On the Cauchy Problem for Nonlinear Hyperbolic Systems

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Abstract. This paper consider various examples of metrics which are contractive w.r.t. an evolution semigroup, and discusses the possibility of an abstract O.D.E. theory on metric spaces, with applications to hyperbolic systems. In particular, using a recently introduced definition of Viscosity Solutions, it is shown how a strictly hyperbolic system of conservation laws can be reformulated as an abstract evolution equation on a closed domain of BV functions, with the L^1 distance.

1 - Introduction.

Aim of this paper is to review some recent theorems concerning the uniqueness and continuous dependence of entropy weak solutions to the Cauchy problem

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar u(x), \tag{1.2}$$

and to show how some of these results can be recast in an abstract theory of evolution equations in metric spaces [29].

We assume that the $n \times n$ system of conservation laws (1.1) is strictly hyperbolic, with each characteristic field either linearly degenerate or genuinely nonlinear [24, 34]. In this setting, a basic existence problem can be formulated as follows.

(EP1) Show that there exists some (nontrivial) positively invariant domain $\mathcal{D} \subset \mathbf{L}^1$ such that, for every initial data $\bar u \in \mathcal{D}$, the Cauchy problem (1.1)-(1.2) has at least one global, entropy-admissible, weak solution $u : [0,\infty[\mapsto \mathcal{D}$.

We recall that, for smooth initial data $\bar u$, the local existence of a solution is well known [23, 31]. Due to genuine nonlinearity, however, this solution may lose its regularity. More precisely, its gradient u_x may become infinite within finite time [21]. For this reason, in general, solutions can be constructed globally in time only within a space of discontinuous functions. A natural choice is the space **BV** of integrable functions $u : I\!R \mapsto I\!R^n$ with bounded variation. The global existence problem **(EP1)** was solved in the fundamental paper of Glimm [19], on a domain $\mathcal{D}$ consisting of functions with suitably small total variation. The positively invariant domain $\mathcal{D}$ can be defined as follows. Consider a piecewise constant function u, say with jumps at the points $x_1 < \cdots < x_N$. For $\alpha = 1, \ldots, N$, call $\sigma^1_\alpha, \ldots, \sigma^n_\alpha$ the strengths of the waves in the standard self-similar solution of the Riemann problem with data $u(x_\alpha-)$, $u(x_\alpha+)$. The total strength of waves in u and the potential for future wave interactions are defined respectively as

$$V(u) \doteq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} |\sigma^i_\alpha|, \qquad\qquad Q(u) \doteq \sum_{\mathcal{A}} \big|\sigma^i_\alpha \sigma^j_\beta\big|, \tag{1.3}$$

where the second sum ranges over all couples of approaching waves. One then defines

$$\mathcal{D} \doteq cl\Big\{u \in \mathbf{L}^1(I\!R;\, I\!R^n), \quad u \text{ is piecewise constant}, \;\; V(u) + \kappa_0 Q(u) < \delta_0\Big\}, \tag{1.4}$$

for suitable constants $\kappa_0, \delta_0 > 0$, where cl denotes closure in $\mathbf{L}^1$.

Concerning initial value problems with large **BV** data, some global existence results have been obtained in [32, 33]. Moreover, for various 2×2 systems, the method of compensated compactness yields existence of weak solutions in a class of $\mathbf{L}^\infty$ functions [18]. The existence problem with arbitrary **BV** data, however, is still poorly understood. In particular, if F is a smooth function defined on an open set $\Omega \subseteq I\!R^n$, and if every Riemann problem with data $u^-, u^+ \in \Omega$ has an entropy weak solution taking values inside Ω, it is not clear which additional conditions will guarantee that the set

$$\big\{u : I\!R \mapsto \Omega; \quad u \in \mathbf{BV}\big\}$$

is positively invariant for a flow generated by (1.1). Indeed, the total variation may approach infinity in finite time.

Concerning uniqueness, we remark that for smooth solutions (1.1) is equivalent to the quasilinear system

$$u_t + A(u)u_x = 0. \tag{1.5}$$

Here $A(u) = DF(u)$ is the Jacobian matrix of F at u. If u, v are both solutions of (1.1), their difference $w = u - v$ satisfies

$$w_t + A\big(u(t,x)\big)w_x + \big[DA(t,x)\cdot w\big]v_x(t,x) = 0, \tag{1.6}$$

where

$$DA(t,x) \doteq \int_0^1 DA\big(\theta u(t,x) + (1-\theta)v(t,x)\big)\, d\theta$$

and $DA(u)$ is the differential of the map $u \mapsto A(u)$. Observe that (1.6) is a linear homogeneous hyperbolic system for w, with coefficients depending on t, x. If the functions u, v are Lipschitz continuous, then these coefficients have some degree of regularity, which allows one to derive a priori estimates on the norm of w. In particular, from $w(0) = 0$ we deduce $w(t) = 0$ for all $t > 0$, hence $u \equiv v$, proving uniqueness.

If the solutions u, v are discontinuous, however, a direct estimate of the difference $w = u - v$ becomes much more difficult. Some results along these lines were obtained in [17, 20, 28, 32].

Progress has been achieved recently by a new approach, based on the preliminary construction of a semigroup of solutions of (1.1). Instead of tackling the uniqueness question directly, it is convenient to consider first a more comprehensive existence problem.

(EP2) Show that there exists a (nontrivial) domain $\mathcal{D} \subset \mathbf{L}^1$ and a continuous flow $S : \mathcal{D}\times[0,\infty[\,\mapsto \mathcal{D}$ with the properties:

(i) $S_0 u = u, \qquad S_s S_t u = S_{s+t} u,$

(ii) $\|S_t u - S_s v\| \le L \cdot \big(|t-s| + \|u - v\|\big),$

(iii) every trajectory $t \mapsto S_t u$ is a weak, entropy-admissible solution of the partial differential equation (1.1).

Observe that (i) is the semigroup property, while (ii) states that the flow is globally Lipschitz continuous, both with respect to time and to the initial data.

The main difference between the problems **(EP1)** and **(EP2)** is that, in the latter, we wish to construct solutions simultaneously for all $\bar{u} \in \mathcal{D}$, continuously depending on the initial data. Establishing the existence of a semigroup S satisfying (i)–(iii) is thus a more difficult task than proving the existence of one single solution. This was achieved in [3] for the linearly degerate case, in [8] for genuinely nonlinear 2×2 systems, and in [9] for general $n \times n$ systems. More precisely, we have

Theorem 1. *Let the system (1.1) be strictly hyperbolic in a neighborhood of the origin, and assume that each characteristic field is either linearly degenerate or genuinely nonlinear. Then there exist a domain $\mathcal{D}$ as in (1.4) and a continuous semigroup $S : \mathcal{D} \times [0,\infty[\,\mapsto \mathcal{D}$ satisfying the above conditions (i)–(iii).*

The uniqueness of the semigroup, on the other hand, can be proved rather easily. In this connection, it is convenient to introduce a new condition, stating that the flow is consistent with the standard solutions of the Riemann problem:

(iv) If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for all $t > 0$ sufficiently small the function $u(t, \cdot) = S_t\bar{u}$ coincides with the solution of (1.1)-(1.2) obtained by piecing together the standard self-similar solutions of the Riemann problems determined by the jumps in $\bar{u}$.

In [6], the following uniqueness result was proven:

Theorem 2. *Under the same assumptions as in Theorem 1, for a given domain $\mathcal{D}$ of the form (1.4), the semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ satisfying (i), (ii) and (iv) is unique. Moreover, it also satisfies the condition (iii).*

In the above setting, we call S the *Standard Riemann Semigroup* generated by the system (1.1).

Having established the existence and uniqueness of the semigroup, we are in a much better position to study the uniqueness of the solution to a particular Cauchy problem, and its dependence on the initial data. Indeed, the analysis relies on the following simple lemma.

