On the Alspach conjecture

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Abstract

It has been conjectured by Alspach [2] that given integers n and $m_1, \ldots, m_t$ with $3 \leq m_i \leq n$ and $\sum_{i=1}^t m_i = \binom{n}{2}$ (n odd) or $\sum_{i=1}^t m_i = \binom{n}{2} - \frac{n}{2}$ (n even) then one can pack K_n (n odd) or K_n minus a 1-factor (n even) with cycles of lengths $m_1, \ldots, m_t$. In this paper we show that if the cycle lengths m_i are bounded by some linear function of n and n is sufficiently large then this conjecture is true.

1 Introduction

The following is a conjecture of Brian Alspach [2].

Conjecture 1 *If $m_1, \ldots, m_t$ are integers with $3 \leq m_i \leq n$ and $\sum_{i=1}^t m_i = \binom{n}{2}$ (n odd) or $\binom{n}{2} - \frac{n}{2}$ (n even) then one can pack K_n (n odd) or K_n minus a 1-factor I (n even) with cycles of lengths $m_1, \ldots, m_t$.*

If we just require some circuits (Eulerian subgraphs) of sizes $m_1, \ldots, m_t$ then the result is much easier and has been proved [5], however Conjecture 1 is far from being proved in general.

A number of special cases of this conjecture have been studied. The case when all the cycle lengths are the same has been investigated by Alspach, Gavlas and Marshall [3, 4]. Various cases when there are combinations of two or three distinct special cycle lengths have also be studied by Adams, Bryant, Khodkar and Fu [1, 6] and by Heinrich, Horak and Rosa [10]. Häggkvist has dealt with the case when all cycles are of even length and there are an even number of cycles of each length [9]. The conjecture has also been verified for $n \leq 10$ by Rosa [12]. More recently, the author has verified the conjecture by computer for all $n \leq 14$. In all these cases there are very strong assumptions made about the cycle lengths that can occur or the value of n.

More generally, Caro and Yuster [7] have shown Conjecture 1 is true when $n \geq N(L)$, where L is the maximum length of the cycles and $N(L)$ is a function of L (see [7] and Theorem 19 below). Unfortunately the function $N(L)$ is a very large and extremely rapidly increasing function of L, in particular it grows faster than exponentially in L. Our aim in this paper is to prove the conjecture when n is greater than some *linear* function of L. For this we shall prove the following three theorems.

Theorem 1 *Assume* $n \equiv 2 \bmod 144$*. If* $m_1, \ldots, m_t$ *are integers with* $72 \le m_i \le \lfloor \frac{n+37}{20} \rfloor$ *and* $\sum_{i=1}^t m_i = \binom{n}{2} - \frac{n}{2}$ *then* $K_n - I$ *can be packed with cycles of lengths* $m_1, \ldots, m_t$*.*

Theorem 2 *Assume* $n \ge N_1$ *and* $n \equiv 2 \bmod 144$*. If* $m_1, \ldots, m_t$ *are integers with* $3 \le m_i \le \lfloor \frac{n+37}{20} \rfloor$ *and* $\sum_{i=1}^t m_i = \binom{n}{2} - \frac{n}{2}$ *then* $K_n - I$ *can be packed with cycles of lengths* $m_1, \ldots, m_t$*.*

Theorem 3 *Assume* $n \ge N_2$*. If* $m_1, \ldots, m_t$ *are integers with* $3 \le m_i \le \lfloor \frac{n-112}{20} \rfloor$ *and* $\sum_{i=1}^t m_i = \binom{n}{2}$ *(n odd) or* $\binom{n}{2} - \frac{n}{2}$ *(n even), then one can pack* K_n *(n odd) or* $K_n - I$ *(n even) with cycles of lengths* $m_1, \ldots, m_t$*.*

The easiest result to prove is Theorem 1 and this forms the basis for the other two results. In Theorem 2 we have removed the lower bound on the lengths of the cycles and in Theorem 3 we have removed the congruence condition on n, however in both cases n must be larger than N_1 or N_2 which are (very large) absolute constants. These theorems use the result in [7] mentioned above, but only for cycle lengths less than 72. All cycles of length at least 72 are handled separately, so the lower bound on n is now linear in the size of the longest cycle. It is worth mentioning that it should be possible to improve the linear bound on the cycle lengths, perhaps to as much as about $n/2$. However, new ideas will be needed to pack cycles of lengths much closer to n. All three theorems require extensive computer verifications, which were performed using Visual Basic on a 150 MHz Pentium based PC.

We shall give a proof of Theorem 1 first. The ideas used are similar to those used to pack circuits in [5]. Our strategy is to pack cycles into sequences of linked octahedra. This is done in Section 2. The octahedra are then packed into $K_n - I$ in such a way that any $\lfloor \frac{n+37}{40} \rfloor$ consecutive octahedra are packed so that non-adjacent octahedra are vertex-disjoint. This last result is proved in Section 3 by finding "self-avoiding" trails of triangles in a Steiner Triple System in $K_{n/2}$ and doubling each vertex. Theorem 1 is then proved at the end of Section 3.

Theorem 1 is not the best possible with our methods. In particular the conditions $72 \le m_i$ and $n \equiv 2 \bmod 144$ can be weakened considerably without including any "sufficiently large" condition on n. However, improving this result introduces more technicalities than we wish to include here. Instead, we shall proceed with the proofs of Theorem 2 and Theorem 3 where we make the extra assumption that n is very large. By including more packings and using the result of Caro and Yuster [7] we prove Theorem 2. This is done in Section 4. In Section 5 we remove the congruence condition on n and prove Theorem 3. Finally in Section 6 we conclude by discussing possible improvements to these results.

2 Packing trails of Octahedra

As usual, write K_n for a complete graph and C_n for a cycle on n vertices. Define K'_n to be K_n if n is odd and K_n minus a one-factor I when n is even. Note that every vertex of K'_n has even degree and that K'_n is the graph with the largest number of edges on n vertices for which this is true. Write $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the set of natural numbers including zero.

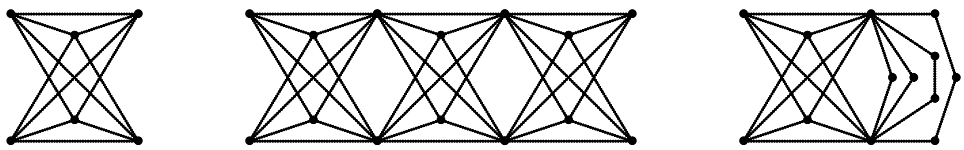
We shall define for some graphs *initial* and *final links.* These will be disjoint (ordered) pairs of vertices. If we have two such graphs G_1 and G_2, write $G_1.G_2$ for the edge-disjoint union of G_1 and

G_2 obtained by identifying the final link vertices of G_1 with the initial link vertices of G_2 (in the same order). The graph $G_1.G_2$ is undefined if an edge occurs in both these links. All other vertices of G_1 will remain distinct from the vertices of G_2 so that $|V(G_1.G_2)| = |V(G_1)| + |V(G_2)| - 2$. Define the initial link of $G_1.G_2$ to be that of G_1 and the final link to be that of G_2. This makes the operation "." into an associative operation on such graphs when defined. Write $G^{\cdot n}$ for $G.G \ldots G$, where there are n copies of G.

Define $O = K_6'$ to be the graph of an octahedron. This graph can also be written as $K_{2,2,2}$, the complete tripartite graph with vertex classes of size two. Define the initial link to be the first vertex class and the final link to be the last vertex class. In fact by symmetry it does not matter which vertex classes are chosen, or the order of the vertices in either link.

Define $\{\mathbf{a}\} = \{a_1, a_2, a_3, a_4\}$ to be a graph with four independent paths of lengths $a_1, \ldots, a_4$ joining the same two distinct vertices. Define the initial link to be these two vertices. We shall not define a final link for this graph.

The following pictures show the "linking" operation on these graphs. The pictures are drawn so that the initial link vertices appear on the left and the final link vertices appear on the right (when defined).



O $\qquad$ $O^{\cdot 3} = O.O.O$ $\qquad$ $O.\{2,2,3,4\}$

Theorem 4 *If $\sum_{i=1}^{t} m_i = 12N$, $12N \geq 40L$ and $72 \leq m_i \leq L$ for $i = 1, \ldots, t$, then we can pack $O^{\cdot N}$ with cycles of lengths $m_1, \ldots, m_t$.*

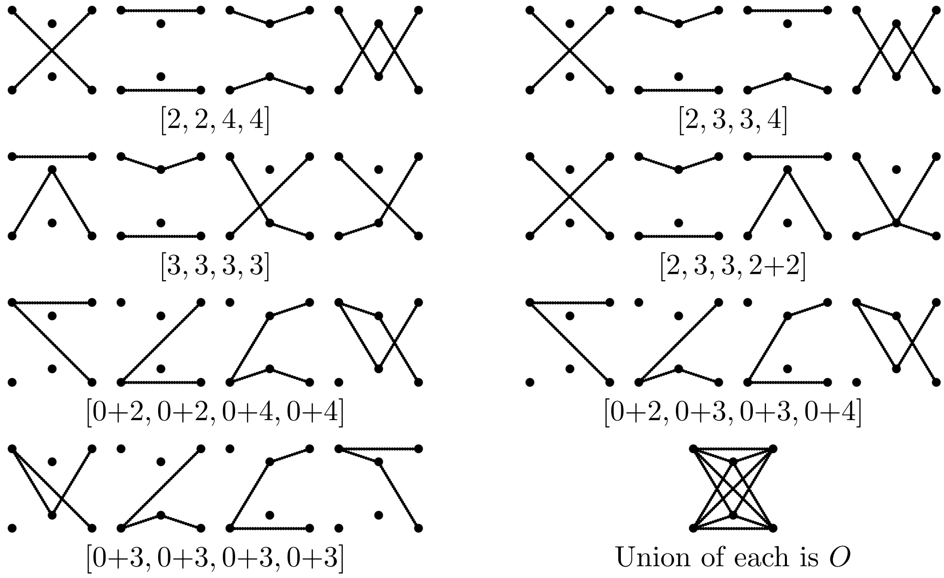
To prove this we shall first pack a single octahedron O with various paths. Define packings $[\mathbf{s}] = [s_1, s_2, s_3, s_4]$ with $\sum_{i=1}^{4} s_i = |E(O)| = 12$ as eight edge disjoint paths in O as follows. For each $i = 1, \ldots, 4$ there are two vertex-disjoint paths each connecting an initial vertex of O to a final vertex of O with the sum of the lengths of these two paths equal to s_i. Since the two paths are vertex-disjoint, each of the four link vertices of O will be an endpoint of one of the two paths.

We also define packings where some or all of the s_i are replaced by sums $p_i{+}q_i$. In these cases there is one path of length p_i joining the two initial link vertices and one path of length q_i joining the two final link vertices (and as a special case, 0 denotes no path). The two paths do not need to be vertex-disjoint in this case. For simplicity we shall not use all possible packings at this stage. The lower bound of 72 in Theorem 1 and Theorem 4 can be reduced by using other packings as well (see Sections 4 and 6).

Lemma 5 *The following packings of an octahedron exist:*

$$[2,2,4,4], \quad [2,3,3,4], \quad [3,3,3,3], \quad [2,3,3,2{+}2],$$
$$[0{+}2,0{+}2,0{+}4,0{+}4], \quad [0{+}2,0{+}3,0{+}3,0{+}4], \quad [0{+}3,0{+}3,0{+}3,0{+}3].$$

Proof. The result follows by inspection of the following diagrams:



[2, 2, 4, 4] [2, 3, 3, 4]

[3, 3, 3, 3] [2, 3, 3, 2+2]

[0+2, 0+2, 0+4, 0+4] [0+2, 0+3, 0+3, 0+4]

[0+3, 0+3, 0+3, 0+3] Union of each is O

□

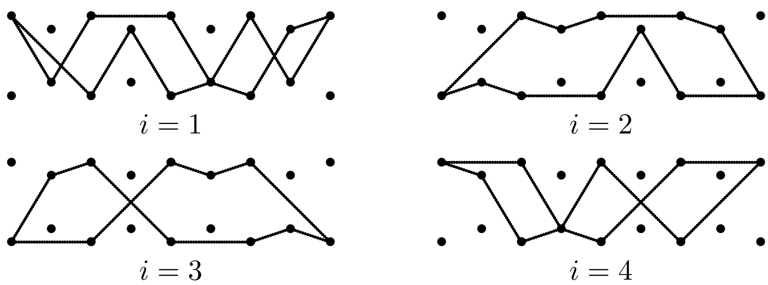
Clearly any permutation (such as [4, 2, 2, 4]) or reversal (such as [2+0, 2+0, 4+0, 4+0]) also exists by symmetry. Note that there is no packing of the form [2, 2, 4, 2+2].

Define $\mathcal{S}$ to be the set of 4-tuples $\mathbf{s} = (s_1, s_2, s_3, s_4)$ for which $\sum_{i=1}^{4} s_i = 12$ and $2 \le s_i \le 4$ for all i. These are just the permutations of either (2, 2, 4, 4), (2, 3, 3, 4) or (3, 3, 3, 3). Also, write $\mathbf{0} = (0, 0, 0, 0)$. Lemma 5 implies that packings of the form $[\mathbf{s}]$ and $[\mathbf{0}+\mathbf{s}] = [0+s_1, 0+s_2, 0+s_3, 0+s_4]$ exist for all $\mathbf{s} \in \mathcal{S}$.

By packing the octahedra using Lemma 5 and joining up the paths we shall get cycles of various lengths. This is best described by an example. Suppose we packed four linked octahedra as

$$O^{\cdot 4} = [0+3, 0+3, 0+3, 0+3].[3, 3, 2, 2+2].[2+2, 3, 3, 2].[4+0, 3+0, 3+0, 2+0]$$

For each $i = 1, \ldots, 4$ join up the paths corresponding to the components s_i or p_i+q_i of each packing. By linking the $i = 1$ paths (0+3, 3, 2+2, 4+0) we get two cycles of lengths $3+3+2 = 8$ and $2+4 = 6$ respectively. For $i = 2$ (0+3, 3, 3, 3+0) we get one cycle of length 12, for $i = 3$ (0+3, 2, 3, 3+0) we get one cycle of length 11 and for $i = 4$ (0+3, 2+2, 2, 2+0) we get two cycles of lengths 5 and 6 respectively. The result is a packing of $C_5 + 2C_6 + C_8 + C_{11} + C_{12}$ into $O^{\cdot 4}$.



$i = 1$ $i = 2$

$i = 3$ $i = 4$

Example of cycles packing $O^{\cdot 4}$

We now need to describe an algorithm for choosing the packings for each octahedron in $O^{\cdot N}$ so that we get the cycle lengths $m_1, \dots, m_t$ specified in the statement of the theorem. We shall construct the packing one octahedron at a time, so that at each step we may have some incompletely packed cycles. The unpacked parts of these cycles will form a graph $\{\mathbf{a}\}$ linked to the final link of the last octahedron packed so far. For example, after the first step in the example above we shall have a packing of $C_8 + C_{12} + C_{11} + C_5$ into $O.\{5, 9, 8, 2\}$. The next step packs this graph and one C_6 into $O^{\cdot 2}.\{2, 6, 6, 4\}$. The next step packs this graph and the second C_6 into $O^{\cdot 3}.\{4, 3, 3, 2\}$. The last step then packs this graph into $O^{\cdot 4}$.

For $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ define the following functions:

$$\min \mathbf{a} = \min_i a_i, \qquad \max \mathbf{a} = \max_i a_i, \qquad \sum \mathbf{a} = \sum_i a_i \qquad \text{and} \qquad k(\mathbf{a}) = 2\max \mathbf{a} - 3\min \mathbf{a}.$$

The function $k(\mathbf{a})$ estimates the unevenness of the lengths and will be used later.

