_i^{-1} \) for all \( i \) and
\[
\det(I - BD) \leq \det(I - B_1D_1) \cdots \det(I - B_kD_k), \tag{3.1}
\]
where \( B_1, \ldots, B_k \) are the diagonal blocks of \( B \). Since \( (I - BD)(I + C^{-1}D) = I \), the left hand side of (3.1) is \( 1/\det(I + C^{-1}D) \). Meanwhile, fix \( i \) and let \( S \subset \{1, 2, \ldots, n\} \) be the index set such that \( B_i = B(S) \) (thus \( C_i = C(S) \) and \( D_i = D(S) \)). Then \( B_i^{-1} \preceq B^{-1}(S) = C_i + D_i \) by Lemma 5 and
\[
\det((I - B_iD_i)^{-1}) = \frac{\det(B_i^{-1})}{\det(B_i^{-1} - D_i)}
= \frac{\det(B_i^{-1} - D_i + D_i)}{\det(B_i^{-1} - D_i)}
= \frac{\det(I + (B_i^{-1} - D_i)^{-1}D_i)}{\det(B_i^{-1} - D_i)}
\geq \det(I + C_i^{-1}D_i).
\]
Therefore (c) holds. A similar argument is applied to (d). Assume \( D \preceq C \) and let \( B = C - D \). Without loss of generality, we may assume \( D \preceq C \). By (c),
\[
\det(I + B^{-1}D) \geq \det(I + B_1^{-1}D_1) \cdots \det(I + B_k^{-1}D_k), \tag{3.2}
\]
where \( B_i = C_i - D_i \) for \( i = 1, \ldots, k \). Since \( (I + B^{-1}D)(I - C^{-1}D) = I \), the left hand side of (3.2) is \( 1/\det(I - C^{-1}D) \). Moreover, since
\[
\det((I + B_i^{-1}D_i)^{-1}) = \frac{\det(B_i + D_i - D_i)}{\det(B_i + D_i)} = \frac{\det(C_i - D_i)}{\det(C_i)} = \det(I - C_i^{-1}D_i),
\]
inequality (d) holds. \( \square \)
REFERENCES


(Received January 28, 2015)