Lemma 1. *Let $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ be a continuous flow satisfying the properties (i)-(ii) in* **(EP2)**. *Then for every Lipschitz continuous map $w : [0, T] \mapsto \mathcal{D}$ one has the estimate*

$$\left\| w(T) - S_T w(0) \right\|_{\mathbf{L}^1} \leq L \cdot \int_0^T \left\{ \liminf_{h \to 0+} \frac{\left\| w(t+h) - S_h w(t) \right\|_{\mathbf{L}^1}}{h} \right\} dt. \tag{1.7}$$

Observe that the only relevant assumption of the lemma is the Lipschitz continuity of the Semigroup. No reference is here made to the particular equation (1.1). Our main application, however, will be in connection with the Cauchy problem (1.1)-(1.2). Let $w = w(t)$ be any approximate solution, Lipschitz continuous as a function of time, taking the correct initial condition $w(0) = \bar{u}$. If (iii) holds, the exact solution is then provided by the function $t \mapsto S_t\bar{u}$. At time T, the $\mathbf{L}^1$ distance between the approximate and the exact solution is estimated by (1.7). Observe that the integrand here determines the instantaneous error rate. The estimate (1.7) thus states that these local errors are magnified at most by a factor of L, and summed over the interval $[0, T]$. Formula (1.7) has two main applications.

1. Let $u = u(t, x)$ be a weak solution of (1.1), Lipschitz continuous as a function of time with values in $\mathbf{L}^1$. If one can show that

$$\liminf_{h \to 0+} \frac{\left\| w(t+h) - S_h w(t) \right\|_{\mathbf{L}^1}}{h} = 0 \qquad \text{for a.e. } t > 0, \tag{1.8}$$

it then follows that $w(t) = S_t w(0)$ for all t. Since we already know that the semigroup S is unique, this yields a new method for proving uniqueness of solutions to a given Cauchy problem. This approach was pursued in [6], establishing the uniqueness of limit solutions obtained by the Glimm scheme [19, 26] or by wave-front tracking approximations [2, 4, 16, 30]. Using similar ideas, a more general uniqueness result for entropy-weak solutions was proved in [11], provided that the total variation of u is suitably bounded along segments in the t-x plane.

2. Let $w = w(t, x)$ be an approximate solution of (1.1) constructed by a wave-front tracking method. For every $T > 0$, the distance between $w(T)$ and the exact solution $S_T w(0)$ can then be estimated by computing the integrand in (1.7). More precisely, assume that at a generic time t the function w is piecewise constant, with jumps located at points $x_1(t) < \cdots < x_N(t)$, travelling with speed $\dot{x}_\alpha$, $\alpha = 1, \ldots, N$. Call $\omega_\alpha = \omega_\alpha(t, x)$ the self-similar solution of the Riemann problem with data $w(t, x_\alpha -)$, $w(t, x_\alpha +)$. Letting $\hat{\lambda}$ be an upper bound for all wave speeds and choosing $h_0 > 0$ suitably small, we can write

$$\lim_{h \to 0+} \frac{\big\| w(t+h) - S_h w(t) \big\|_{\mathbf{L}^1}}{h} = \sum_{\alpha=1}^{N} \frac{1}{h_0} \int_{-\hat{\lambda} h_0}^{\hat{\lambda} h_0} \Big| w\big(t + h_0,\ x_\alpha(t) + y\big) - \omega_\alpha(h_0,\ y) \Big| \, dy. \tag{1.9}$$

In other words, the instantaneous error rate in (1.9) is measured by the differences between the travelling fronts of w and the exact self-similar solutions of the corresponding Riemann problems. This provides an explicitly computable error bound for the wave-front tracking method [6]. By a more refined analysis, error estimates for the Glimm scheme were obtained in [13].

In Section 2 of this paper we consider some examples of metrics which are contractive w.r.t. the flow generated by an evolution equation. The interesting cases are those which extend the classical construction of a Riemann metric on a differential manifold. For hyperbolic systems of conservation laws, the construction of a "Riemannian" contractive metric on the positively invariant set $\mathcal{D} \subset \mathbf{BV}$ is outlined in Sections 3-4.

In the remaining Sections 5-8 we show that the evolution problem (1.1) can be reformulated as an abstract O.D.E. in a metric space, relying on the definition of *Viscosity Solution* introduced in [6]. In this setting, the uniqueness of the semigroup and the well posedness of the Cauchy problem can be obtained from general principles, relying only on the metric structure of the domain $\mathcal{D}$ in (1.4).

2 - Examples of contractive metrics.

By the remarks in the previous section, in order to establish the well posedness of the Cauchy problem (1.1)-(1.2), a key step is the construction of a Lipschitz semigroup of solutions. Toward this goal, one needs to construct a sequence of approximations $u_\nu = u_\nu(t, x)$, carefully controlling:

(1) The total variation Tot.Var.$\big(u_\nu(t, \cdot)\big)$ of each approximate solution.

(2) The distance $\big\| u_\nu(t, \cdot) - v_\nu(t, \cdot) \big\|_{\mathbf{L}^1}$ between any two approximate solutions.

For small **BV** initial data, the Glimm scheme [19, 26] and the wave-front tracking algorithms [2, 4, 16, 30] provide control of the total variation, in terms of a wave interaction potential. Unfortunately, none of the algorithms found in the standard literature yields estimates on the dependence on the initial data. In particular, all convergence proofs are based on a compactness argument, which does not provide informations on the uniqueness of the limit. New constructive algorithms must therefore be devised.

A useful technique for estimating the distance between solutions is the introduction of a new distance d_*, which is equivalent to the old distance and nonexpansive w.r.t. the flow generated by (1.1). In connection with hyperbolic systems, this approach was first introduced in [3], then extended in [8, 9] to more general systems. To familiarize the reader with this method, we collect here a few simple examples of dynamical systems

$$\frac{d}{dt} u(t) = \Phi\big(u(t)\big), \tag{2.1}$$

whose flow $(\bar{u}, t) \mapsto S_t \bar{u}$ is non-expansive for a suitable distance d_*.

A well known class of evolution problems generating a contractive semigroup are those of the form

$$0 \in \frac{du}{dt} + B(u), \tag{2.2}$$

where B is a *hyperaccretive* operator on a Banach space E (see Chapter 3 in [15] for details). Scalar conservation laws, even in several space dimensions, fit into this framework [14, 22]. In this case, the flow is contractive w.r.t. the standard distance $d(u, v) = \|u - v\|$. Below, we consider situations where the flow is contractive not for d, but only for some equivalent distance d_*

In some easy cases, a contractive distance can be defined explicitly.

Example 1. On the domain $\mathcal{D} \doteq]0, \infty[$, consider the differential equation

$$\dot{x} = cx. \tag{2.3}$$

Of course, the corresponding flow is $S_t x = e^{ct} x$. The Euclidean distance $|S_t u - S_t v|$ between two solutions thus increases in time. However, one easily checks that this flow is non-expansive w.r.t. the distance

$$d_*(x, y) \doteq \big| \log(x/y) \big|. \tag{2.4}$$

Example 2. Let ϕ be a smooth scalar function satisfying $C^{-1} \leq \phi(x) \leq C$ for some constant $C > 1$. Then the evolution equation

$$u_t + \phi(x) u_x = 0.$$

generates a linear semigroup on the space $\mathbf{L}^1$, which is non-expansive for the distance

$$d_*(u, v) \doteq \int \frac{\big|u(x) - v(x)\big|}{\phi(x)}\, dx.$$

By the assumptions on ϕ, the above d_* is uniformly equivalent to the standard $\mathbf{L}^1$ distance.

In other cases, the distance is defined in terms of a Riemann-type metric, taking the point of view of differential geometry. More precisely, let $\mathcal{D}$ be a closed domain in a Banach space E. The construction of a metric d_* on $\mathcal{D}$ will involve:

- A dense subset $\mathcal{D}' \subset \mathcal{D}$.
- At each point $u \in \mathcal{D}'$, a space T_u of tangent vectors, equipped with a weighted norm $\|\mathbf{v}\|_u$.
- A family $\mathcal{RP}$ of suitably regular paths $\gamma : [a, b] \mapsto \mathcal{D}'$.

We assume that the family $\mathcal{RP}$ is closed under concatenation of paths. Moreover, for each $\gamma \in \mathcal{RP}$, we require that the differential $\mathbf{v} = D\gamma = d\gamma(\theta)/d\theta$ is a well defined tangent vector in $T_{\gamma(\theta)}$, for almost every $\theta \in [a, b]$. The weighted length of a regular path γ can thus be defined as

$$\|\gamma\|_* \doteq \int_a^b \big\|D\gamma(\theta)\big\|_{\gamma(\theta)}\, d\theta. \tag{2.5}$$

Let now $u, w \in \mathcal{D}'$. Call $\Sigma_{u,w}$ the family of all paths $\gamma \in \mathcal{RP}$ joining u with w. By setting

$$d_*(u,w) \doteq \inf\big\{\|\gamma\|_*,\quad \gamma \in \Sigma_{u,w}\big\}, \tag{2.6}$$

one defines a distance on the dense subset $\mathcal{D}' \subset \mathcal{D}$. By continuity, d_* can then be extended to the whole domain $\mathcal{D}$.