Lemma 6 *There exists a non-empty set $\mathcal{A} \subseteq \mathbb{N}^4$ such that for any $\mathbf{a} \in \mathcal{A}$ and any $m \geq 27$ we can pack $O^{\cdot r}.\{\mathbf{a}\}$ together possibly with the cycle C_m into some $O^{\cdot r+1}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}$. Moreover, $\mathbf{a} \in \mathcal{A}$ for all $\mathbf{a}$ with $\min \mathbf{a} > 12$ and $\max \mathbf{a} > 25$.*

Proof. Without loss of generality assume $a_1 \leq a_2 \leq a_3 \leq a_4$. We shall either pack the graph as $O^{\cdot r}.[\mathbf{s}].\{\mathbf{b}\}$ with $b_i = a_i - s_i$ or as $O^{\cdot r}.[2{+}2, s_2, s_3, s_4].\{\mathbf{b}\}$ with $b_i = a_i - s_i$ for $i \geq 2$, $a_1 = 2$ and $b_1 = m - 2$. In each case we use one of the packings of Lemma 5. The proof is by a computer verification. For this consider the a_i as elements of $F = \{2, \dots, n-1, n^*\}$ where n is some fixed positive integer and n^* represents any number $\geq n$. We construct sets $\mathcal{A}_j$ as a subset of F^4. Initially define $\mathcal{A}_0$ as all $\mathbf{a} \in F^4$ with $a_1 \leq a_2 \leq a_3 \leq a_4$ and inductively define $\mathcal{A}_{j+1}$ as all sequences $\mathbf{a} \in \mathcal{A}_j$ that can be packed as above with $\mathbf{b} \in \mathcal{A}_j$ for any $m \geq 27$. In other words, if $a_1 > 2$ then check if some permutation of $(a_1 - s_1, a_2 - s_2, a_3 - s_3, a_4 - s_4)$ lies in $\mathcal{A}_j$ for some $\mathbf{s} \in \mathcal{S}$. If $a_1 = 2$ then for each $m \geq 27$ check if some permutation of $(m-2, a_2 - s_2, a_3 - s_3, a_4 - s_4)$ lies in $\mathcal{A}_j$ for some $[2{+}2, s_2, s_3, s_4]$ packing of O. If $a_i = n^*$ and $b_i = a_i - s_i$ we need to check that $\mathbf{b} \in \mathcal{A}_j$ for each choice of values $b_i = n - s_1, n - s_1 + 1, \dots, n-1, n^*$. If $m - 2 \geq n$ then $m - 2$ is replaced by n^*, hence only finitely many values of m need to be considered. The process terminates when $\mathcal{A}_{j+1} = \mathcal{A}_j$. Since $\mathcal{A}_{j+1} \subseteq \mathcal{A}_j$ and $\mathcal{A}_0$ is finite, this will occur in finite time. If the final value of $\mathcal{A}_j$ is non-empty then we can take $\mathcal{A}$ as the set of $\mathbf{a} \in \mathbb{N}^4$ which are permutations of 4-tuples $\mathbf{a} \in \mathcal{A}_j$ where any value $a_i \geq n$ is taken to be n^*. For $n = 26$ this process does indeed terminate with a non-empty $\mathcal{A}_j$ and we are done. The largest value of a_1 with $\mathbf{a} \notin \mathcal{A}_j$ and $a_4 = 26^*$ is $a_1 = 12$ and occurs when $\mathbf{a} = (12, 12, 12, 26^*)$. Hence the final part also follows. □

Lemma 6 gives us the main inductive step, however we still need to solve two remaining problems—how to start and how to stop. Starting is easy as the following lemma shows.

Lemma 7 *If $m_1, \dots, m_4 \geq 26$ then we can pack cycles $C_{m_1}, \dots, C_{m_4}$ into some graph of the form $O.\{\mathbf{a}\}$ with $\mathbf{a} \in \mathcal{A}$. Here $\mathcal{A}$ is the set constructed in Lemma 6.*

Proof. By Lemma 6, $\mathbf{m} = (m_1, m_2, m_3, m_4) \in \mathcal{A}$, and by the proof of Lemma 6, $\mathbf{m} = \mathbf{a} + \mathbf{s}$ for some $\mathbf{a} \in \mathcal{A}$ and $\mathbf{s} \in \mathcal{S}$. Using Lemma 5 we can pack the cycles into $[0{+}s_1, 0{+}s_2, 0{+}s_3, 0{+}s_4].\{\mathbf{a}\} = O.\{\mathbf{a}\}$. □

Stopping however is much more difficult. We need to arrange the cycles and the packings so that the last four cycles finish simultaneously. For this we need the a_i to be reasonably similar in length as the next two results show.

For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^4$ write $\mathbf{a} \to \mathbf{b}$ if there exists $\mathbf{s} \in \mathcal{S}$ such that

1. $\mathbf{a} = \mathbf{b} + \mathbf{s}$, i.e., $a_i = b_i + s_i$ for $i = 1, \ldots, 4$,
2. if $b_i < 0 \le a_i$ then $b_i = -2$, $a_i = 2$ and $\mathbf{s}$ is a permutation of $(2, 3, 3, 4)$,
3. if $b_i = 0$ for some i then $\mathbf{b} = (0, 0, 0, 0)$.

The use of negative numbers and conditions 2 and 3 above will become apparent in the proof of Lemma 12. Note that $b_i < 0 \le a_i$ can hold for at most one value of i since when this occurs $s_i = 4$ and $\mathbf{s}$ is a permutation of $(2, 3, 3, 4)$. In general we say $\mathbf{a} \Rightarrow \mathbf{b}$ if there is a sequence $\mathbf{a} = \mathbf{a}_0, \ldots, \mathbf{a}_s = \mathbf{b}$ for some $s \ge 0$ with $\mathbf{a}_i \to \mathbf{a}_{i+1}$ for all $0 \le i < s$. Recall that $\mathbf{0} = (0, 0, 0, 0)$.

Lemma 8 *For all $\mathbf{a} \in \mathbb{Z}^4$, $\mathbf{a} \Rightarrow \mathbf{0}$ if and only if $\sum \mathbf{a} = 12n$, $\min \mathbf{a} \ge 2n$ and $\max \mathbf{a} \le 4n$ for some integer $n \ge 0$.*

Proof. The "only if" is clear, since $\mathbf{a}$ must be the sum of n terms in $\mathcal{S}$ and for each such term $\mathbf{s}$, $2 \le s_i \le 4$ for all i. It remains to prove the "if".
We may assume $a_1 \le a_2 \le a_3 \le a_4$. If $n = 0$ then $\mathbf{a} = \mathbf{0}$ and we are done. If $n = 1$ then $\mathbf{a} \in \mathcal{S}$ and we are done (taking $\mathbf{s} = \mathbf{a}$). Now assume $n \ge 2$. Let $\mathbf{s} = (2, 2, 4, 4)$ and consider $\mathbf{b} = \mathbf{a} - \mathbf{s}$. Clearly $\sum \mathbf{b} = 12(n-1)$. Also, $\min \mathbf{b} \ge 2(n-1)$ unless $a_3 \le 2n+1$ and $\max \mathbf{b} \le 4(n-1)$ unless $a_2 \ge 4n-1$. However, if $a_3 \le 2n+1$ then $12n = \sum \mathbf{a} \le 3(2n+1) + (4n) = 10n+3$, so $n < 2$, a contradiction. On the other hand, if $a_2 \ge 4n-1$ then $12n = \sum \mathbf{a} \ge (2n) + 3(4n-1) = 14n-3$, so again $n < 2$, a contradiction. Since $\min \mathbf{b} > 0$ and $\mathbf{a} = \mathbf{b} + \mathbf{s}$ we have $\mathbf{a} \to \mathbf{b}$. Also $\mathbf{b} \Rightarrow \mathbf{0}$ by induction on n, and so $\mathbf{a} \Rightarrow \mathbf{0}$. □

Corollary 9 *If $\sum \mathbf{a} = 12n > 0$ and $k(\mathbf{a}) \le 0$ then $\mathbf{a} \Rightarrow \mathbf{0}$. In particular $O^{\cdot r}.\{\mathbf{a}\}$ can be packed into $O^{\cdot r+n}$.*

Proof. Assume $a_1 \le a_2 \le a_3 \le a_4$, so that $k(\mathbf{a}) = 2a_4 - 3a_1 \le 0$. Hence $6a_1 \ge 4a_4 \ge \sum \mathbf{a} = 12n$ and so $a_1 \ge 2n$. Also $3a_4 \le a_4 + 3a_1 \le \sum \mathbf{a} = 12n$, so $a_4 \le 4n$. The result follows from Lemma 8. The last part is clear since if $\mathbf{a} \Rightarrow \mathbf{0}$ then $\mathbf{a} = \mathbf{s}_1 + \ldots + \mathbf{s}_n$ and we can pack $O^{\cdot r}.\{\mathbf{a}\}$ as $O^{\cdot r}.[\mathbf{s}_1].[\mathbf{s}_2] \ldots [\mathbf{s}_n{+}\mathbf{0}] = O^{\cdot r+n}$. □

It remains to pack the cycles in a suitable order so that once we have used all the cycles we will have packed some $O^{\cdot r}.\{\mathbf{a}\}$ with $k(\mathbf{a}) \le 0$. The next three lemmas show that if we pack four cycles of similar lengths then we can make the value of $k(\mathbf{a})$ decrease.

For $\mathbf{a} \in \mathbb{Z}^4$ define $\operatorname{ord} \mathbf{a} \in \mathbb{Z}^4$ to be the non-decreasing rearrangement of $\mathbf{a}$. In other words, $\operatorname{ord} \mathbf{a}$ is the 4-tuple obtained by putting the components of $\mathbf{a}$ in non-decreasing order, so for example $\operatorname{ord}(5, 2, -7, 2) = (-7, 2, 2, 5)$.

Lemma 10 *If $\mathbf{a} \to \mathbf{b}$ then $\operatorname{ord} \mathbf{a} \to \operatorname{ord} \mathbf{b}$.*

Proof. Since $\mathbf{a} \to \mathbf{b}$, $\mathbf{a} = \mathbf{b} + \mathbf{s}$ for some $\mathbf{s} \in \mathcal{S}$. Assume $a_j < a_k$ and $b_j > b_k$, so that $b_k < b_j < a_j < a_k$. Then $s_k \geq s_j + 2$ so $s_k = 4$, $s_j = 2$ and $a_j = a_k - 1$, $b_j = b_k + 1$. If $b_k \leq 0 \leq a_k$ then either $b_k = 0$ or $b_k < 0 \leq a_k$. In the first case $\mathbf{b} = \mathbf{0}$, contradicting the assumption $b_j > b_k$. In the second case $a_k = 2$, $b_k = -2$. But then $b_j = -1 < 0 < a_j = 1$ contradicting part 2 of the definition of $\to$. Hence a_j, a_k, b_j and b_k are all non-zero and have the same sign. Define $\mathbf{b}'$ to be $\mathbf{b}$ with b_j and b_k swapped. Then $\mathbf{a} = \mathbf{b}' + \mathbf{s}'$ where $\mathbf{s}'$ is $\mathbf{s}$ with s_j and s_k replaced by 3. Clearly $\mathbf{s}' \in \mathcal{S}$ and so $\mathbf{a} \to \mathbf{b}'$. Repeating this process, we can assume the components of $\mathbf{b}$ are in the same order as those in $\mathbf{a}$. Hence there is a permutation π of the numbers $1, \ldots, 4$ which simultaneously makes $(\mathbf{a}_{\pi_1}, \mathbf{a}_{\pi_2}, \mathbf{a}_{\pi_3}, \mathbf{a}_{\pi_4}) = \operatorname{ord} \mathbf{a}$ and $(\mathbf{b}_{\pi_1}, \mathbf{b}_{\pi_2}, \mathbf{b}_{\pi_3}, \mathbf{b}_{\pi_4}) = \operatorname{ord} \mathbf{b}$. By replacing $\mathbf{s}$ with $(\mathbf{s}_{\pi_1}, \mathbf{s}_{\pi_2}, \mathbf{s}_{\pi_3}, \mathbf{s}_{\pi_4})$ we get $\operatorname{ord} \mathbf{a} \to \operatorname{ord} \mathbf{b}$. □

Lemma 11 *If $\mathbf{a} \in \mathcal{A}$ then there exists some $\mathbf{b} \in \mathbb{Z}^4$ with $\mathbf{a} \Rightarrow \mathbf{b}$, $\max \mathbf{b} = -2$ and $k(\mathbf{b}) \leq \max(38, 41 + k(\mathbf{a}))$.*

Proof. The proof is in two parts. The first is a computer verification of this lemma for all $\mathbf{a}$ with $\max \mathbf{a} \leq 43$. We may assume $2 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 43$. Construct an array A of all $\mathbf{b}$ with $-31 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 43$. (The values 31 and 43 are just the smallest values for which the proof works, larger values may of course be used.) Set $A(\mathbf{b}) = k(\mathbf{b})$ for each element of this array with $b_4 = -2$. All other values are initially set to $+\infty$ (or some suitably large integer). Now, for each element of this array with $b_4 > -2$ in turn, set $A(\mathbf{b}) = \min A(\mathbf{b}')$ where the minimum is over all $\mathbf{b}'$ with $\mathbf{b} \to \mathbf{b}'$ and $A(\mathbf{b}')$ defined. If the elements of A are processed in a suitable order (such as increasing lexicographical order), then the value of $\min A(\mathbf{b}')$ will not be changed in any subsequent modification of A. Finally, the inequality $A(\mathbf{a}) \leq \max(38, 41 + k(\mathbf{a}))$ can be checked for each $\mathbf{a}$ with $\min \mathbf{a} \geq 2$, $\max \mathbf{a} \leq 43$ and $\mathbf{a} \in \mathcal{A}$, where the set $\mathcal{A}$ is constructed as in Lemma 6. Clearly $A(\mathbf{a})$ is an upper bound on the value of $k(\mathbf{b})$ for all $\mathbf{b}$ with $\mathbf{a} \Rightarrow \mathbf{b}$ and $\max \mathbf{b} = -2$.
The second part of the proof is an inductive proof to remove the upper bound of 43 on $\max \mathbf{a}$. Assume therefore that $a_1 \leq a_2 \leq a_3 \leq a_4$ and $a_4 \geq 44$. We consider the case $a_1 \geq 21$ first.
If $a_2 \geq a_1 + 2$, then let $\mathbf{a}' = \mathbf{a} - (2, 4, 3, 3)$. Now $\min \mathbf{a}' = a_1 - 2 > 12$, and $\max \mathbf{a}' = a_4 - 3 > 25$, so $\mathbf{a} \to \mathbf{a}'$, $\mathbf{a}' \in \mathcal{A}$ and $k(\mathbf{a}') = 2(a_4 - 3) - 3(a_1 - 2) = k(\mathbf{a})$. By induction on $\max \mathbf{a}$, $\mathbf{a} \to \mathbf{a}' \Rightarrow \mathbf{b}$ with $\max \mathbf{b} = -2$ and $k(\mathbf{b}) \leq \max(38, 41 + k(\mathbf{a}')) = \max(38, 41 + k(\mathbf{a}))$ and we are done. If $a_3 \geq a_2 + 2$ then use the same argument with $\mathbf{a}' = \mathbf{a} - (2, 2, 4, 4)$. If $a_4 \geq a_3 + 4$ then use the same argument with $\mathbf{a}' = \mathbf{a} - (8, 8, 8, 12)$. Note that $(8, 8, 8, 12) = (3, 3, 2, 4) + (3, 2, 3, 4) + (2, 3, 3, 4)$ is a sum of elements of $\mathcal{S}$ so $\mathbf{a} \Rightarrow \mathbf{a}'$. If none of these conditions hold then $a_4 \leq a_1 + 5$. In this case set $\mathbf{a}' = \mathbf{a} - (3, 3, 3, 3) \in \mathcal{A}$ and note that $k(\mathbf{a}') = 2(a_4 - 3) - 3(a_1 - 3) \leq 13 - a_1 < -3$. Now $\mathbf{a} \to \mathbf{a}' \Rightarrow \mathbf{b}$ with $k(\mathbf{b}) \leq \max(38, 41 + k(\mathbf{a}')) = 38 \leq \max(38, 41 + k(\mathbf{a}))$.
Now assume $a_1 \leq 20$ and $a_4 \geq 44$. If $a_2 \geq 30$ then set $\mathbf{a}' = \mathbf{a} - (0, 4, 3, 3)$. Now $\mathbf{a}'$ differs from $\mathbf{a}$ only in terms that are at least 26. Hence by the proof of Lemma 6, $\mathbf{a}' \in \mathcal{A}$. By induction $\mathbf{a}' \Rightarrow \mathbf{b}'$ with $\max \mathbf{b}' = -2$ and $k(\mathbf{b}') \leq \max(38, 41 + k(\mathbf{a}'))$. Since the components of $\mathbf{a}'$ are in non-decreasing order, we can replace each term $\mathbf{a}'_i$ in $\mathbf{a}' = \mathbf{a}'_0 \to \mathbf{a}'_1 \to \ldots \to \mathbf{a}'_s = \mathbf{b}'$ with $\operatorname{ord} \mathbf{a}'_i$ and $\mathbf{b}'$ with $\operatorname{ord} \mathbf{b}'$. Clearly this new $\mathbf{b}'$ still satisfies $\max \mathbf{b}' = -2$ and $k(\mathbf{b}') \leq \max(38, 41 + k(\mathbf{a}'))$. Since at each step the components are in non-decreasing order and since at most one component can jump across zero at each step, there must be some $\mathbf{a}'' = \mathbf{a}'_i$ with $\mathbf{a}' \Rightarrow \mathbf{a}'' \Rightarrow \mathbf{b}'$ and $a''_1 < 0 < a''_2$. Hence $\mathbf{a} \Rightarrow \mathbf{a}'' + (0, 4, 3, 3) \to \mathbf{a}'' - (2, 0, 0, 0) \Rightarrow \mathbf{b} = \mathbf{b}' - (2, 0, 0, 0)$. However $k(\mathbf{a}') = k(\mathbf{a}) - 6 \geq 2(44) - 3(20) - 6 > 0$ and $k(\mathbf{b}) = k(\mathbf{b}') + 6$, so $k(\mathbf{b}) \leq (41 + k(\mathbf{a}')) + 6 \leq \max(38, 41 + k(\mathbf{a}))$ as required. A similar argument applies when $a_2 \leq 29$, $a_3 \geq 33$ with $\mathbf{a}' = \mathbf{a} - (0, 0, 4, 4)$ and

$\mathbf{b} = \mathbf{b}' - (2,2,0,0)$ and when $a_3 \le 32$, $a_4 \ge 44$ with $\mathbf{a}' = \mathbf{a} - (0,0,0,12)$ and $\mathbf{b} = \mathbf{b}' - (8,8,8,0)$. Since $a_4 \ge 44$ at least one of these conditions hold and we are done. □

Lemma 12 *Assume* $72 \le m_1 \le m_2 \le m_3 \le m_4$*,* $\max \mathbf{a} \le m_1 - 2$*,* $\mathbf{a} \in \mathcal{A}$ *and* $3m_4 < 4m_1$*. Then we can pack* $O^{\cdot r}.\{\mathbf{a}\}$ *and cycles* $C_{m_1}, \ldots, C_{m_4}$ *into some* $O^{\cdot r'}.\{\mathbf{a}'\}$ *with* $\mathbf{a}' \in \mathcal{A}$ *and* $k(\mathbf{a}') \le \max(0, k(\mathbf{a}) - m_1/3)$*.*

Proof. Use the previous lemma to find $\mathbf{b}$ with $\max \mathbf{b} = -2$, $\mathbf{a} \Rightarrow \mathbf{b}$ and $k(\mathbf{b}) \le \max(38, 41 + k(\mathbf{a}))$. Now $k(\mathbf{a}) \le 2(m_1 - 2) - 3(2) = 2m_1 - 10$, so $k(\mathbf{b}) \le 2m_1 + 31$. Hence $\min \mathbf{b} = -(k(\mathbf{b}) + 4)/3 \ge (m_1 - 35)/3 - m_1 > 12 - m_1$. The method of packing is as follows. Let $\mathbf{a} \to \mathbf{a}_0 \to \mathbf{a}_1 \to \ldots \to \mathbf{a}_s = \mathbf{b}$ and assume the components of each $\mathbf{a}_i$ are in non-decreasing order. As long as the components are positive, we pack $O^{\cdot r+i}.\{\mathbf{a}_i\}$ as $O^{\cdot r+i}.[\mathbf{s}].\{\mathbf{a}_{i+1}\}$ as before where $\mathbf{s} = \mathbf{a}_i - \mathbf{a}_{i+1}$. However, when a component in $\mathbf{a}_i$ becomes negative in $\mathbf{a}_{i+1}$ we start a new cycle C_{m_j}. We start C_{m_4} when $(\mathbf{a}_{i+1})_1$ becomes negative, C_{m_3} when $(\mathbf{a}_{i+1})_2$ becomes negative and so on. By the definintion of $\to$, only one of these occurs at a time and we can use (a permutation of) the $[2{+}2, 2, 3, 3]$ packing at the point when the j^{th} component goes from $+2$ to -2 and we start the new cycle $C_{m_{5-j}}$. If we define $\mathbf{a}'_i$ to be $\mathbf{a}_i$ with m_{5-j} added to each negative component $(\mathbf{a}_i)_j$, then we have inductively defined packings into $O^{\cdot r+k}.\{\mathbf{a}'_i\}$ for $0 \le k \le s$. When we finish we have a packing of $O^{\cdot r}.\{\mathbf{a}\}$ and all four cycles into $O^{\cdot r'}.\{\mathbf{a}'\}$ where $\mathbf{a}' = \mathbf{a}'_s = (b_1 + m_4, b_2 + m_3, b_3 + m_2, b_4 + m_1)$ and $r' = r + s$. Since $\min \mathbf{b} > 12 - m_1$, all the terms in $\mathbf{a}'$ are positive (so we did not run out of edges in the cycles), and indeed, $\min \mathbf{a}' > 12$, $\max \mathbf{a}' \ge m_1 + b_4 = m_1 - 2 > 25$ so $\mathbf{a}' \in \mathcal{A}$. It remains to check the value of $k(\mathbf{a}')$. Assume $\min \mathbf{a}' = b_i + m_{5-i}$ and $\max \mathbf{a}' = b_j + m_{5-j}$. If $i < j$ then

$$\begin{aligned} k(\mathbf{a}') &= 2b_j - 3b_i + 2m_{5-j} - 3m_{5-i} \\ &\le k(\mathbf{b}) - m_1 \\ &\le \max(38 - m_1, k(\mathbf{a}) + 41 - m_1) \\ &\le \max(0, k(\mathbf{a}) - m_1/3), \end{aligned}$$

where in the last inequality we have used $m_1 \ge 72$. If $i > j$ then

$$\begin{aligned} k(\mathbf{a}') &= 2b_j - 3b_i + 2m_{5-j} - 3m_{5-i} \\ &\le k(\mathbf{m}) - b_1 \\ &\le 2(4m_1/3) - 3m_1 + (k(\mathbf{b}) + 4)/3 \\ &\le (k(\mathbf{b}) + 4 - m_1)/3 \\ &\le \max(0, (k(\mathbf{a}) + 45 - m_1)/3). \end{aligned}$$

If $k(\mathbf{a}) \le m_1/3$ then $k(\mathbf{a}) + 45 - m_1 \le 0$ and so $k(\mathbf{a}') \le 0$. If $k(\mathbf{a}) \ge m_1/3$ then

$$(k(\mathbf{a}) + 45 - m_1)/3 \le k(\mathbf{a}) - 2m_1/9 + 15 - m_1/3 \le k(\mathbf{a}) - m_1/3.$$

In all cases $k(\mathbf{a}') \le \max(0, k(\mathbf{a}) - m_1/3)$. □

Proof. of Theorem 4
Order the cycle lengths in decreasing order. Look at the first (longest) four cycles. If the length of the fourth cycle is less than or equal to three quarters the length of the first, discard the first cycle. Otherwise remove the first four cycles and group them as $\mathbf{m}_1$. Repeat this process until

there are fewer than four cycles remaining or until we have six groups $\mathbf{m}_1, \ldots, \mathbf{m}_6$. We now have $\mathbf{m}_1, \ldots, \mathbf{m}_k$ ($k \leq 6$) groups of cycles each consisting of four cycles of similar lengths, a set of discarded cycles $m_1, \ldots, m_s$, say, and the remaining cycles $m_{s+1}, \ldots, m_u$, say. First calculate the maximum length of discarded cycles. Since $m_{i+3} \leq 3m_i/4$ for $1 \leq i \leq s-3$ and $m_1, m_2, m_3 \leq L$, it is easy to see that the total length of discarded cycles is bounded by an infinite geometric series with sum $3L/(1-\frac{3}{4}) = 12L$. The total length of the grouped cycles and remaining cycles is at most $4kL$ and $(u-s)L$ respectively. If there are at most three remaining cycles then $12N < 12L + 4kL + 3L \leq 39L$ contradicting the assumption that $12N \geq 40L$. Hence $k = 6$ and $u \geq 4 + s \geq 4$. Pack all the cycles $m_1, \ldots, m_u$ into some $O^{\cdot r}.\{\mathbf{a}\}$ with $\mathbf{a} \in \mathcal{A}$ using first Lemma 7 and then inductively using Lemma 6 until all the cycles have been used. These cycles should be packed in order of decreasing length. We shall now show that $\max \mathbf{a} \leq \min \mathbf{m}_6 - 2$. If the path of length a_i comes from packing one of the remaining cycles C_{m_j}, $(j > s)$, then $a_i \leq m_j - 2 \leq \min \mathbf{m}_6 - 2$, so assume it comes from packing a discarded cycle C_{m_j}, $(j \leq s)$. By considering the geometric series above, the total of all the cycle lengths packed before C_{m_j} is at most $12L - 12m_j$ if j is 1 mod 3 and hence at most $12L - 10m_j$ in general. Therefore there are at most $L - 5m_j/6$ octahedra that have been fully packed before cycle C_{m_j} is started. Also, $\sum \mathbf{a} \leq 4L$ so $12r + 4L \geq |E(O^{\cdot r}.\{\mathbf{a}\})| \geq 12N - 4kL \geq 16L$ and hence $r \geq L$. Since a_i must reduce by at least two for each extra octahedron packed, $a_i \leq m_j - 2(r - L + 5m_j/6) \leq 2(L-r) - 2m_j/3 < 0$. This is clearly impossible, so no path a_i is part of a discarded cycle.
Now $k(\mathbf{a}) < 2 \max \mathbf{a} < 2 \min \mathbf{m}_6$. Pack the $\mathbf{m}_i$ in reverse order ($\mathbf{m}_6$ first) using Lemma 12. At each stage $\max \mathbf{a} \leq \min \mathbf{m}_i - 2$ and $k(\mathbf{a})$ is reduced by at least $\min \mathbf{m}_i/3 \geq \min \mathbf{m}_6/3$. Hence after six steps we have a packing of all the original cycles into some graph of the form $O^{\cdot r'}.\{\mathbf{a}\}$ with $\mathbf{a} \in \mathcal{A}$ and $k(\mathbf{a}) \leq 0$. Since $\sum \mathbf{a}$ is clearly divisible by 12, we can now pack this into $O^{\cdot N}$ by Corollary 9 as desired. □

It is worth noting here that the proof of Theorem 4 can be strengthened to allow the packing of some $\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}$ at the start of the trail of octahedra:

Corollary 13 *Assume $\mathbf{b} \in \mathcal{A}$, $\max \mathbf{b} \leq L$, $\sum \mathbf{b} + \sum_{i=1}^{t} m_i = 12N$, $72 \leq m_i \leq L$ and $\sum_{i=1}^{t} m_i \geq 40L$. Then $\{\mathbf{b}\}$ and the cycles $C_{m_1}, \ldots, C_{m_t}$ can be packed into $O^{\cdot N}$ with the initial link of $\{\mathbf{b}\}$ packed into the initial link of $O^{\cdot N}$.*

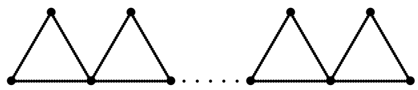
Proof. The proof is almost identical except when we show that no discarded cycle m_j remains partially packed when we start the $\mathbf{m}_i$. For this let $L' = \sum \mathbf{b}$. The total of all the cycle lengths packed before C_{m_j} is now at most $L' + 12L - 10m_j$. Therefore there are at most $L'/12 + L - 5m_j/6$ octahedra that have been fully packed before cycle m_j is started. As before $12r + 4L \geq 12N - 4kL$, but now $12N - 4kL \geq L' + 16L$ and hence $r \geq L + L'/12$. Since a_i must reduce by at least two for each extra octahedron packed, $a_i \leq m_j - 2(r - L - L'/12 + 5m_j/6) \leq 2(L + L'/12 - r) - 2m_j/3 < 0$, which is a contradiction as before. Similarly, since $r \geq L$ and $\max \mathbf{b} \leq L$, none of the original paths in $\{\mathbf{b}\}$ will be left unpacked at this point. □

3 Self-Avoiding Trails of Triangles

Define a *k-self-avoiding trail* of triangles in K_n to be a sequence of triangles $T_1, \ldots, T_N$ in K_n such that

1. the triangles $T_1, \dots, T_N$ are edge-disjoint,
2. for $1 \le i < N$, $V(T_i) \cap V(T_{i+1}) \neq \emptyset$,
3. for all i, j, $V(T_i) \cap V(T_j) \neq \emptyset$ implies either $|i-j| \le 1$ or $|i-j| \ge k$.

In other words, the triangles are linked together so that any subsequence containing at most k consecutive triangles forms a graph isomorphic to one of the form



A *completely self-avoiding trail* of triangles is a k-self-avoiding trail with k equal to the total number of triangles. We say that a k-self-avoiding trail *packs* K_n if the triangles also pack K_n. In other words, the triangles form a Steiner Triple System for K_n. The aim of this section is to prove the existence of self-avoiding trails with reasonably large k that pack K_n, at least for some values of n.