At this stage, the relevant question is whether the metric d_* is non-expansive in connection with the flow of (2.1). Clearly, this is the case provided that, for every $\gamma \in \mathcal{RP}$, the weighted length of the path $S_t\gamma : \theta \mapsto S_t\big(\gamma(\theta)\big)$ is a non-increasing function of time. To check this condition, for each solution u of (2.1) we consider the corresponding linearized evolution equation for tangent vectors

$$\frac{d}{dt}\mathbf{v}(t) = D\Phi\big(u(t)\big)\cdot \mathbf{v}(t). \tag{2.7}$$

A natural assumption now is:

(A1) For every solution $u = u(t)$ of (2.1) taking values inside $\mathcal{D}'$, and every solution $\mathbf{v} = \mathbf{v}(t)$ of the variational equation (2.7), the weighted norm of the tangent vector

$$t \mapsto \big\|\mathbf{v}(t)\big\|_{u(t)} \tag{2.8}$$

is a non-increasing function of time.

Condition **(A1)** is "almost sufficient" for the contractivity of the flow of (2.1). A formal proof goes as follows. Let $\bar u, \bar w \in \mathcal{D}'$ and $\varepsilon > 0$ be given. Call $u(t) \doteq S_t\bar u$, $w(t) \doteq S_t\bar w$ the trajectories of (2.1) with initial data $\bar u, \bar w$, respectively. Choose a regular path $\gamma : [a,b] \mapsto \mathcal{D}'$ joining $\bar u$ with $\bar w$, such that

$$\|\gamma\|_* \le d_*(\bar u, \bar w) + \varepsilon. \tag{2.9}$$

Define

$$u^\theta(t) \doteq S_t\big(\gamma(\theta)\big), \qquad\qquad \mathbf{v}^\theta(t) \doteq \frac{\partial}{\partial\theta} S_t\big(\gamma(\theta)\big).$$

Each u^θ is thus a solution of (2.1), while $\mathbf{v}^\theta$ solves the corresponding linearized variational equation (2.7). Using **(A1)** we now compute

$$\begin{aligned} d_*\big(u(t),\ w(t)\big) &\le \|S_t\gamma\|_* \\ &= \int_a^b \big\|\mathbf{v}^\theta(t)\big\|_{u^\theta(t)}\, d\theta \\ &\le \int_a^b \big\|\mathbf{v}^\theta(0)\big\|_{u^\theta(0)}\, d\theta \\ &\le d_*(\bar u, \bar w) + \varepsilon. \end{aligned} \tag{2.10}$$

Since $\varepsilon > 0$ was arbitrary, this achieves the proof. In the previous argument, we tacitly assumed that the regularity of every path is preserved by the flow:

(A2) For every $\gamma \in \mathcal{RP}$ and every $t > 0$, the path $S_t\gamma$ still belongs to the family $\mathcal{RP}$ of regular paths.

In many cases involving finite dimensional evolution equations, the above assumption is trivially satisfied. In connection with the system of conservation laws (1.1), however, **(A2)** fails, due to the possible loss of regularity in piecewise Lipschitz solutions. This technical difficulty will be addressed in Section 4.

We conclude this section with two more examples of flows which are contractive w.r.t. a Riemann-type distance.

Example 3. For the evolution $\dot x = cx$ considered in Example 1, introduce the Riemann metric

$$\|\mathbf{v}\|_x \doteq \frac{1}{x}\cdot|\mathbf{v}|.$$

Along any solution $x = x(t)$ of (2.3), we now have

$$\dot{\mathbf{v}} = c\mathbf{v}, \qquad \frac{d}{dt}\|\mathbf{v}(t)\|_{x(t)} = 0.$$

It is thus clear that the corresponding distance

$$d_*(x,y) = \left|\int_x^y \frac{1}{s}\,ds\right| \qquad\qquad x,y>0,$$

is non-expansive for the flow of (2.3). Actually, we have just carried out an alternative construction of the same distance d_* in (2.4).

Example 4. On the space $E = \mathbf{L}^1\big([0,1];\mathbb{R}\big)\times\mathbf{L}^1\big([0,1];\mathbb{R}\big)$, consider the discontinuous evolution equation

$$\begin{cases} u_t(x) = \phi\Big(\mathrm{meas}\big\{y;\ v(y) < u(x)\big\}\Big),\\ v_t(y) = \psi\Big(\mathrm{meas}\big\{x;\ u(x) > v(y)\big\}\Big),\end{cases} \tag{2.11}$$

where ϕ,ψ are smooth functions satisfying

$$\phi(s) > 1,\quad \psi(s) < -1, \qquad \left|\frac{d\phi}{ds}\right| < \kappa,\quad \left|\frac{d\phi}{ds}\right| < \kappa \qquad\qquad \forall s\in[0,1],$$

for some constant $\kappa > 0$. The corresponding flow is then contractive w.r.t. the weighted distance d_* on E, defined in terms of the Riemannian metric

$$\big\|(\mathbf{w},\mathbf{z})\big\|_{(u,v)} \doteq \int_0^1 \big|\mathbf{w}(x)\big|\cdot\exp\Big(-\kappa\cdot\mathrm{meas}\big\{y;\ v(y)<u(x)\big\}\Big)dx$$
$$+\int_0^1 \big|\mathbf{z}(y)\big|\cdot\exp\Big(-\kappa\cdot\mathrm{meas}\big\{x;\ u(x)<v(y)\big\}\Big)dx.$$

It is interesting to observe that, if u,v solve (2.11), then the two functions

$$U(t,\xi)\doteq \mathrm{meas}\big\{x;\ u(t,x)>\xi\big\}, \qquad V(t,\xi)\doteq \mathrm{meas}\big\{x;\ v(t,y)<\xi\big\},$$

provide a solution to the linearly degenerate, strictly hyperbolic system

$$U_t + \phi(V)U_\xi = 0, \qquad V_t + \psi(U)V_\xi = 0.$$

Another example of a Riemann-type metric on $\mathbb{R}^n$, contractive for the flow of a discontinuous differential equation, can be found in [7].

3 - A contractive metric for hyperbolic systems.

In this section we outline the consruction of a distance, equivalent to the $\mathbf{L}^1$ metric, which is contractive w.r.t. the flow generated by a system of conservation laws.

Consider the domain $\mathcal{D}$ of functions with small total variation, defined as in (1.4). Let $\mathcal{D}_{PL}$ be the dense subset of Piecewise Lipschitz functions.

We recall below the definition of generalized differential of a path $\gamma : [a,b] \mapsto \mathbf{L}^1$, introduced in [14]. For any $u \in \mathbf{L}^1$, on the family Σ_u of all continuous paths $\gamma : [0,\theta_0] \mapsto \mathbf{L}^1$ such that $\gamma(0) = u$, consider the equivalence relation

$$\gamma \sim \gamma' \qquad \text{iff} \qquad \lim_{\theta\to 0+} \frac{1}{\theta}\big\|\gamma(\theta) - \gamma'(\theta)\big\|_{\mathbf{L}^1} = 0 \qquad \qquad \big(\gamma,\gamma' \in \Sigma_u\big). \tag{3.1}$$

Now assume that u is piecewise Lipschitz, say with jumps at the points $x_1 < \cdots < x_N$. The space of *generalized tangent vectors* at u is then defined as $T_u \doteq \mathbf{L}^1 \times \mathbb{R}^N$. To each $(v,\xi) \in T_u$, with $\xi = (\xi_1, \ldots, \xi_N)$, we associate the path $\gamma_{(v,\xi;u)} \in \Sigma_u$ defined by

$$\gamma_{(v,\xi;u)}(\theta) \doteq u + \theta v + \sum_{\xi_\alpha<0} \big[u(x_\alpha^+) - u(x_\alpha^-)\big]\,\chi_{[x_\alpha+\theta\xi_\alpha,\,x_\alpha]} - \sum_{\xi_\alpha>0} \big[u(x_\alpha^+) - u(x_\alpha^-)\big]\,\chi_{[x_\alpha,\,x_\alpha+\theta\xi_\alpha]}. \tag{3.2}$$

More generally, we say that a path $\gamma \in \Sigma_u$ *generates the generalized tangent vector* $(v,\xi) \in T_u$, if γ is equivalent to $\gamma_{(v,\xi;u)}$, under the relation (3.1).

In other words, for small values of θ, the function $u^\theta \doteq \gamma(\theta)$ can be obtained from u by adding θv and by shifting the positions of the jumps from x_α to $x_\alpha + \theta\xi_\alpha$. As $\theta \to 0+$, this procedure yields a first order approximation to u^θ, with an error $o(\theta)$ in the $\mathbf{L}^1$ norm, with $o(\theta)/\theta \to 0$ as $\theta \to 0$. In connection with the above differential structure, one can define a kind of continuous differentiability property for maps $\gamma : \theta \mapsto u^\theta \in \mathbf{L}^1$, with piecewise Lipschitz values. Following [9, 12], we say that a map $\gamma :]a,b[\mapsto \mathbf{L}^1$ is a *regular path* if there exists an integer N such that:

(i) Every function $u^\theta \doteq \gamma(\theta)$ is piecewise Lipschitz continuous with jumps at points $x_1^\theta < \cdots < x_N^\theta$ continuously depending on θ. Outside the jumps, each u^θ is continuous with a Lipschitz constant L independent of θ. All functions u^θ coincide outside some interval $[-M, M]$.

(ii) The map $\theta \mapsto u_x^\theta$ is continuous with values in $\mathbf{L}^1$.

(iii) There exists a continuous map $\theta \mapsto (v^\theta, \xi^\theta) \in \mathbf{L}^1 \times \mathbb{R}^N$ such that for every θ

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\big\|\gamma(\theta+\varepsilon) - \gamma_{(v^\theta,\xi^\theta;u^\theta)}(\varepsilon)\big\|_{\mathbf{L}^1} = 0. \tag{3.3}$$

More generally, we say that a continuous map $\gamma : [a,b] \mapsto \mathbf{L}^1$ is a *piecewise regular path* if there exist points $a = \theta_0 < \theta_1 < \cdots < \theta_\nu = b$ such that the restriction of γ to each open subinterval $]\theta_{j-1}, \theta_j[$ is a regular path.

Now consider any $u \in \mathcal{D}_{PL}$, say with jumps at $x_1 < \cdots < x_N$. For simplicity, assume that for every $\alpha = 1, \ldots, N$, the jump at x_α determines a single shock in the k_α-th family, of strength

σ_α. Given a generalized tangent vector $(v,\xi)\in T_u=\mathbf{L}^1\times I\!R^N$, recalling (2.16) we then define its weighted norm as

$$\big\|(v,\xi)\big\|_u \doteq \sum_{\alpha=1}^N |\sigma_\alpha\xi_\alpha| W^u_{k_\alpha}(x_\alpha)+\sum_{i=1}^n\int_{-\infty}^\infty \big|v_i(x)\big|W_i^u(x)\,dx, \tag{3.4}$$

Here $v_i(x)\doteq l_i\big(u(x)\big)\cdot v(x)$ is the i-th component of v. The weight functions W_i^u are defined by

$$W_i^u(x)\doteq 1+\kappa_1 R_i^u(x)+\kappa_1\kappa_2 Q(u), \tag{3.5}$$

$$R_i^u(x)\doteq\left[\sum_{j\le i}\int_x^\infty+\sum_{j\ge i}\int_{-\infty}^x\right]|u_x^j(y)|dy+\left[\sum_{\substack{k_\alpha\le i\\ x_\alpha>x}}+\sum_{\substack{k_\alpha\ge i\\ x_\alpha<x}}\right]\cdot|\sigma_\alpha|,$$

$$Q(u)\doteq\sum_{i\le j}\iint_{x<y}\big|u_x^j(x)\big|\big|u_x^i(y)\big|\;dxdy+\sum_{k_\alpha\le k_\beta,\,x_\alpha>x_\beta}|\sigma_\alpha\sigma_\beta| \tag{3.6}$$

$$+\sum_\alpha|\sigma_\alpha|\left[\sum_{i\le k_\alpha}\int_{x_\alpha}^\infty\big|u_x^i(x)\big|dx+\sum_{i\ge k_\alpha}\int_{-\infty}^{x_\alpha}\big|u_x^i(x)\big|dx\right],$$

while κ_1,κ_2 are suitably large constants. Intuitively, $R_i^u(x)$ can be regarded as the total strength of all waves in u which approach an infinitesimal i-shock located at x. Finally, let $\gamma:\theta\mapsto u^\theta$ be a piecewise regular path defined on $[a,b]$, and let (v^θ,ξ^θ) be its generalized tangent vector at u^θ. In analogy with (2.5), we define the *weighted length* of γ as

$$\|\gamma\|_*\doteq\int_a^b\big\|(v^\theta,\xi^\theta)\big\|_{u^\theta}d\theta. \tag{3.7}$$

The formula (2.6) now defines a weighted distance d_* on $\mathcal{D}_{PL}$. Due to the choice of the weights in (3.5), it is not difficult to check that d_* is uniformly equivalent to the standard $\mathbf{L}^1$ distance. By continuity, it can thus be extended to the whole domain $\mathcal{D}$.

4 - Construction of the semigroup.

In the previous section we defined a weighted distance d_* on the domain $\mathcal{D}$ in (2.4), uniformly equivalent to the $\mathbf{L}^1$ distance. At this stage, we are still a long way from proving that d_* is contractive w.r.t. a semigroup generated by the system (1.1). Indeed, the very existence of the semigroup still needs to be established.

The main clue that d_* may be contractive for the flow generated by the conservation laws is the fact that **(A1)** holds. More precisely, let $u=u(t,x)$ be a weak solution of (1.1), with $u(t,\cdot)\in\mathcal{D}_{PL}$ for every t. Assume that all shocks in u are entropy-admissible and interect at most two at a time. Then, in analogy with (2.7), the generalized tangent vectors $\big(v(t,\cdot),\ \xi(t)\big)\in T_{u(t)}$ satisfy a linearized evolution equation, say

$$\frac{d}{dt}(v,\xi)=\Lambda\big(u(t)\big)\cdot(v,\xi). \tag{4.1}$$

The precise form of Λ and the restarting conditions at times of shock interactions were derived in [9].

In this setting, for a suitable choice of the constants δ_0, κ_i in (1.4) and (3.5), the main result in [5] states that the map

$$t \mapsto \left\| \big(v(t), \xi(t)\big) \right\|_{u(t)} \tag{4.2}$$

is a non-increasing function of time, also in the presence of shock interactions.

An immediate consequence is the following. Let $\gamma : \theta \mapsto \bar{u}^\theta$ be a regular path of initial data. Assume that, for each $\theta \in [a, b]$, the solution u^θ of the corresponding Cauchy problem remains regular (i.e., piecewise Lipschitz). Then the weighted length of the path $S_t\gamma : \theta \mapsto u^\theta(t)$ is a non-increasing function of time.

Exploiting the above results, we now try to construct a semigroup generated by (1.1). A naive attempt goes as follows. Start with a dense family of sufficiently regular solutions, say piecewise Lipschitz. For any two solutions u, w, and $\varepsilon > 0$, consider a regular path γ_0 joining $u(0)$ with $w(0)$, such that

$$\|\gamma_0\|_* \leq d_*\big(u(0),\ w(0)\big) + \varepsilon.$$

Let $t \mapsto u^\theta(t)$ be the solution of (1.1) with initial data $u^\theta(0) = \gamma_0(\theta)$. For any $\tau > 0$, consider the path $\gamma_\tau : \theta \mapsto u^\theta(\tau)$. We then have

$$d_*\big(u(\tau),\ w(\tau)\big) \leq \|\gamma_\tau\|_* \leq \|\gamma_0\|_* \leq d_*\big(u(0),\ w(0)\big). \tag{4.3}$$

Since ε is arbitrary, this shows that the flow of regular solutions is contractive w.r.t. the weighted distance d_*. This flow can thus be extended by continuity to a unique semigroup S defined on the whole domain $\mathcal{D}$.

This approach was carried out in [3], in connection with systems whose characteristic fields are linearly degenerate. In the general case, a major technical difficulty arises. Indeed, piecewise Lipschitz initial data may lose their regularity in two ways:

- The number of jumps becomes infinite in finite time, due to repeated shock interactions.