Theorem 14 *For all* $n \equiv 1 \bmod 72$ *then there exists a* $\lfloor \frac{n+18}{20} \rfloor$*-self-avoiding trail of triangles which packs* K_n*.*

Proof. Write $n = 24r + 1$ and define the following set of triples of integers

$$\begin{aligned} \mathcal{T} \;=\; & \{(2t, 10r - t, 10r + t) : t = 1 \dots 2r\} \\ & \cup\{(1, 5r, 5r + 1)\} \\ & \cup\{(2t - 1, 6r - t + 1, 6r + t) : t = 2 \dots r - 1\} \\ & \cup\{(2r - 1, 6r, 8r - 1)\} \\ & \cup\{(2t - 1, 6r - t, 6r + t - 1) : t = r + 1 \dots 2r - 1\} \\ & \cup\{(4r - 1, 6r + 1, 10r)\} \end{aligned}$$

By inspection it is clear that for each triple $(a, b, c) \in \mathcal{T}$, a, b and c are distinct, $a + b = c$ and each integer from 1 to $12r$ inclusive occurs in precisely one triple in $\mathcal{T}$. As a result it is easy to see that

$$\mathcal{S} = \{(j, b + j, c + j) : j \in \mathbb{Z}/n\mathbb{Z},\ (a, b, c) \in \mathcal{T}\}$$

is a Steiner Triple System on the integers mod $n = 24r + 1$. (This construction is based on a *Skolem sequence.* See [14, 11, 13] for further details and other similar constructions.) Write the triangle $(j, b+j, c+j)$ as $T_{a,j}$ where $(a, b, c) \in \mathcal{T}$. Such $T_{a,j}$ exist for all $a = 1, \dots, 4r$ and $j \in \mathbb{Z}/n\mathbb{Z}$. Write $r = 3m$ (so $n = 72m + 1$). We now construct a trail of these triangles (starting with T_2) as follows

$$\begin{array}{llll} T_{6i+2} & = T_{6i+2,i} & = (i, 10r - 2i - 1, 10r + 4i + 1) & i = 0, \dots, 2m - 1 \\ T_{6i+3} & = T_{6i+3+\delta,10r-2i-1} & = (10r - 2i - 1, 16r - 5i - 2 - \delta, 16r + i + 1) & i = 0, \dots, 2m - 1 \\ T_{6i+4} & = T_{6i+4,16r+i+1} & = (16r + i + 1, 2r - 2i - 2, 2r + 4i + 2) & i = 0, \dots, 2m - 1 \\ T_{6i-1} & = T_{6i-1+\delta,2r-2i} & = (2r - 2i, 8r - 5i + 1 - \delta, 8r + i) & i = 1, \dots, 2m - 1 \\ T_{6i} & = T_{6i,8r+i} & = (8r + i, 18r - 2i, 18r + 4i) & i = 1, \dots, 2m - 1 \\ T_{6i+1} & = T_{6i+1+\delta,18r-2i} & = (18r - 2i, -5i - 1 - \delta, i) & i = 1, \dots, 2m - 1 \end{array}$$

Here $\delta = 0$ if $i < m$ and $\delta = 2$ if $i \geq m$. The triangles include one triangle of the form $T_{a,j}$ for each a other than $a = 1$, $2r-1$ and $4r-1$. Now add three more triangles corresponding to these values of a.

$$\begin{array}{lll} T_{12m-1} & = T_{4r-1,4m} & = (2m, 20m+1, 32m) \quad \text{Note: } 2m = 2r - 2(2m-1) - 2 \\ T_{12m} & = T_{2r-1,14m} & = (14m, 32m, 38m-1) \\ T_{12m+1} & = T_{1,-m} & = (-m, 14m, 14m+1) \end{array}$$

In the last triangle $-m$ is relatively prime to $n = 72m+1$, so if we define T_{i+12m} to be T_i translated by $-m$ (i.e., if $T_i = T_{a,j}$ then $T_{i+12m} = T_{a,j-m}$), then the triangles $T_2, \dots, T_{12nm+1}$ will be a permutation of the triangles $T_{a,j}$ of the Steiner Triple System $\mathcal{S}$. It remains to show that it is $\lfloor \frac{n+18}{20} \rfloor$-self avoiding. It is clearly enough by symmetry to show this for the subsequence $T_2, \dots, T_{24m-2}$. By inspection each T_i meets T_{i+1} at one vertex. The triangles $T_2, \dots, T_{12m-2}$ (and by symmetry $T_{12m+2}, \dots, T_{24m-2}$) are completely self-avoiding since in the triangles above,

$$\begin{array}{rcl@{\quad}rcl} -5i-1-\delta & \in & [-10m+2, -6] & i & \in & [0, 2m-1] \\ 2r-2i, 2r-2i-2 & \in & [2m, 6m-2] & 2r+4i+2 & \in & [6m+2, 14m-2] \\ 8r-5i+1-\delta & \in & [14m+4, 24m-4] & 8r+i & \in & [24m+1, 26m-1] \\ 10r-2i-1 & \in & [26m+1, 30m-1] & 10r+4i+1 & \in & [30m+1, 38m-3] \\ 16r-5i-2-\delta & \in & [38m+1, 48m-2] & 16r+i+1 & \in & [48m+1, 50m-1] \\ 18r-2i & \in & [50m+2, 54m-2] & 18r+4i & \in & [54m+4, 62m-4] \end{array}$$

and all these intervals are disjoint mod $n = 72m+1$. We now consider the intersections between T_j and $T_{12m+j'}$, $2 \leq j, j' \leq 12m-2$. Since $T_{12m+j'}$ is a shift by $-m$ of $T_{j'}$ there are only a few possible intersections which can occur. One way in which they can intersect is when the vertex of T_j is of the same "type" as the one in $T_{12m+j'}$. For example, if the vertex $18r-2i$ of some T_j meets vertex $(18r-2i')-m$ of some $T_{12m+j'}$. Clearly if $i < i'$ then the separation of the triangles is at least $12m$ and the closest the triangles can get is when $i = i' + m/2$ (which can happen only in the $2r-2i$, $10r-2i-1$ and $18r-2i$ cases). The separation between the triangles T_j and $T_{12m+j'}$ is therefore at least $12m + j' - j \geq 12m - 6m/2 - 1$ (the -1 is because some vertices occur in two successive triangles). Another way two such triangles can intersect is if the vertex of $T_{12m+j'}$ is of the type immediately after the type of the vertex of T_j in the above table since the $-m$ shift can move a vertex into the preceeding interval. This occurs whenever there is a solution to one of the following equations. Together with each equation is listed the solution in which the triangles are closest together in the trail and their separation in this case.

Equation	Closest Triangles	Minimum Separation
$-5i-1-\delta = i'-m$	$i = (m-1)/5, i' = 0$	$12m - 6(m-1)/5$
$i = 2r-2i'-m$	$i = 2m-1, i' = (3m+1)/2$	$12m - 6(m-3)/2 - 4$
$2r-2i-2 = 2r+4i'+2-m$	$i = (m-4)/2, i' = 0$	$12m - 6(m-4)/2 - 1$
$2r+4i+2 = 8r-5i'+1-\delta-m$	$i = 2m-1, i' = (9m+1)/5$	$12m - 6(m-6)/5 - 5$
$8r-5i+1-\delta = 8r+i'-m$	$i = m/5, i' = 1$	$12m - 6(m-5)/5$
$8r+i = 10r-2i'-1-m$	$i = 2m-1, i' = 3m/2$	$12m - 6(m-2)/2 + 2$
$10r-2i-1 = 10r+4i'+1-m$	$i = (m-2)/2, i' = 0$	$12m - 6(m-2)/2 - 1$
$10r+4i+1 = 16r-5i'-2-\delta-m$	$i = 2m-1, i' = (9m-1)/5$	$12m - 6(m-4)/5 + 1$
$16r-5i-2-\delta = 16r+i'+1-m$	$i = (m-3)/5, i' = 0$	$12m - 6(m-3)/5$
$16r+i+1 = 18r-2i'-m$	$i = 2m-1, i' = 3m/2$	$12m - 6(m-2)/2 - 4$
$18r-2i = 18r+4i'-m$	$i = (m-4)/2, i' = 1$	$12m - 6(m-6)/2 - 1$
$18r+4i = n-5i'-1-\delta-m$	$i = 2m-1, i' = (9m+2)/5$	$12m - 6(m-7)/5 + 1$

It remains to check the intersections of T_{12m-1}, T_{12m} and T_{12m+1} with the other triangles. The vertices of these three triangles which have not been dealt with already are $\{14m, 14m+1, 20m+1, 32m, 38m-1\}$. The possible intersections are listed below along with the smallest separations.

Equation	Closest Triangles	Minimum Separation
$14m = 8r - 5i' + 1 - \delta - m$	$i' = (9m-1)/5$	$6(9m-1)/5 - 2$
$14m + 1 = 8r - 5i' + 1 - \delta - m$	$i' = (9m-2)/5$	$6(9m-2)/5 - 2$
$20m + 1 = 8r - 5i + 1 - \delta$	$i = 4m/5$	$12m - 6(4m)/5$
$20m + 1 = 8r - 5i' + 1 - \delta - m$	$i' = 3m/5$	$6(3m)/5$
$32m = 10r + 4i + 1$		No solutions
$32m = 10r + 4i' + 1 - m$	$i' = (3m-1)/4$	$6(3m-1)/4 + 2$
$38m - 1 = 16r - 5i' - 2 - \delta - m$	$i' = (9m-3)/5$	$6(9m-3)/5 + 3$

For $m \geq 1$, the smallest separation in any of the cases above is $6(3m)/5$. Hence the trail is $\lceil 6(3m)/5 \rceil = \lfloor \frac{n+18}{20} \rfloor$-self-avoiding. □

Finally, we can now give the proof of Theorem 1.

Proof. of Theorem 1.
Let $n = 144m + 2$. Construct a $\lfloor \frac{n/2+18}{20} \rfloor$-self-avoiding trail of triangles that pack $K_{n/2}$ using Theorem 14. Now double up each vertex. Each edge is replaced by four edges, triangles are replaced by octahedra and $K_{n/2}$ becomes $K_n - I$. The trail of triangles become linked octahedra with the property that any k consecutive octahedra with $k \leq \lfloor \frac{n/2+18}{20} \rfloor$ form a graph isomorphic to $O^{\cdot k}$. The total number of octahedra is $N = n(n-2)/24 = 6mn$. Since $72 \leq m_i \leq L = \lfloor \frac{n+37}{20} \rfloor < n$ and $12N \geq 72n > 40L$ we can pack this trail of octahedra with the cycles using Theorem 4. A cycle of length m_i must be packed into at most $\lfloor \frac{m_i}{2} \rfloor$ consecutive octahedra since at least two edges of the cycle must occur in each of these octahedra. Since $\lfloor \frac{L}{2} \rfloor = \lfloor \frac{n/2+18}{20} \rfloor$ the vertices of the cycle will remain distinct when the octahedra are packed into $K_n - I$. □

4 More Packing Results

We shall first generalise the notation used to indicate packings of the octahedron in Lemma 5. We shall extend the notation to cover other graphs where we have defined disjoint pairs of initial and final link vertices. Also, we shall no longer necessarily have exactly four terms of the form s_i or $p_i + q_i$ and we shall include the possibility of whole cycles being packed in addition to the paths. So for example the packing $[2, 4] + 2C_3$ will denote a packing with two C_3's and two pairs of paths from initial to final links (each pair being vertex disjoint, the first pair of total length 2 and the second of total length 4).

We now stengthen Lemma 5 to include more packings of the octahedron. In fact we shall include *all* useful packings.

Lemma 15 *The following packings of an octahedron exist. Indeed, in any packing of $O^{\cdot N}$ with* *cycles, each octahedra is packed in one of these forms (up to permutation and reversal).*

$[2, 2, 4, 4]$	$[2, 3, 3, 4]$	$[3, 3, 3, 3]$	$[2{+}2, 2, 3, 3]$
$[0{+}2, 0{+}2, 0{+}4, 0{+}4]$	$[0{+}2, 0{+}3, 0{+}3, 0{+}4]$	$[0{+}3, 0{+}3, 0{+}3, 0{+}3]$	$[0{+}2, 0{+}2, 0{+}3, 0{+}5]$
$[0{+}2, 0{+}2, 4, 4]$	$[0{+}2, 0{+}3, 3, 4]$	$[0{+}2, 0{+}4, 3, 3]$	$[0{+}3, 0{+}3, 2, 4]$
$[0{+}3, 0{+}3, 3, 3]$	$[0{+}3, 0{+}4, 2, 3]$	$[0{+}4, 0{+}4, 2, 2]$	$[0{+}2, 0{+}5, 2, 3]$
$[2{+}4, 2{+}4]$	$[2{+}3, 3{+}4]$	$[2{+}2, 4{+}4]$	$[2{+}3, 4{+}3]$
$[3{+}3, 3{+}3]$	$[2{+}2, 3{+}5]$	$[4{+}5, 3]$	$[5{+}5, 2]$
$[2{+}2, 0{+}2, 0{+}2, 3]$	$[2{+}2, 0{+}3, 0{+}3, 2]$	$[2{+}2, 0{+}2, 0{+}4, 2]$	$[2{+}2, 2{+}2, 2, 2]$
$[2{+}2, 2{+}2, 0{+}2, 0{+}2]$			
$[] + 2C_6$	$[] + C_6 + 2C_3$	$[] + C_5 + C_4 + C_3$	$[] + 3C_4$
$[] + 4C_3$	$[2, 2] + 2C_4$	$[0{+}2, 0{+}2] + 2C_4$	$[0{+}2, 0{+}2]{+}C_5{+}C_3$
$[2, 3] + C_4 + C_3$	$[0{+}2, 0{+}3]{+}C_4{+}C_3$	$[2, 4] + C_6$	$[2, 4] + 2C_3$
$[3, 3] + C_6$	$[3, 3] + 2C_3$	$[0{+}2, 0{+}4] + C_6$	$[0{+}2, 0{+}4] + 2C_3$
$[0{+}3, 0{+}3] + C_6$	$[0{+}3, 0{+}3] + 2C_3$	$[2{+}2, 2] + C_6$	$[2{+}2, 2] + 2C_3$
$[3, 4] + C_5$	$[0{+}3, 0{+}4] + C_5$	$[0{+}2, 0{+}5] + C_5$	$[2{+}2, 3] + C_5$
$[2{+}3, 2] + C_5$	$[4, 4] + C_4$	$[0{+}3, 0{+}5] + C_4$	$[0{+}4, 0{+}4] + C_4$
$[2{+}3, 3] + C_4$	$[2{+}4, 2] + C_4$	$[3{+}3, 2] + C_4$	$[0{+}2, 0{+}2, 2, 2] + C_4$
$[0{+}2, 0{+}2, 0{+}2, 0{+}2]{+}C_4$	$[2{+}2, 2{+}2] + C_4$	$[3{+}3, 3] + C_3$	$[3{+}4, 2] + C_3$
$[2{+}4, 3] + C_3$	$[2{+}3, 4] + C_3$	$[2{+}2, 2{+}3] + C_3$	

Proof. We make the following observation. If G is some graph with initial and final links defined as disjoint pairs of vertices and if $G^{\cdot n}$ is packed with cycles, then the intersection H of a cycle with some component G must be one of the following:

1. A cycle in G,
2. A path connecting the two initial link vertices of G,
3. A path connecting the two final link vertices of G,
4. Two vertex-disjoint paths both connecting an initial link vertex to a final link vertex.

We shall denote these possibilities by C_n, $n{+}0$, $0{+}n$ and n respectively, where n is the total number of edges in H. The set of all possible H can be constructed by computer as follows:

- List all subgraphs of G with maximum degree at most 2, even degree at all non-link vertices and the same parity of degree at the two initial link vertices and the same parity at the two final link vertices.
- Remove all H which strictly contain some other graph H' on this list and for which the set of vertices of degree one in H' is a subset of the vertices of degree 1 in H. (This eliminates graphs that are unions of several possible H's.)
- Classify each H as of "type C_n" if there are no degree 1 vertices, "type $0{+}n$" if the final link vertices only have degree 1, "type $n{+}0$" if the initial link vertices only have degree 1 and "type n" if all link vertices have degree 1. In each case n is the total number of edges in H.

The set of all packings can then be constructed as the set of all partitions of the edges of G into graphs on this list. We define the *type* of the packing as the collection of types of each component graph H. There are many possible partitions, but many are of the same type. For the $G = O$ we get just 109 types of decomposition. Of these, 27 are symmetric under reversal and the others form 41 pairs under reversal. This gives 68 types up to permutations and reversal, which we have listed above. For ease of notation, n+0 and 0+m components have been combined as n+m in the table above. □

The first 29 of the packings in Lemma 15 do not involve embedded cycles C_n and these can be used to strengthen Lemma 6. First we generalize some more of our notation. In the notation $\{a_1, a_2, a_3, a_4\}$ we shall include the possibility that some or all of the a_i are zero. In this case the paths corresponding to a_i do not exist. We also define variants of $\to$ and $\Rightarrow$ to correspond the the additional packings that we are using. We shall denote these modified versions as $\stackrel{S}{\to}$ and $\stackrel{S}{\Rightarrow}$ where S denotes the set of packings used. The relation $\stackrel{S}{\to}$ will be defined as follows. If $\mathbf{a} \stackrel{S}{\to} \mathbf{b}$ then there must be some packing in S that can be written as four terms $[t_1, t_2, t_3, t_4]$, each term t_i being of the form s or p+q where $s > 0$ and $p, q \geq 0$. As before, if p or q is zero then they represent no path. Moreover,

1. If $t_i = s$ with $s > 0$ then $a_i = b_i + s$ and either $a_i, b_i > 0$ or $a_i, b_i < 0$.
2. If $t_i = p$+q then $a_i = p$ and $b_i = -q$ (either or both p and q may be 0).

If S contains the [$\mathbf{s}$], [$\mathbf{s}$+$\mathbf{0}$] and (all permutations of) the [2+2, 2, 3, 3] packings of Lemma 5 then $\stackrel{S}{\to}$ and $\stackrel{S}{\Rightarrow}$ are the same as $\to$ and $\Rightarrow$ respectively. (Note that we must not include the [$\mathbf{0}$+$\mathbf{s}$] packings since by definition $\mathbf{0} \to \mathbf{a}$ is false for all $\mathbf{a}$.) If G is a graph for which we have defined disjoint initial and final links, then $\stackrel{G}{\to}$ and $\stackrel{G}{\Rightarrow}$ will denote $\stackrel{S}{\to}$ and $\stackrel{S}{\Rightarrow}$ with S equal to the set of all packings of G which do not include complete cycles. Note that $\stackrel{O}{\to}$ only differs from $\to$ when some of the components become or cross zero, however when this happens there are many more $\stackrel{O}{\to}$ relations than $\to$ relations. For example,

$(0, 0, 0, 0) \stackrel{O}{\to} (-2, -2, -3, -5)$	Using $[0{+}2, 0{+}2, 0{+}3, 0{+}5]$	
$(0, 0, 2, 3) \stackrel{O}{\to} (-4, -3, 0, 0)$	Using $[0{+}4, 0{+}3, 2{+}0, 3{+}0]$	(a permutation of $[2{+}4, 3{+}3]$)
$(0, 0, 5, 9) \stackrel{O}{\to} (-4, 0, 0, 6)$	Using $[0{+}4, 0{+}0, 5{+}0, 3]$	(a reversal of $[4{+}5, 3]$)

In order to simplify some of the proofs we shall define the following notation and algorithm. Let $A \subseteq \mathbb{N}^4$, $\mathcal{L} \subseteq \mathbb{N}$ and let S be a set of packings of a graph G with disjoint initial and final links. Let $\mathcal{A}(S, A, \mathcal{L})$ be the set of all elements $\mathbf{a} \in \mathbb{N}^4$ such that for any choice of cycle lengths $m_1, \dots, m_4 \in \mathcal{L}$ the graph $O^r.\{\mathbf{a}\}$ can be packed together with cycles $m_1, \dots, m_s$ for some $0 \leq s \leq 4$ into some graph of the form $O^{\cdot r}.G.\{\mathbf{b}\}$ with $\mathbf{b} \in A$. As in Lemma 6 we convert this into a finite problem suitable for computer calculation by identifying all numbers greater than or equal to some fixed number n. Let u be the largest integer that occurs in a packing in S as some v+u. For the octahedra this is at most 5. Let $F = \{-u, \dots, n-1, n^*\}$ and let $\mathrm{Fin}\colon \mathbb{N}^4 \to F^4$ be the function that sends components $a_i \geq n$ to n^*. With this we define the function $\mathcal{A}_n(S, A, \mathcal{L})$ via the following algorithm.

- Let A_n be the (finite) subset of F^4 consisting of all $\mathbf{a} \in F^4$ such that $\mathrm{Fin}^{-1}(\mathbf{a}) \subseteq A$.

- For each $\mathbf{a} \in F^4$ let s be the number of negative components a_i of $\mathbf{a}$. Now for each choice of $m_1, \ldots, m_s \in \mathcal{L}$ check whether one can obtain an element of A_n by adding m_i to the negative components of $\mathbf{a}$ in some order (so that when adding m_i we obtain strictly positive components). As usual, any term $\geq n$ is regarded as n^*. Since all numbers $\geq n$ are indentified, this is a finite check. Define $A^{\text{ext}}_{n,\mathcal{L}}$ as the subset of all $\mathbf{a} \in F^4$ for which this is possible. Note that $A_n \subseteq A^{\text{ext}}_{n,\mathcal{L}}$ since if $\mathbf{a} \in A_n$ then no component of $\mathbf{a}$ is negative.

- Let $\mathbf{a} \in F^4$. Then for each packing p in S (distinct permutations being regarded as distinct packings), list all the corresponding values of $\mathbf{b} = \mathbf{b}_{p,1}, \ldots, \mathbf{b}_{p,n_p}$ where $\mathbf{a} \stackrel{\{p\}}{\to} \mathbf{b}$. There will be at most one such value unless one or more of the components in $\mathbf{a}$ is n^*, in which case we must enumerate the different possible choices $a_i - s_i = n - s_1, \ldots, n-1, n^*$ as in Lemma 6.

- If there exists some p for which $n_p > 0$ and $\mathbf{b}_{p,k} \in A^{\text{ext}}_{n,\mathcal{L}}$ for all $1 \leq k \leq n_p$, then any $\mathbf{a}'$ with $\text{Fin}(\mathbf{a}') = \mathbf{a}$ is in the set $\mathcal{A}_n(S, A, \mathcal{L})$, otherwise none of these $\mathbf{a}'$ are in $\mathcal{A}_n(S, A, \mathcal{L})$.

Note that this algorithm ensures a slightly stonger condition than we really need since we have assumed the packing used (and hence the $\mathbf{b}$) depends only on $\mathbf{a}$ and does not depend on the choice of cycle lengths $m_1, \ldots, m_s$. In practice this makes very little difference and the algorithm above is easier to implement on a computer. The computer program works using the finite set $\text{Fin}(\mathcal{A}_n(S, A, \mathcal{L}))$ at all times, but this set clearly determines $\mathcal{A}_n(S, A, \mathcal{L})$. Note that $\mathcal{A}_n(S, A, \mathcal{L}) \subseteq \mathcal{A}(S, A, \mathcal{L})$ for all n. As with $\stackrel{G}{\to}$ we define $\mathcal{A}_n(G, A, \mathcal{L})$ as $\mathcal{A}_n(S, A, \mathcal{L})$ with S equal to the set of all suitable packings of G. We shall also write $\mathcal{A}_n(S, A, \geq m)$ when the set $\mathcal{L} = \{m, m+1, \ldots\}$.

Lemma 16 *There exists a non-empty set $\mathcal{A}' \subseteq \mathbb{N}^4$ such that for any $\mathbf{a} \in \mathcal{A}'$ and any $m_1, \ldots, m_4 \geq 10$ we can pack $O^{\cdot r}.\{\mathbf{a}\}$ together with cycles $C_{m_1}, \ldots, C_{m_s}$ for some $0 \leq s \leq 4$ into some $O^{\cdot r+1}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}'$. Moreover, $\mathbf{0} \in \mathcal{A}'$ and $\mathbf{a} \in \mathcal{A}'$ for all $\mathbf{a}$ with $\min \mathbf{a} > 4$ and $\max \mathbf{a} > 12$.*

Proof. The proof is similar to Lemma 6. The main difference is that some of the packings used start more than one new cycle at a time, so we may need up to four new cycles. Note in particular that we include the [**0**+**s**] and [**s**+**0**] of Lemma 5 or Lemma 15 which start or stop four cycles simultaneously. Define $\mathcal{A}'_0 = \mathbb{N}^4$ and then inductively define $\mathcal{A}'_j = \mathcal{A}_n(O, \mathcal{A}'_{j-1}, \geq 10)$. Eventually $\mathcal{A}'_j = \mathcal{A}'_{j+1}$ and for this j we define $\mathcal{A}' = \mathcal{A}'_j$. As a result, $\mathcal{A}' \subseteq \mathcal{A}(O, \mathcal{A}', \geq 10)$ and hence by definition we can pack $O^{\cdot r}.\{\mathbf{a}\}$ together with some given cycles of lengths at least 10 into some $O^{\cdot r+1}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}'$. A computer calculation shows that the set $\mathcal{A}'$ is non-empty when $n = 13$. The $\mathcal{A}'$ constructed by this method also satisfies the last conditions in the statement of the lemma so the result is proved. □

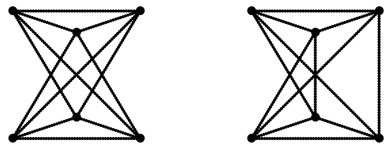
Rather than prove a stronger analogue of Lemma 12, we shall show that with longer cycles we can go from graphs with $\mathbf{a} \in \mathcal{A}'$ to graphs with $\mathbf{a} \in \mathcal{A}$ where $\mathcal{A}'$ is the set constructed in Lemma 16 and $\mathcal{A}$ is the set constructed in Lemma 6.

Lemma 17 *If $\mathbf{a} \in \mathcal{A}'$ and $m_1, \ldots, m_4 \geq 72$ then we can pack $O^{\cdot r}.\{\mathbf{a}\}$ and cycles $C_{m_1}, \ldots, C_{m_s}$ for some $0 \leq s \leq 4$ into a graph of the form $O^{\cdot r+7}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}$.*

Proof. Using the set $\mathcal{A}$ constructed in Lemma 6 set $\mathcal{A}_0 = \mathcal{A}$. Then for each $j > 0$ define $\mathcal{A}_j = \mathcal{A}_n(O, \mathcal{A}_{j-1}, \geq 72)$. Repeat this process until $\mathcal{A}' \subseteq \mathcal{A}_j$. This indeed occurs when $j = 7$ and

$n = 26$ and hence shows that for any $\mathbf{a} \in \mathcal{A}'$ we can pack $O^{\cdot r}.\{\mathbf{a}\}$ together with some cycles of length at least 72 into some graph of the form $O^{\cdot r+7}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}$. Finally, it is clear that at most four cycles will be used since a cycle of length at least 72 cannot be packed completely into just seven linked octahedra. The result follows. □

We now need to deal with cycles of lengths less than 10. Define $\widetilde{O}$ to be the octahedron $K_{2,2,2}$ but with different link vertices. The initial link will be the first class of this tripartite graph as before, but the final link will consist of two vertices one from each of the other two classes. As a result the final link will contain an edge.



O $\qquad$ $\widetilde{O}$

Lemma 18 *If* $3 \leq m_1, \ldots, m_s \leq 9$ *and* $m'_1, \ldots, m'_4 \geq 10$ *then we can pack cycles of lengths* $m_1, \ldots, m_s, m'_1, \ldots, m'_{s'}$ *for some* $0 \leq s' \leq 4$ *into some graph of the form* $O^{\cdot r}.\{\mathbf{a}\}$ *or* $O^{\cdot r-1}.\widetilde{O}.\{\mathbf{a}\}$ *with* $\mathbf{a} \in \mathcal{A}'$.

Proof. Once again the result uses computer verification. We shall constuct packings of $O^{\cdot r}$ of the form $[\mathbf{0}+\mathbf{b}] + \sum C_{l_i}$. We put these packings into two lists, the first will contain the packings with $\mathbf{b} \neq \mathbf{0}$ and the second will contain the packings of complete cycles only ($\mathbf{b} = \mathbf{0}$). For $r = 1$ we use the packings of Lemma 15. For $r > 1$ we take each such packing of $O^{\cdot r-1}$ in turn and for each packing of O given by Lemma 15, we combine the two packings in every possible way to get packings of $O^{\cdot r}$. Throw away any resulting packings that involve cycles or paths of lengths more than 9 and remove any repetitions of packings already achieved. We also discard the packing if the completed cycles $\sum C_{l_i}$ contain a subset that can be packed completely into some $O^{\cdot s}$ with $1 \leq s < r$ (and hence occur already in the second list). Any remaining packings are added to one of the two lists depending on whether $\mathbf{b} = \mathbf{0}$ or not. This process eventually terminates since for large r all the packings contain subsets of cycles that can be packed exactly (this is not meant to be obvious, but running the computer program shows it to be true). It can now be checked that we have exact packings into some $O^{\cdot r}$ of four C_3's, three C_4's, twelve C_5's, two C_6's, twelve C_7's six C_8's, or four C_9's respectively. Hence by packing any of these combinations we can assume we have at most eleven of each type of cycle. If our original collection of cycles contains a subset that can be packed exactly into some $O^{\cdot s}$, then we pack these at the begining of the trail of octahedra. Hence by induction we can assume that no subset of cycles packs some $O^{\cdot s}$ exactly. Now list each possible combination of cycles of length ≤ 9 for which there is no subset that can be packed exactly into some $O^{\cdot s}$, $s \geq 1$. This list is clearly finite. For each of these find some packing $[\mathbf{0}+\mathbf{b}] + \sum C_{l_i}$ constucted above for which the C_{l_i}'s are a subset of the C_{m_i}'s. Check if the remaining C_{m_i} can packed by adding a graph $\{\mathbf{b}'\}$ to $O^{\cdot r}$, in other words if the lengths of the remaining cycles are the set of values $b_i + b'_i$ for $b'_i > 0$. Now check that $\mathbf{b} + \mathbf{b}'$ lies in the set $A^{\text{ext}}_{13,\geq 10}$ constructed from $\mathcal{A}'$ above. If it does, then we can add some of the cycles of length ≥ 10 to get a packing into some $O^{\cdot r}.\{\mathbf{a}\}$. Running this algorithm shows that this succeeds except for the following combinations of small cycles.

$(3,3,3,9)$	$(3,3,3)$	$(3,6,9)$	$(3,6)$	$(3,7,7,7)$	$(3,7,7,8)$
$(3,7,7)$	$(3,7,8)$	$(3,7,9,9,9)$	$(3,7,9,9)$	$(3,7,9)$	$(3,7)$
$(3,8,8,9,9)$	$(3,8,8,9)$	$(3,8,8)$	$(3,8,9,9,9)$	$(3,8,9,9)$	$(3,8,9)$
$(3,8)$	$(3,9,9,9)$	$(3,9,9)$	$(3,9)$	(3)	

For these we modify the last octahedron to $\widetilde{O}$. Once this is done, these exceptional combinations can also be packed into $\widetilde{O}.\{\mathbf{a}\}$. Indeed, we only require the following two packings of $\widetilde{O}$ which are easily seen to exist.

$$[0{+}1, 0{+}2] + 3C_3, \qquad [0{+}1, 0{+}2, 0{+}3, 0{+}3] + C_3$$

□

We now need the following theorem, which is an immediate consequence of a result of Caro and Yuster (Theorem 4.1 of [7]). This result is proved using a theorem of Gustavsson [8] which in turn is a generalization of Wilson's Decomposition theorem [16]. The upper bound on $\epsilon(L)$ in Theorem 19 is just for convienience of use in the next corollary. In practice the $\epsilon(L)$ given by [7] is extremely small. Write $\delta(G)$ for the minimum degree of the vertices in G.