- The Lipschitz constant becomes infinite, due to genuine nonlinearity.

If any one of the solutions u^θ loses its regularity at some time $t > 0$, then for all $\tau > t$ the length of γ_τ can no longer be computed by (2.5), and the whole estimate breaks down. Still, the construction of the semigroup can be accomplished by means of two techniques.

1. The system of conservation laws (1.1) is approximated by an auxiliary evolution equation, not necessarily in conservation form. In particular, for the new system, shock and rarefaction curves will locally coincide.

2. Given a family of solutions u^θ, at the first time τ_1 when one or more of these functions loses its regularity, a restarting procedure is performed. The path $\gamma_{\tau_1} : \theta \mapsto u^\theta(\tau_1)$ is thus replaced by a new regular path γ_{τ_1+}, close to the old one and with almost the same length. The corresponding solutions will remain regular up to a time $\tau_2 > \tau_1$ when a second restarting is performed, etc... In a finite number of steps, a path of approximate solutions is constructed on any given interval $[0, T]$. The basic estimate (4.3) can still be recovered.

The first approximation technique was introduced in [8]. Alone, it suffices to handle the case of 2×2 systems. The restarting procedure was first used in the paper [9], to which we refer for all details.

Remarks. When B is a hyperaccretive operator [15], the contractive semigroup generated by (2.2) can be obtained as limit of the approximating flows

$$\frac{du}{dt} + B_\lambda(u) = 0.$$

For each $\lambda > 0$, the Lipschitz continuous operator B_λ is here defined as

$$B_\lambda \doteq \lambda^{-1}(I - R_\lambda), \qquad R_\lambda \doteq (I + \lambda B)^{-1}.$$

On the other hand, for semigroups which are contractive w.r.t. a Riemann-type distance d_*, no general approximation procedure is known.

An alternative attempt to construct the semigroup generated by a system of conservation laws is to extend the formula (3.7). Namely, one can consider more general paths $\gamma : \theta \mapsto u^\theta$, with each u^θ not necessarily piecewise Lipschitz. A first step in this direction was taken in [10] by introducing a notion of *shift differential*, for paths taking values in the wider class of **BV** functions.

5 - Evolution equations in metric spaces.

Let E be a metric space with distance $d(\cdot,\cdot)$. To define an evolution equation on E, one can adopt the approach of differential geometry. At each point $u \in E$, a tangent space T_u is defined, consisting of equivalence classes of paths γ originating from u. A generalized differential equation on E is then obtained by assigning a function $u \mapsto \mathbf{v}(u)$, mapping each point u into some tangent vector $\mathbf{v}(u) \in T_u$.

More precisely, the construction goes as follows. Fix any $u \in E$ and consider the set Σ_u of all continuous paths $\gamma : [0, \bar\theta] \mapsto E$, for some $\bar\theta > 0$, with $\gamma(0) = u$. On this set, consider the equivalence relation

$$\gamma \sim \gamma' \qquad \text{iff} \qquad \lim_{\theta \to 0+} \frac{d\big(\gamma(\theta),\ \gamma'(\theta)\big)}{\theta} = 0 \qquad \big(\gamma, \gamma' \in \Sigma_u\big). \tag{5.1}$$

We shall denote by T_u the set of all these equivalence classes.

A map $\mathbf{v} : E \mapsto \cup_{u \in E} T_u$ such that

$$\mathbf{v}(u) \in T_u \qquad \text{for all } \ u \in E \tag{5.2}$$

will be called a *generalized vector field* on E. We say that a Lipschitz continuous function $u : [0, T] \mapsto E$ is a *solution of the quasidifferential equation*

$$\frac{du}{dt} = \mathbf{v}(u) \tag{5.3}$$

if, for each $t \in [0, T[$, the path

$$\theta \mapsto \gamma(\theta) \doteq u(t + \theta)$$

lies in the equivalence class $\mathbf{v}\big(u(t)\big) \in T_{u(t)}$. Together with (5.3) we shall often consider an initial condition

$$u(0) = \bar u. \tag{5.4}$$

Example 5. If E is a Banach space and $f : E \mapsto E$ is a bounded vector field on E, the differential equation $du/dt = f(u)$ can be recast in the form (5.3) by taking $\mathbf{v}(u)$ as the equivalence class of the path $\gamma(\theta) \doteq u + \theta f(u)$.

The above construction is essentially the same as in [29]. For a different approach to evolution equations on metric spaces, see [1]. In general, without suitable regularity assumptions on the generalized vector field $\mathbf{v}$, both the existence and the uniqueness of solutions to the Cauchy problem (5.3)-(5.4) may fail. The question of existence will not be addressed in the present paper. Our aim here is to show that, if a Lipschitz semigroup of solutions does exists, then a general uniqueness result can be proved.

6 - Lipschitz flows and uniqueness.

In the following, we assume that there exists a flow $S : E\times[0,\infty[\mapsto E$ and a Lipschitz constant L with the following properties:

(S1) $S_0\bar u = \bar u, \qquad S_sS_t\bar u = S_{s+t}\bar u,$

(S2) $d\big(S_t\bar u,\ S_s\bar v\big) \leq L\cdot\big(|t-s| + d(\bar u,\bar v)\big),$

(S3) every trajectory $t \mapsto S_t\bar u$ provides a solution to the generalized Cauchy problem (5.3)-(5.4).

Under these assumptions, all Cauchy problems have a unique solution, coinciding with the corresponding trajectory of the semigroup. To prove this claim, we first establish an error estimate extending (1.4) to metric spaces.

Lemma 2. *Let E be a metric space and let $S : E\times[0,\infty[\mapsto E$ be a continuous flow satisfying the properties* **(S1)-(S2)**. *Then for every Lipschitz continuous map $w : [0,T]\mapsto E$ one has*

$$d\big(w(T),\ S_T w(0)\big) \leq L\cdot\int_0^T \left\{\liminf_{h\to 0+}\frac{d\big(w(t+h),\ S_h w(t)\big)}{h}\right\}\,dt. \tag{6.1}$$

Proof. As a preliminary, observe that the integrand

$$\phi(t) \doteq \liminf_{h\to 0+}\frac{d\big(w(t+h),\ S_h w(t)\big)}{h}$$

in (6.1) is measurable. Indeed, for every $h > 0$, the function $f_h(t) \doteq d\big(w(t+h),\ S_h w(t)\big)/h$ is continuous. By continuity of the maps $h \mapsto f_h(t)$, we have

$$\phi(t) = \lim_{\varepsilon\to 0+}\ \inf_{h\in\mathbf{Q}\cap]0,\varepsilon]} f_h(t),$$

where the infimum is taken only over rational values of h. Hence ϕ is Borel measurable.

Next, let $\varepsilon > 0$ be given. To fix the ideas, let $M > L$ be a Lipschitz constant for w, so that

$$d\big(w(t),\ w(s)\big) \leq M|t-s|, \qquad\qquad t,s\in[0,T]. \tag{6.2}$$

By Lusin's theorem, there exists a compact set $J \subset [0,T]$ and a continuous function ψ such that

$$\mathrm{meas}(J) > T-\varepsilon, \qquad \int_0^T \big|\psi(s)-\phi(s)\big|\, ds < \varepsilon, \qquad \psi(x)=\phi(x) \ \text{ for all } \ x\in J. \tag{6.3}$$

For $t\in[0,T]$, define

$$\Psi_\varepsilon(t) \doteq L\cdot \int_0^t \big[\varepsilon+\psi(s)\big]\, ds + 2ML\cdot \mathrm{meas}\big([0,t]\setminus J\big), \tag{6.4}$$

$$\tau \doteq \sup\Big\{t\in[0,T];\quad d\big(S_{T-t}w(t),\ S_T w(0)\big)\le \Psi_\varepsilon(t)\Big\}.$$

We claim that $\tau=T$. Indeed, assume the contrary. By continuity, at $t=\tau$ we have

$$d\big(S_{T-\tau}w(\tau),\ S_T w(0)\big) = \Psi_\varepsilon(\tau). \tag{6.5}$$

If $\tau\notin J$, since J is closed we can choose $h>0$ such that $[\tau,\tau+h]\cap J=\emptyset$. By (6.5) and (6.2) this implies

$$\begin{aligned} d\big(S_{T-\tau-h}w(\tau+h),\ S_T w(0)\big) &\le d\big(S_{T-\tau}w(\tau),\ S_T w(0)\big) + L\cdot d\big(w(\tau+h),\ S_h w(\tau)\big)\\ &\le \Psi_\varepsilon(\tau)+L\cdot 2Mh\\ &\le \Psi_\varepsilon(\tau+h). \end{aligned}$$

In the case $\tau\in J$, by (6.3) and the continuity of ψ, there exists $h>0$ such that

$$\frac{d\big(w(\tau+h),\ S_h w(\tau)\big)}{h} \le \psi(\tau)+\frac{\varepsilon}{2} \le \min_{s\in[\tau,\tau+h]}\psi(s)+\varepsilon.$$

By (6.5), this yields

$$\begin{aligned} d\big(S_{T-\tau-h}w(\tau+h),\ S_T w(0)\big) &\le d\big(S_{T-\tau}w(\tau),\ S_T w(0)\big) + L\cdot d\big(w(\tau+h),\ S_h w(\tau)\big)\\ &\le \Psi_\varepsilon(\tau)+L\cdot\int_\tau^{\tau+h}\big[\psi(s)+\varepsilon\big]ds\\ &\le \Psi_\varepsilon(\tau+h), \end{aligned}$$

again in contradiction with the maximality of τ. Therefore, $\tau=T$. Recalling (6.3) and the definition (6.4) of Ψ_ε, we deduce

$$\begin{aligned} d\big(w(T),\ S_T w(0)\big) &\le \Psi_\varepsilon(T)\\ &= \int_0^T \psi(t)dt + L\varepsilon T + 2ML\cdot \mathrm{meas}\big([0,T]\setminus J\big)\\ &\le L\cdot\int_0^T \phi(t)dt + L(\varepsilon+\varepsilon T)+2ML\varepsilon. \end{aligned}$$

Since $\varepsilon>0$ is arbitrary, the lemma is proved.

Corollary 1. *In connection with the evolution problem (5.3), assume that a Lipschitz semigroup* $S : E \times [0, \infty[\mapsto E$ *exists, satisfying the properties* **(S1)-(S3)**. *Then, for every initial data* $\bar u \in E$, *the corresponding Cauchy problem (5.3)-(5.4) has the unique solution* $u(t) = S_t \bar u$.

Indeed, let $w : [0, \tau] \mapsto E$ be any solution of the Cauchy problem (5.3)-(5.4). By definition of solution, the integrand in (6.1) is identically zero. Hence $d\big(w(t),\ S_t\bar u\big) = 0$ for all $t \geq 0$.

Example 6. On the real line, define the semigroup $S : I\!\!R \times [0, \infty[\mapsto I\!\!R$ as follows. For each $\bar x$, the trajectory $t \mapsto S_t\bar x$ is the unique strictly increasing solution of the differential equation $\dot x = \sqrt{|x|}$ with initial data $x(0) = \bar x$. Then S is continuous but not Lipschitz. The constant function $w(t) \equiv 0$ satisfies

$$\lim_{h\to 0+} \frac{d\big(w(t+h),\ S_h w(t)\big)}{h} = 0 \qquad\qquad t \geq 0$$

but $0 = w(t) \neq S_t w(0) = t^2/4$ for all $t > 0$.

Example 7. Let S be a Lipschitz semigroup on a metric space E, satisfying **(S1)-(S2)**. For each $u \in E$, define the generalized tangent vector $\mathbf{v}(u) \in T_u$ as the equivalence class of the map $\theta \mapsto S_\theta u$. Then, by Lemma 2, for each $\bar u \in E$, the only solution to the Cauchy problem (5.3)-(5.4) is the trajectory $t \mapsto S_t\bar u$.

7 - Construction of tangent vectors.

In the standard theory of conservation laws, the equation (1.1) is regarded as a partial differential equation. Its solutions are functions of two variables $u = u(t, x)$ such that the vector $\big(u,\ f(u)\big)$ has zero divergence, i.e.:

$$\iint \phi_t u + \phi_x f(u)\ dxdt = 0 \qquad\qquad \text{for all} \ \ \phi \in \mathcal{C}^1_c. \tag{7.1}$$

We want to show here that the equation (1.1), together with the standard entropy conditions [25], can be reformulated as a quasidifferential equation on a metric space of **BV** functions with the usual $\mathbf{L}^1$ distance. For each $u \in \mathbf{BV}$, our task is thus to define a suitable tangent vector $\mathbf{v}(u) \in T_u$ such that the entropy-weak solutions of (1.1) coincide with the solutions of the evolution equation (5.3). Observe that, if the distributional derivative $f(u)_x$ is in $\mathbf{L}^1$, one could simply define $\mathbf{v}(u)$ as the equivalence class of the map $\theta \mapsto u + \theta \cdot f(u)_x$. If u is discontinuous, however, the tangent vector $\mathbf{v}(u)$ requires a more careful definition.

In this section, given $\bar u \in \mathbf{L}^1(I\!\!R;\ I\!\!R^n)$, we outline a general procedure for defining a tangent vector $\mathbf{v} \in T_{\bar u}$, in terms of local approximations. This construction is strongly motivated by the definition of Viscosity Solutions introduced in [6].

Let a constant $\hat\lambda > 0$ be given, together with a positive, finite Radon measure μ on $I\!\!R$. For each open interval $I \doteq]a, b[$ and $\theta \in \big[0,\ (b-a)/2\big[$, define

$$I_\theta \doteq]a + \hat\lambda\theta,\ b - \hat\lambda\theta[\,. \tag{7.2}$$

Lemma 3. *Assume that, for each* $\xi \in I\!\!R$, *two continuous functions* $U^\sharp_\xi, U^\flat_\xi : [0,\ \infty[\mapsto \mathbf{L}^1$ *are given, with* $U^\sharp_\xi(0) = U^\flat_\xi(0) = \bar u$. *Let the following compatibility conditions hold:*

$$\frac{1}{\theta}\int_{I_\theta\cap I'_\theta} \Big| U^\sharp_\xi(\theta, x) - U^\sharp_{\xi'}(\theta, x)\Big|\ dx \leq \mu\big(I \cap I'\big), \tag{7.3}$$

$$\frac{1}{\theta}\int_{I_\theta\cap I'_\theta}\Big|U^\sharp_\xi(\theta,x)-U^\flat_{\xi'}(\theta,x)\Big|\,dx\le\mu\big(I\cap I'\setminus\{\xi\}\big)+\mu\big(I'\cap\{\xi\}\big)\cdot\mu(I'),\tag{7.4}$$

$$\frac{1}{\theta}\int_{I_\theta\cap I'_\theta}\Big|U^\flat_\xi(\theta,x)-U^\flat_{\xi'}(\theta,x)\Big|\,dx\le\mu(I\cap I')\cdot\mu(I\cup I'),\tag{7.5}$$

valid for all intervals I,I' and points $\xi\in I$, $\xi'\in I'$. Then there exists a continuous map $\theta\mapsto u^\theta$ with values in $\mathbf{L}^1$ such that

$$\limsup_{\theta\to0+}\frac{1}{\theta}\int_{I_\theta}\Big|U^\sharp_\xi(\theta,x)-u^\theta(x)\Big|\,dx\le3\mu\big(I\setminus\{\xi\}\big),\tag{7.6}$$

$$\limsup_{\theta\to0+}\frac{1}{\theta}\int_{I_\theta}\Big|U^\flat_\xi(\theta,x)-u^\theta(x)\Big|\,dx\le2\mu^2(I),\tag{7.7}$$

for each open interval I and every point $\xi\in I$. Moreover, if $\theta\mapsto w^\theta\in\mathbf{L}^1$ is another map satisfying the corresponding estimates (7.6)-(7.7), then

$$\lim_{\theta\to0+}\frac{\|u^\theta-w^\theta\|_{\mathbf{L}^1}}{\theta}=0.\tag{7.8}$$

According to (4.8), the conditions (7.6)-(7.7) thus determine a unique equivalence class $\mathbf{v}\in T_u$. The proof of Lemma 3 is given in several steps.

1. For convenience, we first introduce a definition. Given $\varepsilon>0$, by an ε-*covering* of the real line we mean a family

$$\mathcal{F}\doteq\big\{I_1,\ldots,I_N,\ I'_1,\ldots,I'_M\big\}\tag{7.9}$$

of open intervals which cover $I\!R$ such that:

(i) The intervals I_α, $\alpha=1,\ldots,N$ are mutually disjoint; moreover, no point $x\in I\!R$ lies inside more than two distinct intervals I'_β.