Theorem 19 *There exist constants $N(L)$ and $\frac{1}{8} > \epsilon = \epsilon(L) > 0$ depending only on L such that if G is a graph with $|V(G)| \geq N(L)$, $\delta(G) \geq (1-\epsilon)|V(G)|$ and every vertex is of even degree and if $C_{m_1}, \dots, C_{m_t}$ is a collection of cycles with $3 \leq m_i \leq L$ and $\sum_{i=1}^{t} m_i = |E(G)|$ then there exists a packing of $C_{m_1}, \dots, C_{m_t}$ into G.*

Corollary 20 *If $n \geq N(L)$, $\sum_{i=1}^{t} m_i = |E(K'_n)|$, $\sum_{m_i > L} m_i \leq \frac{n}{8}(\epsilon(L)n - 4)$ and $3 \leq m_i \leq \frac{n}{2}$ then we can pack cycles $C_{m_1}, \dots, C_{m_t}$ into K'_n*

Proof. We start by packing the large cycles C_{m_i} with $m_i > L$ into K'_n in such a way that no vertex is used too often. To be precise, the remaining degree at each vertex after removing these cycles will be at least $(1-\epsilon)n$ where $\epsilon = \epsilon(L)$. We use a greedy algorithm. Assume we have packed some cycles already and we now need to pack C_{m_i}. Let S be the set of vertices at which the degree is at least $(1-\epsilon)n + 2$ after removing the cycles already packed. Since each of the other vertices must meet more than $\frac{1}{2}(\epsilon n - 4)$ of the packed cycles, $\frac{n}{8}(\epsilon n - 4) \geq \sum_{m_i > L} m_i \geq \frac{1}{2}(n - |S|)(\epsilon n - 4)$. Hence $|S| \geq 3n/4$. Pick any vertex v_1 in S and inductively choose $v_r \in S$ so that the vertex v_r and edge $v_{r-1}v_r$ have not been used yet. At each stage there are at least $|S| - (r-1) - (\epsilon n - 4) > \frac{n}{8} + 5$ choices for v_r. Finally for v_{m_i} we also need the edge $v_1 v_{m_i}$ to be unused. For this there are $|S| - (m_i - 1) - 2(\epsilon n - 4) > 9$ choices. Add the cycle $v_1, \dots, v_{m_i}$ to our packing. Repeat this process until no more cycles are left. Now use Theorem 19 to pack the cycles of length $\leq L$ into the the remaining edges of K'_n. □

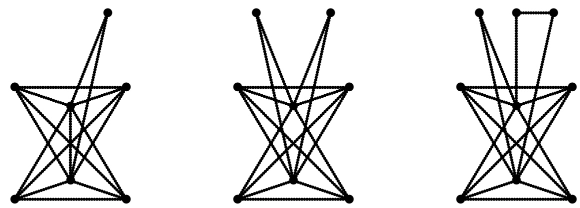
Proof. of Theorem 2.
If $n \geq 4$ then $m_i \leq \lfloor \frac{n+37}{20} \rfloor \leq \frac{n}{2}$. By Corollary 20 with $L = 71$ we can assume there are more than $\frac{n}{8}(\epsilon(71)n - 4)$ edges to be packed in cycles of length at least 72. As in Theorem 1 we pack a trail of octahedra. First we pack the cycles of length less than 10 using Lemma 18. We then use Lemma 16 to inductively pack those cycles of length between 10 and 71. By using at most four cycles of length ≥ 72, we can ensure that all cycles of length less than 72 are used. (Since

Lemma 16 can use up to four cycles, we may need some longer cycles to guarantee the last few short cycles are packed, also, if there are no cycles of length between 10 and 71 we may need up to four longer cycles in Lemma 18.) Using four more cycles and Lemma 17 we get a packing of cycles into some $O^{\cdot r}.\{\mathbf{a}\}$ with $\mathbf{a} \in \mathcal{A}$ (or a similar graph in which one octahedron is replaced with $\widetilde{O}$). Since we have used at most eight cycles of length ≥ 72 and all cycles are of length at most $\frac{n}{2}$, the remaining cycles will have total length of at least $\frac{n}{8}(\epsilon(71)n-4)-8(\frac{n}{2})$. Now use Corollary 13 to pack the remaining octahedra. This works provided $\frac{n}{8}(\epsilon(71)n-4)-8(\frac{n}{2}) \geq 40(\frac{n}{2})$ or equivalently $n \geq 196/\epsilon(71)$. As in the proof of Theorem 1, packing the octahedra into K'_n gives the required packing of cycles. The only remaining complication is when one of the octahedra is replaced with $\widetilde{O}$ in Lemma 18. The only two edges that are in $\widetilde{O}$ but not O are edges joining doubled points, so are in the 1-factor I in $K'_n = K_n - I$. Hence by slightly modifying this 1-factor we can pack one $\widetilde{O}$ in place of one of the O's. The value of N_1 can be taken as $\max(N(71), 196/\epsilon(71))$. □

5 Proof of Theorem 3

In this section we shall remove the congruence condition on n to obtain Theorem 3. To do this we will divide the vertices V of K'_n as $V = V_0 \cup V_1$ where $|V_0| = 2n_0 \equiv 2 \bmod 144$ and $|V_1| = n_1$ is small. For technical reasons some small values of n_1 are not allowed and so we shall insist that $6 \leq n_1 \leq 149$. The edges of K'_n now consist of the edges of K'_{2n_0}, the edges of K'_{n_1}, the edges of a bipartite graph K_{2n_0,n_1} and (if n is odd) the missing 1-factor I of K'_{2n_0}.

Define the graphs $O_{[a,b]}$ as O with two paths of lengths a and b joining the two non-link (middle) vertices of O.



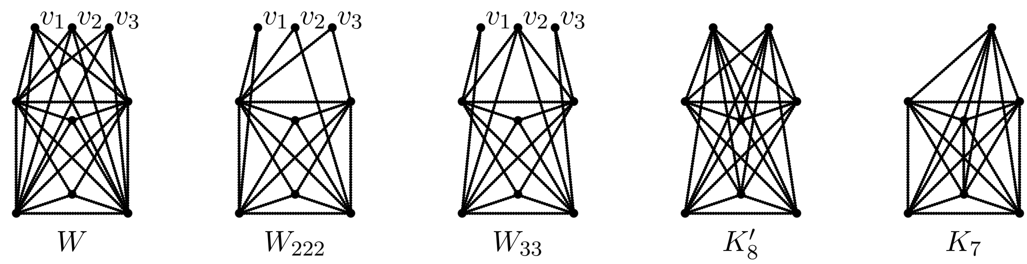
$O_{[1,2]}$ $O_{[2,2]}$ $O_{[2,3]}$

The general strategy is as follows. To include the edges not in K'_{2n_0} we shall add "detours" to the octahedra packing K'_{2n_0}. If n_1 is odd we shall replace some octahedra with $O_{[1,2]}$. The paths of length 1 give us the missing 1-factor of K'_{2n_0} and the paths of length 2 join vertices in K'_{2n_0} to one of the vertices of K'_{n_1}. Replacing some more octahedra with $O_{[2,3]}$ we can use up the edges in K'_{n_1}. Each path of length 3 will use one edge in K'_{n_1} and two edges in the bipartite graph. Finally, replacing yet more octahedra with $O_{[2,2]}$ will use up the remaining edges of the bipartite graph. Hence, provided we choose the modified octahedra carefully, we shall be able to use up all the edges in K'_n. Unfortunately, we now need to pack cycles into these modified octahedra, and for very short cycles we shall need to introduce several alternative modifications to the octahedra.

Lemma 21 *Assume $(a,b) = (1,2)$, $(2,2)$ or $(2,3)$. If $\mathbf{a} \in \mathcal{A}'$ and $m_1, \dots, m_{32} \geq 12$ then we can pack $O^{\cdot r}.\{\mathbf{a}\}$ and cycles $C_{m_1}, \dots, C_{m_s}$ for some $0 \leq s \leq 32$ into a graph of the form $O^{\cdot r+7}.O_{[a,b]}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}'$.*

Proof. First we construct the packings of $O_{[a,b]}$ corresponding to the table in Lemma 15 above for O. The algorithm used is identical to that of Lemma 15. We shall not list all the packings here as there are somewhat more of them than for O. Using the set $\mathcal{A}'$ constructed in Lemma 16, construct $\mathcal{A}''_0 = \mathcal{A}_n(O_{[a,b]}, \mathcal{A}', \geq 12)$. Then for $j = 1, \ldots, 7$ define $\mathcal{A}''_j = \mathcal{A}_n(O, \mathcal{A}''_{j-1}, \geq 12)$. Finally for $n = 15$ and each choice of (a, b) it can be checked by computer that $\mathcal{A}' \subseteq \mathcal{A}''_7$. As a result, we can pack $O^{\cdot r}.\{\mathbf{a}\}$ and cycles C_{m_i} into $O^{\cdot r+7}.O_{[a,b]}.\{\mathbf{b}\}$ for some $\mathbf{b} \in \mathcal{A}'$. Clearly at most 32 cycles C_{m_i} will be required. The result follows. □

We now need to deal will cycles of length less than 12. Since the $O_{[2,3]}$'s are rare, we shall only need to pack $O_{[1,2]}$'s and $O_{[2,2]}$'s. Unfortunately we shall also need to consider some other graphs when the cycle lengths are very short. These other graphs are shown below. In each of these the initial link is the leftmost pair of vertices and the final link is the rightmost pair. The graph W is obtained by adding three vertices v_1, v_2, v_3 to O and joining them to the initial and final link vertices. Two additional edges are also added, one between the two initial link vertices and one between the two final link vertices. The graphs W_{222} and W_{33} are obtained by removing a C_6 from W. The graph W_{222} has the missing C_6 meeting each of the vertices v_1, v_2, v_3 so that there are paths of length 2 missing from v_1 to v_2, from v_2 to v_3 and from v_3 to v_1. The graph W_{33} has two paths of length 3 removed between v_1 and v_3 say. The graphs W and W_{33} are symmetric under reversal (interchange of initial and final links), however W_{222} is not. When we write W_{222} in the following lemma we mean than some choice of either W_{222} or its reversal will make the result true.



Define initial and final links of $K'_8 = K_{2,2,2,2}$ as two distinct vertex classes. By symmetry it does not matter which two classes are used or the order of the two vertices in each link. Define initial and final links of K_7 as any two disjoint pairs of vertices. Once again, by symmetry it does not matter which pairs are chosen.

Lemma 22 *Let $3 \leq m \leq 11$. Assume G is a graph of the form $G_1.G_2 \ldots G_s$ and one of the following conditions hold:*

1. *for all i, $G_i \in \{O, O_{[2,2]}, O_{[1,2]}\}$ and $m \in \{9, 10, 11\}$,*
2. *for all i, $G_i \in \{O, O_{[2,2]}, W_{222}, W_{33}\}$ and $m \in \{4, 6, 8\}$,*
3. *for all i, $G_i \in \{O, O_{[2,2]}, W\}$ and $m \in \{6, 7\}$,*
4. *for all i, $G_i \in \{O, K_7, K'_8\}$ and $m \in \{3, 5\}$.*

Assume also that if $G_i \neq O$ then $i \geq 6$ and if $G_i, G_j \neq O$ then either $i = j$ or $|i - j| \geq 6$ (or $i \geq 12$ and $|i - j| \geq 12$ respectively when $m = 11$), so the non-O graphs are well separated from each other and from the begining of the sequence. If $|E(G)|$ is divisible by $\gcd(m, 12)$, then we can pack some graph $G.O^{\cdot r}$ with $0 \leq r \leq 10$ completely with cycles of length m. Also, if $m \in \{3, 4, 6\}$ we can take $r = 0$.

Proof. For each m we define two finite sets $\mathcal{A}_0$ and $\mathcal{A}_+$ with $\mathcal{A}_0 \subseteq \mathcal{A}_+ \subseteq \mathbb{N}^4$ as follows:

m	$\mathcal{A}_+$, (underlined elements lie in $\mathcal{A}_0$)
11	$\{\underline{(0,0,0,0)}, (8,8,8,8), (5,5,5,5), (2,2,2,2), (0,0,9,9), (7,7,7,7), (4,4,4,4), (0,0,2,2), (9,9,9,9), (6,6,6,6), (3,3,3,3)\}$
10	$\{\underline{(0,0,0,0)}, (6,6,8,8), (2,2,6,6), (4,4,8,8), (2,2,4,4), \underline{(0,0,2,3)}, (0,0,6,7), (2,3,8,8), (0,0,4,5), (2,3,6,6)\}$
9	$\{\underline{(0,0,0,0)}, (6,6,6,6), (3,3,3,3), \underline{(0,0,2,2)}, (7,7,7,7), (4,4,4,4), \underline{(0,0,4,4)}, (0,0,7,7), (3,3,7,7)\}$
8	$\{\underline{(0,0,0,0)}, (4,4,6,6), (0,0,4,4), (2,2,4,4)\}$
7	$\{\underline{(0,0,0,0)}, (4,4,4,4), (0,0,2,2), (5,5,5,5), (2,2,2,2), (0,0,5,5), (3,3,3,3)\}$
6	$\{\underline{(0,0,0,0)}, \underline{(0,0,2,2)}, \underline{(0,0,4,4)}\}$
5	$\{\underline{(0,0,0,0)}, (2,2,2,2), (0,0,3,3), (0,0,2,2), (3,3,3,3)\}$
4	$\{\underline{(0,0,0,0)}\}$
3	$\{\underline{(0,0,0,0)}\}$

The set $\mathcal{A}_0$ is defined as the subset of $\mathcal{A}_+$ which occurs underlined in the above table. We first check that for each m and any $\mathbf{a} \in \mathcal{A}_+$, the graph $O^{\cdot t}.\{\mathbf{a}\}$ can be packed with possibly some C_m's into some graph $O^{\cdot t+1}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}_+$. In terms of the notation above, it is enough that $\mathcal{A}_+ \subseteq \mathcal{A}_m(O, \mathcal{A}_+, \{m\})$. This holds for $m \geq 7$, however, for $m \leq 6$ we also need to use some packings with embedded C_m's in the octahedra. Define $S_{O,m}$ to be the set of packings of O together with the set of packings of O with one or more C_m's removed. So, for example, since there is a $[2,2] + 2C_4$ packing of O, the packing $[2,2]$ will occur in $S_{O,4}$. It is now clearly sufficient that $\mathcal{A}_+ \subseteq \mathcal{A}_m(S_{O,m}, \mathcal{A}_+, \{m\})$. This holds for all m and $\mathcal{A}_+$ in the table above. We also check that with enough additional C_m's we can pack $O^{\cdot t}.\{\mathbf{a}\}$ into some $O^{\cdot t+r}.\{\mathbf{b}\}$ with $\mathbf{b} \in \mathcal{A}_0$ and $0 \leq r \leq 10$. Both these facts can be verified by hand easily from the above table and the packings of Lemma 15.