(ii) For every α, there exists $\xi_\alpha\in I_\alpha$ such that $\mu\big(I_\alpha\setminus\{\xi_\alpha\}\big)<\varepsilon/N$.

(iii) For every β, $\mu(I'_\beta)<\varepsilon$.

The existence of an ε-covering is proved as follows. Let $\varepsilon>0$ be given and let $\mathcal{A}\doteq\{\xi_1,\ldots,\xi_N\}$ be the set of all points $\xi\in I\!R$ such that $\mu\big(\{\xi\}\big)\ge\varepsilon$. For each such point ξ_α, choose an open interval I_α such that $\xi_\alpha\in I_\alpha$ and $\mu\big(I_\alpha\setminus\{\xi_\alpha\}\big)<\varepsilon/N$. By possibly shrinking their size, we can assume that the intervals I_α are mutually disjoint. The construction is then completed by covering the closed set $K\doteq I\!R\setminus\cup I_\alpha$ with finitely many open intervals I'_β satisfying (i) and (iii).

2. To construct the function $\theta\mapsto u^\theta$, fix a sequence ε_n strictly decreasing to zero. For each n, let $\{I_1,\ldots,I_{N(n)},I'_1,\ldots,I'_{M(n)}\}$ be an ε_n-covering of $I\!R$. Recalling (7.2), observe that the family $\{I_{1,\theta},\ldots,I_{N(n),\theta},I'_{1,\theta},\ldots,I'_{M(n),\theta}\}$ is still a covering of $I\!R$ when $\theta\in[0,\theta_n]$ for some $\theta_n>0$ sufficiently small. We can assume that the sequence θ_n strictly decreases to zero. For a fixed n, we choose points $\xi_\alpha\in I_\alpha$ according to (ii), then choose arbitrary points $\xi'_\beta\in I'_\beta$ and define

$$u^\theta_n(x)\doteq\begin{cases}U^\sharp_{\xi_\alpha}(\theta,x) & \text{if}\quad x\in I_{\alpha,\theta}\text{ for some }\alpha,\\[2mm] U^\flat_{\xi'_\beta}(\theta,x) & \text{if}\ \ x\in I'_{\beta,\theta},\quad x\notin\bigcup_\alpha I_{\alpha,\theta},\ \ x\notin\bigcup_{\beta'<\beta}I'_{\beta',\theta}.\end{cases}\tag{7.10}$$

Observe that u_n^θ is well defined for $\theta \in [0, \theta_n]$. The function u is now obtained by convex interpolation:

$$u^\theta \doteq s u^{\theta_n} + (1-s) u^{\theta_{n-1}} \qquad \text{if} \quad \theta = s\theta_{n+1} + (1-s)\theta_n, \quad s \in [0,1]. \tag{7.11}$$

3. We claim that the map u^θ defined by (7.11) satisfies (7.6)-(7.7). Indeed, let any interval I and any point $\xi \in I$ be given. By the construction of u_n^θ, using (7.3)-(7.4) and the properties of an ε_n-covering we obtain

$$\begin{aligned}
&\frac{1}{\theta}\int_{I_\theta}\Big|U_\xi^\sharp(\theta,x) - u_n^\theta(x)\Big|\,dx \\
&\quad \le \frac{1}{\theta}\left[\sum_{\alpha=1}^{N(n)}\int_{I_\theta \cap I_{\alpha,\theta}} + \sum_{\beta=1}^{M(n)}\int_{I_\theta\cap I'_{\beta,\theta}}\right]\Big|U_\xi^\sharp(\theta,x) - u_n^\theta(x)\Big|\,dx \\
&\quad \le \sum_{\xi_\alpha \neq \xi}\mu(I\cap I_\alpha) + \sum_\beta \mu\big(I\cap I'_\beta \setminus \{\xi\}\big) + \sum_\beta \mu\big(I'_\beta \cap \{\xi\}\big)\cdot\mu(I'_\beta) \\
&\quad \le \mu\big(I\setminus\{\xi\}\big) + 2\mu\big(I\setminus\{\xi\}\big) + 2\varepsilon_n^2.
\end{aligned} \tag{7.12}$$

Indeed, there can be at mosts two distinct intervals I'_β containing the point ξ, and both of these have measure $< \varepsilon_n$. Since $\varepsilon_n \to 0$ as $n \to \infty$, from (7.12) we deduce (7.6).

The proof of (7.7) is similar. Using (7.4)-(7.5) and the properties of an ε_n-covering we now obtain

$$\begin{aligned}
&\frac{1}{\theta}\int_{I_\theta}\Big|U_\xi^\flat(\theta,x) - u_n^\theta(x)\Big|\,dx \\
&\quad \le \frac{1}{\theta}\left[\sum_{\alpha=1}^{N(n)}\int_{I_\theta \cap I_{\alpha,\theta}} + \sum_{\beta=1}^{M(n)}\int_{I_\theta\cap I'_{\beta,\theta}}\right]\Big|U_\xi^\flat(\theta,x) - u_n^\theta(x)\Big|\,dx \\
&\quad \le \sum_\alpha \mu\big(I_\alpha \cap I\setminus\{\xi_\alpha\}\big) + \sum_{\xi_\alpha\in I}\mu\big(\{\xi_\alpha\}\big)\cdot\mu(I) + \sum_\beta \mu(I\cup I'_\beta)\cdot\mu(I\cap I'_\beta) \\
&\quad \le \varepsilon_n + \sum_\alpha \mu\big(\{\xi_\alpha\}\big)\cdot\mu(I) + \big(\mu(I)+\varepsilon_n\big)\sum_\beta \mu(I\cap I'_\beta) \\
&\quad \le \varepsilon_n + 2\varepsilon_n\mu(I) + 2\mu^2(I)
\end{aligned} \tag{7.13}$$

for all $\theta \in [0, \theta_n]$. We used here the relations

$$\sum_{\alpha=1}^{N(n)}\mu\big(I_\alpha\setminus\{\xi_\alpha\}\big) \le \varepsilon_n, \qquad \sum_\beta \mu(I\cap I'_\beta) + \sum_\alpha \mu\big(I\cap\{\xi_\alpha\}\big) \le 2\mu(I).$$

Letting $n \to \infty$, from (7.13) it thus follows (7.7).

4. To prove (7.8), fix any $\varepsilon > 0$ and consider an ε-covering $\mathcal{F}$ as in (7.9). For $\theta > 0$ sufficiently small, the intervals $I_{\alpha,\theta}$ and $I'_{\beta,\theta}$ defined as in (7.2) still form a covering of $I\!R$. Using (7.6)-(7.7), then the properties (i)–(iii) of the ε-covering, we thus obtain the estimate

$$\begin{aligned}\limsup_{\theta\to0+}\frac{\|u^\theta-w^\theta\|_{L^1}}{\theta}&\le\limsup_{\theta\to0+}\sum_{\alpha=1}^N\frac{1}{\theta}\int_{I_{\alpha,\theta}}\Big\{\big|U^\sharp_{\xi_\alpha}(\theta,x)-u^\theta(x)\big|+\big|U^\sharp_{\xi_\alpha}(\theta,x)-w^\theta(x)\big|\Big\}\,dx\\&\quad+\limsup_{\theta\to0+}\sum_\beta\frac{1}{\theta}\int_{I'_{\beta,\theta}}\Big\{\big|U^\flat_{\xi'_\beta}(\theta,x)-u^\theta(x)\big|+\big|U^\flat_{\xi'_\beta}(\theta,x)-w^\theta(x)\big|\Big\}\,dx\\&\le\sum_{\alpha=1}^N6\cdot\mu\big(I_{\alpha,\theta}\setminus\{\xi_\alpha\}\big)+\sum_\beta4\mu^2(I'_\beta)\\&\le6N\cdot\frac{\varepsilon}{N}+4\varepsilon\cdot\sum_\beta\mu(I'_\beta)\\&\le6\varepsilon+4\varepsilon\cdot2\mu(I\!R).\end{aligned}\tag{7.14}$$

Since $\varepsilon>0$ was arbitrary, this establishes (7.8).

5 - Application to hyperbolic systems.

We now indicate how the abstract results of the previous section can be applied to systems of conservation laws. As usual, we assume that the system is strictly hyperbolic, and that each characteristic field is either genuinely nonlinear or linearly degenerate. Consider the domain $\mathcal{D}$ defined at (1.4), regarded as a metric space with the usual $\mathbf{L}^1$ distance. For each $u\in\mathcal{D}$, we will define a generalized tangent vector $\mathbf{v}(u)\in T_u$, using Lemma 3 in connection with the functions $U^\sharp, U^\flat$ defined as follows.

Fix a constant $\hat\lambda>0$, strictly bigger than all characteristic speeds. Let $\xi\in I\!R$ be given. Let ω be the self-similar solution [24, 34] of the Riemann problem

$$\omega_\theta+f(\omega)_x=0,\qquad\omega(0,x)=\begin{cases}u(\xi+)&\text{if}\quad x>0,\\u(\xi-)&\text{if}\quad x<0.\end{cases}\tag{8.1}$$

Define the function $U^\sharp_\xi$ by setting

$$U^\sharp_\xi(\theta,x)\doteq\begin{cases}\omega(\theta,\ x-\xi)&\text{if}\quad|x-\xi|\le\hat\lambda\theta,\\u(x)&\text{if}\quad|x-\xi|>\hat\lambda\theta.\end{cases}\tag{8.2}$$

Moreover, call $\widetilde A\doteq A\big(u(\xi)\big)$ and define $U^\flat_\xi$ as the solution to the linear hyperbolic problem with constant coefficients

$$w_\theta+\widetilde Aw_x=0,\qquad\qquad w(0,x)=u(x).\tag{8.3}$$

As measure μ we take a multiple of the measure of total variation of u:

$$\mu(I)\doteq C\cdot\text{Tot.Var.}\{u;\,I\}$$

for every open interval I, where C is a suitably large constant. The estimates (7.3)–(7.5) then clearly hold. For example, if $I\cap I'\ne\emptyset$, then $\big|u(\xi)-u(\xi')\big|\le\text{Tot.Var.}\{u;\,I\cup I'\}$, hence

$$\begin{aligned}\frac{1}{\theta}\int_{I_\theta\cap I'_\theta}\Big|U^\flat_\xi(\theta,x)-U^\flat_{\xi'}(\theta,x)\Big|\,dx&\le C'\cdot\big\|A(u(\xi))-A(u(\xi'))\big\|\cdot\text{Tot.Var.}\{u;\,I\cap I'\}\\&\le C\cdot\text{Tot.Var.}\{u;\,I\cup I'\}\cdot\text{Tot.Var.}\{u;\,I\cap I'\}.\end{aligned}$$

Therefore, by Lemma 3, there exists a unique tangent vector $\mathbf{v}(u)$ characterized by (7.6)-(7.7). The results in [3, 8, 9] yield the existence of a Lipschitz continuous semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ whose trajectories are entropy-weak solutions of (1.1) or, equivalently, solutions of the quasidifferential equation (5.3). Their uniqueness can now be obtained by an application of Lemma 2.

We remark that a major advantage of the formulation (5.3) based on Lemma 3, compared with (1.1), is that it does not require the equations to be in conservation form. For example, one can define a different (nonconservative) evolution equation as follows: in the construction of $U^\sharp$, replace the standard solution ω of the Riemann problem (8.1) with an approximation $\tilde{\omega}$, satisfying

$$\frac{1}{\theta} \int \left| \tilde{\omega}(\theta, x) - \omega(\theta, x) \right| \, dx = O\big(|u^+ - u^-|^2\big).$$

This procedure was used in [8, 9] to define a class of *Viscosity ε-solutions*, whose continuous dependence on the initial data is somewhat easier to establish.

References

[1] J. P. Aubin, Mutational equations in metric spaces, *Set Valued Analysis* **1** (1993), 3-46.

[2] P. Baiti and H. K. Jenssen, On the front tracking algorithm, Preprint S.I.S.S.A., Trieste 1996.

[3] A. Bressan, Contractive metrics for nonlinear hyperbolic systems, *Indiana Univ. Math. J.* **37** (1988), 409-421.

[4] A. Bressan, Global solutions of systems of conservation laws by wave-front tracking, *J. Math. Anal. Appl.* **170** (1992), 414-432.

[5] A. Bressan, A locally contractive metric for systems of conservation laws, *Ann. Scuola Norm. Sup. Pisa* **IV-22** (1995), 109-135.

[6] A. Bressan, The unique limit of the Glimm scheme, *Arch. Rational Mech. Anal.* **130** (1995), 205-230.

[7] A. Bressan and G. Colombo, Existence and continuous dependence for discontinuous O.D.E.'s, *Boll. Un. Matem. Italiana* **4-B** (1990), 295-311.

[8] A. Bressan and R. M. Colombo, The semigroup generated by 2×2 systems of conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), 1-75.

[9] A. Bressan, G. Crasta and B. Piccoli, Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws, preprint S.I.S.S.A., Trieste, 1996.

[10] A. Bressan and G. Guerra, Shift-differentiability of the flow generated by a conservation law, *Discrete and Continuous Dynamical Systems* **3** (1997), 35-58.

[11] A. Bressan and P. LeFloch, Uniqueness of weak solutions to systems of conservation laws, *Arch. Rational Mech. Anal.*, to appear.

[12] A. Bressan and A. Marson, A variational calculus for discontinuous solutions of systems of conservation laws, *Comm. Part. Diff. Equat.* **20** (1995), 1491-1552.

[13] A. Bressan and A. Marson, Error bounds for a deterministic version of the Glimm scheme, *Arch. Rational Mech. Anal.*, to appear.

[14] M. Crandall, The semigroup approach to first-order quasilinear equations in several space variables, *Israel J. Math.* **12** (1972), 108-132.

[15] K. Deimling, "Nonlinear Functional Analysis", Springer-Verlag, Berlin 1985.

[16] R. DiPerna, Global existence of solutions to nonlinear hyperbolic systems of conservation laws, *J. Differential Equations* **20** (1976), 187-212.

[17] R. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana Univ. Math. J.* **28** (1979), 137-188.

[18] R. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Anal.* **82** (1983), 27-70.

[19] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697-715.

[20] A. Heibig, Existence and uniqueness of solutions for some hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal.* **126** (1994), 79-101.

[21] F. John, Formation of singularities in one-dimensional wave propagation, *Comm. Pure Appl. Math.* **27** (1974), 377-405.

[22] S. Kruzkov, First order quasilinear equations with several space variables, *Math. U.S.S.R. Sbornik* **10** (1970), 217-243.

[23] P. Lax, Nonlinear hyperbolic equations, *Comm. Pure Appl. Math.* **6** (1953), 231-258.

[24] P. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-566.

[25] P. Lax, Shock waves and entropy. In *Contributions to Nonlinear Functional Analysis*, E. Zarantonello ed., Academic Press, New York (1971), 603-634.

[26] T. P. Liu, The deterministic version of the Glimm scheme, *Comm. Math. Phys.* **57** (1977), 135-148.

[27] T. P. Liu, Uniqueness of weak solutions of the Cauchy problem for general 2×2 conservation laws, *J. Differential Equations* **20** (1976), 369-388.

[28] O. A. Oleinik, Discontinuous solutions of nonlinear differential equations, *Usp. Mat. Nauk* **12** (1957), 3-73. English Transl. *Amer. Math. Soc. Transl.* Ser. 2, Vol.**26**, 95-172.

[29] A. I. Panasyuk, Quasidifferential equations in a metric space, *Differential Equations* **21** (1985), 914-921.

[30] N. H. Risebro, A front-tracking alternative to the random choice method, *Proc. Amer. Math. Soc.* **117** (1993), 1125-1139.

[31] B. Rozdestvenskii and N. Yanenko, "Systems of Quasi-Linear Equations", *A. M. S. Translations of Mathematical Monographs*, Vol. 55, 1983.

[32] D. Serre, Solutions á variations bornées pour certains systèmes hyperboliques de lois de conservation, *J. Differential Equations* **68** (1987), 137-168.

[33] S. Shochet, Sufficient conditions for local existence via Glimm's scheme for large BV data, *J. Differential Equations* **89** (1991), 317-354.

[34] J. Smoller, "Shock waves and reaction-diffusion equations", Second Edition, *Springer-Verlag*, New York, 1994.