Now let H be one of the other possible graphs listed above. Let $S_{H,m}$ be the set of packings of H, where we are allowed to remove C_m's from H. For example, the $[0{+}4, 0{+}4] + 2C_6$ packing of W_{222} exists, so we can include $[0{+}4, 0{+}4]$ in $S_{H,m}$ when $H = W_{222}$ and $m = 6$. For $H = W_{222}$ we include in $S_{H,m}$ packings of both W_{222} and its reversal. Define $\mathcal{A}'_1 = \mathcal{A}_m(S_{H,m}, \mathcal{A}_+, \{m\})$. Then define $\mathcal{A}'_j = \mathcal{A}_m(S_{O,m}, \mathcal{A}'_{j-1}, \{m\})$ for $j \geq 2$. It can be checked (by computer) that $\mathcal{A}_+ \subseteq \mathcal{A}'_{12}$ always, and $\mathcal{A}_+ \subseteq \mathcal{A}'_6$ except in the case $m = 11$. Note that the cases when $H = W$ are trivial, since one can remove two C_7's from W to get O or one C_6 from W to get either W_{222} or W_{33}.

As a consequence of these results, if $\mathbf{a} \in \mathcal{A}_+$ then we can pack $\{\mathbf{a}\}$ and C_m's into graphs of the form $O.\{\mathbf{b}\}$ and $O^{\cdot r}.H.\{\mathbf{b}\}$ for any $r \geq 5$ ($r \geq 11$ if $m = 11$), any H and some $\mathbf{b} \in \mathcal{A}_+$ with the initial link of $\{\mathbf{a}\}$ mapped to the initial link of these graphs. Hence by induction (and using $\mathbf{0} \in \mathcal{A}_+$) we can pack $G.\{\mathbf{b}\}$ completely with m-cycles for some $\mathbf{b} \in \mathcal{A}_+$. Finally, by adding more octahedra, we can pack $G.O^{\cdot r}.\{\mathbf{b}\}$ with m-cycles for some $\mathbf{b} \in \mathcal{A}_0$. For $m \in \{3,4,6\}$, $\mathcal{A}_0 = \mathcal{A}_+$ so we can take $r = 0$. By assumption, $\gcd(m, 12)$ divides $|E(G)|$. Since $\gcd(m, 12)$ also divides $|E(O)|$ and $|E(C_m)|$, it must divide $|E(\{\mathbf{b}\})| = \sum \mathbf{b}$. By inspection of the table above, it is clear that in each case $\mathbf{b} = \mathbf{0}$ and we are done.

For ease of checking, the following table lists the packings of W, W_{222}, W_{33}, K_7 and K'_8 needed in this proof. They can all be generated by the algorithm of Lemma 15 or constructed by hand.

The required packings of $O_{[2,2]}$, $O_{[1,2]}$ are much more numerous, so are not listed here.

W	$O + 2C_7$ $\quad W_{222} + C_6$ or $W_{33} + C_6$
W_{222}, W_{33}	$[] + 5C_4$ $\quad [0{+}4, 0{+}4] + 2C_6$ $\quad [2{+}2, 2{+}2] + 2C_6$
	$[0{+}4, 0{+}4, 2, 2] + C_8$ $\quad [0{+}4, 0{+}4, 0{+}2, 0{+}2] + C_8$
K_7	$[] + 7C_3$ $\quad [1{+}1, 2{+}2] + 3C_5$
	$[2{+}3, 2{+}3, 0{+}3, 0{+}3] + C_5$ $\quad [0{+}3, 0{+}3] + 3C_5$
K'_8	$[] + 8C_3$ $\quad [0{+}2, 0{+}2] + 4C_5$
	$[3{+}2, 3{+}2, 0{+}2, 0{+}2] + 2C_5$ $\quad [3{+}3, 3{+}3, 3{+}3, 3{+}3]$

□

Note that in Lemma 22, no cycle C_m will meet more than one G_i with $G_i \neq O$ since these G_i are separated by at least $m/2$ octahedra. It is also worth noting that for each m with $3 \leq m \leq 11$, one can pack $12/\gcd(m, 12)$ m-cycles into $O^{\cdot m/\gcd(m,12)}$ except for the case $m = 8$ when we need 6 C_8's to pack $O^{\cdot 4}$.

We now need to modify the octahedra as described at the beginning of this section so that we pack the whole of K'_n. For this we need some extra properties of the self-avoiding trail of triangles constructed in Section 3. In analogy to the situation that occurs when we double vertices, the vertices of each triangle in Theorem 14 that meet the previous or subsequent triangle will be called *link vertices* and the other vertex will be called the *non-link vertex* or *midvertex* of the triangle.

Lemma 23 *There exists a constant c_1 such that for all $n \equiv 1 \bmod 72$ and any vertex v of K_n, any $c_1 n$ consecutive triangles in the trail of triangles constructed in Theorem 14 contains a triangle with v as its midvertex.*

Proof. We shall just consider the midvertices of the triangles $T_{12mj+6i+2}$. These are of the form $10r + 4i + 1 - mj \bmod n$ with $0 \leq i < 2m$. Taking any $12m$ consecutive triangles T_k it is clear that for some j we have triangles of the form $T_{12mj+6i+2}$ for at least m consecutive values of i. Hence any $12m$ consecutive triangles contain every fourth vertex in a contiguous block of length $4m$ as a midvertex of one of these triangles. Since T_{12m+k} is a translate of T_k by $-m$ and $n = 72m+1$, it is now clear that any consecutive set of $72(12m)$ triangles T_k contains at least one out of every four consecutive vertices as a midvertex. But $T_{72(12m)+k}$ is a translate of T_k by $-72m \equiv 1 \bmod n$, so it is now clear that any consecutive sequence of $4(72)(12m) < 48n$ triangles contains every vertex as a midvertex. The result follows with $c_1 = 48$. □

Lemma 24 *Let $T_2, \ldots, T_{12mn+1}$ be the sequence of triangles constructed in Theorem 14 and assume $r \leq \frac{m-9}{6}$. Then there exists disjoint collections $S_1, \ldots, S_r$ of these triangles with the following properties:*

1. *all triangles in $\cup S_i$ lie in the first $4mn$ triangles of the sequence,*
2. *no two triangles in $\cup S_i$ are closer than 6 apart in the sequence,*
3. *for each i, the triangles of S_i are vertex-disjoint,*
4. *there are at least $24m - 144r - 143$ triangles in each S_i.*

Proof. Let i be any one of the r integers $\lfloor\frac{m-6r}{9}\rfloor+1, \lfloor\frac{m-6r}{9}\rfloor+2, \ldots, \lfloor\frac{m+3r}{9}\rfloor$, and consider the translates $T_{12mj+6i+2}$ of the triangle $T_{6i+2} = (i, 30m-2i-1, 30m+4i+1)$ of Theorem 14. All such triangles are at least 6 apart in the sequence, so condition 2 will hold. If we fix i and if two translates $T_{12mj+6i+2}$ and $T_{12m(j+k)+6i+2}$ meet, then one of the following equations must be satisfied:

$$\begin{aligned} 30m-2i-1 &\equiv i \pm mk \mod n, \\ 30m+4i+1 &\equiv i \pm mk \mod n, \\ 30m-2i-1 &\equiv 30m+4i+1 \pm mk \mod n. \end{aligned}$$

Multiplying by 72 and using $72m \equiv -1 \mod n$ we get

$$\pm k \equiv 102 + 72(3i) \text{ or } 42 + 72(3i) \text{ or } n - 144 - 72(6i) \mod n$$

Now $24m - 144r \leq 72(3i) \leq 24m + 72r < n/2 - 102$, so

$$k \geq \min(24m - 144r + 102, 24m - 144r + 42, 24m - 144r - 143) = 24m - 144r - 143$$

Hence if we take $24m - 144r - 143$ consecutive triangles of this form they will be vertex-disjoint. Finally, any $4mn$ consecutive triangles of the trail contain at least $\lfloor n/3\rfloor = 24m$ such triangles, so we can assume they all lie within the first $4mn$ triangles. Since there are r distinct values of i possible, we get r disjoint collections of such triangles. □

Lemma 25 *Let $T_2, \ldots, T_{12mn+1}$ be the sequence of triangles constructed in Theorem 14 and assume $m \geq 36$. Then there exists disjoint subsets S_i of these triangles for $0 \leq i \leq \frac{m-10}{4}$ with the following properties:*

1. *all triangles in $\cup S_i$ lie in the first $6mn + 211n$ triangles of the sequence,*
2. *no two triangles in $\cup S_i$ are closer than 12 apart in the sequence,*
3. *no two triangles in any individual S_i are closer than $5m$ apart in the sequence,*
4. *the midvertices of the triangles in each S_i, $i \geq 1$ are distinct and enumerate all the vertices of K_n,*
5. *all the link vertices of triangles in S_0 are distinct and enumerate all but at most 3827 of the vertices of K_n,*
6. *if v occurs as a link vertex of $T \in S_0$ and as a mid-vertex of $T' \in S_1 \cup S_2$ then T' occurs after T in the sequence.*

Proof. Once again we consider just the translates of the triangles $T_{6i+2} = (i, 30m-2i-1, 30m+4i+1)$. Let $i_0 = \lfloor\frac{9m-28}{8}\rfloor$, $j_0 = 288i_0 - 4n$, $i_1 = \lfloor\frac{m-10}{2}\rfloor$. Define the sets S_i as

$$\begin{aligned} S_i &= \{T_{12mj+6(2i-2)+2} : j = 0, 1, \ldots, j_0 - 1\} \\ &\quad \cup \{T_{12m(j-j_0)+6(i_0+2i-2)+2} : j = j_0, \ldots, n-1\} \qquad (i \geq 1) \\ S_0 &= \{T_{12mj+6i_1+2} : j = 0, \ldots, 36m - 1914\} \end{aligned}$$

2. It is enough to show that the numbers $2i-2$, i_0+2i-2 and i_1 are all at least 2 apart and between 0 and $2m-2$. Since $1 \le i \le r$ it is enough that

$$2r-2 \le i_1-2, \quad i_1 \le i_0-2 \quad \text{and} \quad i_0+2r-2 \le 2m-2.$$

Substituting the definitions of i_0 and i_1 gives the following sufficient conditions

$$r \le \frac{m-10}{4}, \quad 4m-40 \le 9m-44 \quad \text{and} \quad r \le \frac{7m}{16}.$$

All are satisfied when $1 \le r \le \frac{m-10}{4}$.
1. Note that $36m-1264 \le j_0 \le 36m-1012$, so if $T_k \in S_i$ and $i \ge 1$ then

$$k \le 12m \max(j_0+1, n-j_0) \le 12m(36m+1265) \le 6mn+211n.$$

Also, $j_0 > 0$ when $m \ge 36$. The result for S_0 is clear.
3. Is clear for S_0, and is true for S_i, $i > 0$ provided $5m \le 6i_0 \le 12m-5m$. However, $m-4 \le i_0 \le 9m/8$ and $5m \le 6(m-4)$, $6(9m)/8 \le 12m-5m$ when $m \ge 36$.
4. The midvertices are $30m+8(i-1)+1-mj$ for $j < j_0$ or $30m+8(i-1)+4i_0+1-m(j-j_0)$ for $j \ge j_0$. However $4i_0+mj_0 \equiv 0 \bmod n$ so both these expressions are $30+8(i-1)+1-mj \bmod n$. Since m is relatively prime to n, this enumerates all the numbers mod n as j ranges from 0 to $n-1$.
5. The link vertices are i_1-mj and $30m-2i_1-1-mj$. As j runs through the numbers mod n, it is clear that all the vertices i_1-mj are distinct mod n. Similarly the vertices $30m-2i_1-1-mj$ are all distinct mod n. If $i_1-mj \equiv 30m-2i_1-1-m(j\pm k) \bmod n$ then $\pm k \equiv 72(3i_1)+42$. so $\pm k \equiv 72(3\lfloor m/2 \rfloor - m) - 466$ and $|k| \ge 36m-574$. Hence all the link vertices are distinct. Since there are $36m-1913$ triangles, the link vertices enumerate all but $n-2(36m-1913)=3827$ of the vertices of K_n.
6. Consider a triangle in S_0. The link vertices are i_1-mj and $30m-2i_1-1-mj$ and these are the midvertices $30m+8(i-1)+1-mj'$ in S_i for some j' by part 4. If $i_1-mj \equiv 30m+8(i-1)+1-mj'$ and $i=1$ or 2 then it can be checked that $34 \le j'-j-j_0 \le 826$. If $30m-2i_1-1-mj \equiv 30m+8(i-1)+1-mj'$ and $i=1$ or 2 then it can be checked that $1 \le j'-j \le 649$. In either case the triangle in S_1 or S_2 occurs after that of S_0 provided $j < 36m-1913 \le \min(j_0-649, n-j_0-826)$. □

Now we turn to the proof of Theorem 3. The following lemma covers the cases when there are many 3 and 5-cycles.

Lemma 26 *There exist absolute constants c_2 and c_3 such that if $\sum_{m_i\in\{3,5\}} m_i \ge \frac{1}{3}\binom{n}{2} + c_2 n$ and $\sum_{m_i \ge 72} \ge c_3 n$ then the conclusion of Theorem 3 holds.*

Proof. Write $n = 2n_0+n_1$ with $n_0 = 72m+1$ and $6 \le n_1 \le 149$. We shall take $c_3 = 1.4\times 10^9$. Then $\binom{n}{2} \ge c_3 n$, so $n > 2c_3 > 10^9$ and $m > 10^6$. Construct the self-avoiding trail of triangles in K_{n_0} as before using Theorem 14. Let $r = \lceil \frac{n_1}{2} \rceil \le 75$ and write the vertices of K'_{n_1} as r disjoint pairs of vertices P_i, (or one singleton P_1 and $r-1$ pairs $P_2,\dots,P_r$ if n is odd). Since $r < 10^5 < \frac{m-9}{6}$, we can construct r disjoint collections S_i of triangles using Lemma 24. Doubling up each vertex in K_{n_0} gives us a trail of octahedra in K'_{2n_0} as in the proof of Theorem 1. The collections S_i are now collections of octahedra. We shall now modify the octahedra in the trail given by Theorem 14 so

as to include all the additional edges.
First we deal with the edges in K'_{n_1}. Since $n_1 \geq 6$ we can decompose K'_{n_1} into cycles of length at most $\frac{n_1}{2}$. (For $n_1 = 6, 7$ we can decompose K'_{n_1} into triangles, for all larger n_1 we can decompose K'_{n_1} into triangles and squares by [5].) Furthermore, we can ensure that at least one of these cycles is a triangle. For each such cycle C_s pick a pair of non-adjacent vertices u_1 and u_2 in K'_{2n_0}. If the cycle is $v_1, \ldots, v_s, v_{s+1} = v_1$ then we can construct s paths $u_1 v_i v_{i+1} u_2$ of length 3. Now pick a set of vertices $v'_1, \ldots, v'_s$ in K'_{n_1} disjoint from the v_i (possible since $s \leq \frac{n_1}{2}$). Now construct s paths $u_1 v'_i u_2$ of length 2. By Lemma 23, the vertex pair $u_1 u_2$ must occur as non-link vertices of at least one out of any $c_1 n_0$ consecutive octahedra. We can modify one such octahedron by adding one of the length 3 paths and one of the length 2 paths so that it is now isomorphic to $O_{[2,3]}$. Repeat this process with a different octahedron for each of the paths until we run out of paths. Both u_1 and u_2 are now joined to the same set of vertices $\{v_1, \ldots, v_s, v'_1, \ldots, v'_s\}$ in K'_{n_1}. Now repeat this process with each of the other cycles that pack K'_{n_1} in turn until we have used up all the edges of K'_{n_1}. We use a different pair $u_1 u_2$ for each cycle. We still have a lot of choice as to which octahedra are modified this way. We shall choose these octahedra to be near the end of the sequence given by Theorem 14. To be more precise, the first such octahedron will be at least n_0 and at most $(1 + c_1)n_0$ from the end of the sequence. Each successive octahedra above will be at least n_0 and at most $(1 + c_1)n_0$ octahedra before the previous one. Since $m_i \leq \lfloor \frac{n-112}{20} \rfloor < 2n_0$, the modified octahedra will be at sufficient distance from one another so that when cycles are packed into the trail of octahedra, each cycle will encounter at most one modified octahedron (and so the cycle will not accidentally meet itself in K'_{n_1}). Note that we have only modified some of the last $|E(K'_{n_1})|(1 + c_1)n_0$ octahedra and we have also only involved at most $c' = \frac{1}{3}|E(K'_{n_1})|$ pairs $u_1 u_2$. Later on, it may be necessary to pack a single triangle $C_{m_i} = C_3$ into K'_{n_1}. If this happens, pick one of the cycles above with $s = 3$ and replace the paths $u_1 v_i v_{i+1} u_2$ with paths $u_1 v_i u_2$. This changes three $O_{[2,3]}$'s into $O_{[2,2]}$'s and frees up the triangle $v_1 v_2 v_3$ in K'_{n_1} without changing anything else.
Remove octahedra from S_i which meet any of the pairs $u_1 u_2$ used above. Since the octahedra in S_i are vertex disjoint, this removes at most c' octahedra from each S_i. Join P_i to each of the remaining octahedra in S_i. If n is odd and $i = 1$ then we also fill in the edges of the missing 1-factor of O to obtain K_7's. These octahedra now become K'_8's or K_7's and we have used most of the edges joining K'_{2n_0} to K'_{n_1} and most of the missing 1-factor I of $K'_{2n_0} = K_{2n_0} - I$ when n is odd. Each collection S_i of octahedra can miss up to $n_0 - 3(24m - 144r - 143 - c') = 432r + 430 + 3c'$ pairs of vertices in K'_{2n_0}. All the other vertices are joined to P_i and if n is odd and $i = 1$ then all other pairs of vertices in K'_{2n_0} are now joined to each other.
We now continue the algorithm above. If any edges remain then there must be an independent pair $u_1 u_2$ of vertices in K'_{2n_0} that have not been joined yet to all the vertices in K'_{n_1}. Both u_1 and u_2 are joined to the same set of vertices in K'_{n_1}, so there must be some pair v_1, v_2 in K'_{n_1} (or just one v_1) which has not yet been joined to either u_1 or u_2. Find some octahedron between n_0 and $(1 + c_1)n_0$ from the last octahedron modified at the begining of this proof with the pair $u_1 u_2$ as non-link vertices. Add two paths $u_1 v_1 u_2$ and $u_1 v_2 u_2$ (or the edge $u_1 u_2$ if there is no v_2) to the octahedra to get $O_{[2,2]}$ (or $O_{[1,2]}$). Eventually we will have used up all the remaining edges. The trail of octahedra has been modified so that some of the first $4mn_0$ octahedra have been modified to K'_8's or K_7's and some of the last $c'' n_0$ octahedra have been modified to $O_{[a,b]}$ with $(a, b) = (1, 2)$, $(2, 2)$ or $(2, 3)$ and $c'' \leq ((432r + 430 + 3c')r + |E(K'_{n_1})|)(1 + c_1) \leq 1.62 \times 10^8$. In each case the modified octahedra are well separated—at least 6 apart for the K'_8's and K_7's and at least n_0 apart for the $O_{[a,b]}$'s.

Pack the first $4mn_0+s$ octahedra for some $0 \le s \le 10$ with C_3's and C_5's according to Lemma 22. We pack the C_3's first, stopping when we have either packed more than $4mn_0$ octahedra, or if we run out of C_3's. We also stop if we are less than six octahedra from the next K_7 or K'_8 but do not have enough C_3's to pack the next K_7 or K'_8. By Lemma 22 we can stop packing C_3's at any point in the sequence and so there will be at most 27 unpacked C_3's left. Now pack the C_5's using Lemma 22 until we have packed at least the first $4mn_0$ graphs in the sequence and at most 10 more O's after these. This succeeds provided the total length of C_3's and C_5's is at least $12(4mn_0+10)+4rn_0+27(3) \le \frac{1}{3}\binom{n}{2}+c_2n$.
Pack the remaining cycles of length less that 72 using Lemma 18 and Lemma 16. As in the proof of Theorem 2, by using at most four cycles of length at least 72 we can use up all the cycles of length less than 72 and get a packing into some $G.\{\mathbf{a}\}$ with $\mathbf{a} \in \mathcal{A}'$ where G is some initial segment of our trail of modified octahedra. It is possible that we may pack a $\widetilde{O}$ in Lemma 18. To avoid this, remove one triangle try again. In each of the exceptional cases in Lemma 18, the removal of a single triangle will make that case non-exceptional, so we no longer need to use a $\widetilde{O}$. Pack this single triangle into K'_{n_1} instead, modifying three $O_{[2,3]}$'s into $O_{[2,2]}$'s as described above.
Provided the total length of cycles $m_i \ge 72$ is at least $4L+|E(O_{[2,3]})|(c''+1)n_0 \le c_3n$ we will not encounter any of the modified octahedra that occur near the end of the sequence, and we will still have at least $n_0 > 7$ unmodified octahedra remaining before the first of these modified octahedron. Now use Lemma 21 inductively to pack cycles of length at least 72 into the sequence of modified octahedra until we have reached the last modified octahedra. We now have a packing into some $G'.\{\mathbf{a}'\}$ where $\mathbf{a}' \in \mathcal{A}'$ and G is an initial segment of the sequence of octahedra that includes all the modified octahedra, but does not include the last n_0 octahedra in the sequence. Using at most four more cycles and Lemma 17 we can pack $G''.\{\mathbf{a}''\}$ with $\mathbf{a}'' \in \mathcal{A}$. Now use Corollary 13 to complete the packing. In Corollary 13 we need the remaining cycles to be of length at least $40L$. The remaining cycles are of length at least $12n_0 - \sum \mathbf{a}' \ge 12n_0 - 4L - \sum \mathbf{a}'' \ge 12n_0 - 8L$. Hence we require $48L \le 12n_0$. However, $48L \le 48\lfloor\frac{2n_0+37}{20}\rfloor < 12n_0$ when $n_0 \ge 13$. As in Theorem 1, the cycles are packed properly in K'_n since each cycle can meet at most $\lfloor\frac{m_i}{2}\rfloor$ octahedra and the trail of octahedra is $\lfloor\frac{n_0+18}{20}\rfloor$-self-avoiding. Also, the excursions into K'_{n_1} are sufficiently far apart that no cycle meets itself in K'_{n_1}. The result is now proved. A simple calculation shows that we can take $c_3 = 1.4 \times 10^9$ and $c_2 = 150$. □

The following lemma deals with the remaining cases when there are not very many 3 and 5-cycles.

Lemma 27 *There exist absolute constants c_4 and c_5 such that if $\sum_{m_i \in \{3,5\}} m_i \le \frac{1}{2}\binom{n}{2} - c_4n$ and $\sum_{m_i \ge 72} m_i \ge c_5n$ then the conclusion of Theorem 3 holds.*

Proof. Write $n = 2n_0+n_1$ with $n_0 = 72m+1$ and $6 \le n_1 \le 149$. We shall take $c_4 = 8.9 \times 10^7$. Since $\frac{1}{2}\binom{n}{2} \ge c_4n$, $n > 4c_4 > 10^8$ and $m > 10^5$. Construct the self-avoiding trail of triangles in K_{n_0} as before using Theorem 14. Let $r = \lceil\frac{n_1}{2}\rceil \le 75$ and write the vertices of K'_{n_1} as r disjoint pairs of vertices P_i, (or one singleton P_1 and $r-1$ pairs $P_2,\dots,P_r$ if n is odd). Since $r < 10^4 < \frac{m-10}{4}$, we can construct sets $S_0,\dots,S_r$ of triangles as in Lemma 25. Pick an integer j so that $144mj \ge \frac{1}{2}\binom{n}{2} - c_4n \ge \sum_{m_i \in \{3,5\}} m_i$. A simple calculation shows that we can take $j = 36m+76-c_4$. Translate all the vertices of all the triangles of $S_0,\dots,S_r$ by $-mj$ so as to move the triangles along $12mj$ in the sequence of Theorem 14. Clearly properties 2 to 6 of Lemma 25 still hold and the triangles of S_i all occur between the $(12mj+1)^{\text{st}}$ and $(12mj+6mn_0+211n_0)^{\text{th}}$ triangles of the sequence. Doubling up each vertex gives us a trail of octahedra as in Theorem 1 and the sets S_i are now sets of octahedra. We shall now modify the octahedra in the trail so as to include all the additional edges.

We deal with the edges in K'_{n_1} in exactly the same way as in Lemma 26. Once again we will have only modified some of the last $|E(K'_{n_1})|(1+c_1)n_0$ octahedra and have only involved at most $c' = \frac{1}{3}|E(K'_{n_1})|$ non-adjacent vertex pairs u_1u_2.

Remove octahedra from S_i which have these pairs as non-link vertices (when $i > 0$) or link vertices (when $i = 0$). By parts 4 and 5 of Lemma 25, this removes at most c' octahedra from each S_i. Now for each $i \geq 1$ (if n even) or $i \geq 3$ (if n odd) join P_i to each of the non-link vertices of each octahedron in S_i. These octahedra now become $O_{[2,2]}$'s. We now pack most of the C_3's and C_5's into some of the first $12mj$ octahedra. By definition of j we will run out of C_3's and C_5's before the $(12mj+1)^{\text{st}}$ octahedron and so will not encounter any octahedron in any S_i. Since four C_3's can be packed into O and twelve C_5's can be packed into $O^{\cdot 5}$, the remaining C_3's and C_5's will have total length of at most $3(3) + 11(5)$. Now pack the other cycles of length less than 12 using Lemma 22. We first pack as many C_4's as possible. If n is odd, then each time we come to an octahedron in S_0, change the octahedron into the graph W by attaching the link vertices to the three vertices in $P_1 \cup P_2$ and joining the link vertices. Now use the C_4's to pack the subgraph W_{222}. When two such graphs are packed, pack the remaining edges of the two W's with three C_4's by pairing up the missing paths of length two from the first W with those of the second W. We continue until we run out of C_4's. If we have an odd number of W's, then "unmodify" the last one, converting it back to an O. We also change back an octahedron if it occured too early or late in the subset of packed octahedra. (In Lemma 22 we need a few octahedra at the start and end of the sequence to be unmodified.) We shall have at most two unpacked C_4's remaining (three would pack another O) and we may have changed back at most four octahedra. (One each at the start and end due to Lemma 22, one if we have an odd number of W's and possibly one more since changing back octahedra gives us a few more C_4's to pack.) Now continue with C_8's in a similar manner. Once again, if n is odd we convert octahedra in S_0 into W's. This time we pack one W_{222} and three W_{33}'s and use three C_8's to pack the remaining edges. (Make the missing paths of length 3 of the first W_{33} join v_1 and v_2, and those of the next join v_2 and v_3 and those of the last join v_3 and v_1. Now matching up two paths of length 3 and one of length 2 from distinct W's gives three C_8's.) If we have some W_{33}'s or W_{222}'s left over or some modified octahedra occur too early or late in the packed sequence, then change these back into O's. We shall have at most five C_8's remaining (six would pack another $O^{\cdot 4}$) and at most six octahedra changed back. Now continue with C_6's and C_7's packing W's where possible. We shall have at most one C_6 and eleven C_7's remaining and we may need to change back at most three octahedra for the C_7's and five octahedra for the C_6's (up to two $O_{[2,2]}$ may need to be changed back into O's to get the divisibility condition in Lemma 22 for the C_6's).

If n is odd, join P_1 and P_2 to the midvertices of octahedra in S_1 and S_2 respectively that have not been joined already when constructing W's and which are at least $5m$ further on in the sequence than the last W. When joining the singleton P_1 to the midvertices of an octahedron, join these vertices together as well so that the octahedron becomes $O_{[1,2]}$. By part 6 of Lemma 25, all the link vertices of octahedra in S_0 not yet encountered will be joined to $P_1 \cup P_2$ at this point.

Now pack C_9's, C_{10}'s and C_{11}'s, packing O's, $O_{[2,2]}$'s and $O_{[1,2]}$'s as required. Once again, we may need to change back some O's. A simple count shows that the total length of all cycles remaining of length less than 12 is now at most 393 and we have changed back at most 30 modified octahedra. Pack all these remaining cycles into the trail of octahedra together with at most four longer cycles using Lemma 18 for the cycles of length at most 9, and then Lemma 16 for the cycles of lengths

10 and 11. If we encounter an $O_{[a,b]}$ we change it back to O (this will occur at most three times since modified octahedra are separated by a distance of at least 12). If in Lemma 18 we need to use the graph $\widetilde{O}$, then remove a triangle from the set of small cycles being packed and try again. Removing a triangle from the list of exceptional cases in Lemma 18 will never give another exceptional case, so we shall not need the $\widetilde{O}$. The remaining triangle will be packed in K'_{n_1} as in Lemma 26 by changing three $O_{[2,3]}$'s to $O_{[2,2]}$ near the end of the sequence.

Now pack cycles of lengths at least 12 in increasing order of length using Lemma 21 until we have passed the last octahedra in any S_i. We now have a packing into some graph $G.\{\mathbf{a}\}$ with $\mathbf{a} \in \mathcal{A}'$ and G is an initial segment of the sequence of octahedra.

We now estimate the number of edges remaining between K_{2n_0} and K_{n_1}. We may have missed up to $c'r$ squares and triangles avoiding vertex pairs u_1u_2 used at the begining of the proof. We may also have missed at most 34×4 squares and triangles when we changed back some octahedra. Finally we may have up to 3827 missing squares and 3827 missing triangles because S_0 does not cover all of the vertices. We now continue the algorithm at the begining of this proof. If any edges remain then there must be an independent pair u_1u_2 of vertices in K_{2n_0} joined to a pair v_1, v_2 in K'_{n_1} (or to just one v_1). Find some octahedron between n_0 and $(1+c_1)n_0$ from the last octahedron modified at the begining of this proof with the pair u_1u_2 as non-link vertices. Add two paths $u_1v_1u_2$ and $u_1v_2u_2$ (or the edge u_1u_2 if there is no v_2) to the octahedra to get $O_{[2,2]}$ (or $O_{[1,2]}$). Eventually we will have used up all the remaining edges. The octahedra at the end of the sequence that have been modified all lie at most $c''n_0$ from the end where $c'' = (c'r + 3827(2) + 34(4) + |E(K'_{n_1})|)(1+c_1) \le 1.45 \times 10^7$.

We now finish using the same argument as in Lemma 26. Provided the total length of cycles $m_i \ge 72$ is at least $4L + |E(O_{[2,3]})|(c''+1)n_0 \le c_5 n$, and provided $12mj + 6mn_0 + 211n_0 + (c''+1)n_0 \le 12mn_0$, we shall still have at least $n_0 > 7$ unmodified octahedra remaining before encountering the modified octahedra at the end of the sequence and after packing all the cycles of length less than 72. The argument of the end of Lemma 26 will finish the proof when these two inequalities hold. Note that the modified octahedra at the end of the sequence are at least $n_0 > \frac{L}{2}$ from any other modified octahedron, and the octahedra in any S_i are at least $5m > \frac{L}{2}$ apart. Hence no cycle meets itself in K'_{n_1}. The octahedra in S_i may be close to some in S_j, $j \ne i$, but this is not important since the modified octahedra in each S_i will only meet K'_{n_1} in the vertices of P_i and the P_i are disjoint. With $j = 36m + 76 - c_4$ as above, a simple calculation shows that we can take $c_4 = 8.9 \times 10^7$ and $c_5 = 1.3 \times 10^8$. □

Finally we give the proof of Theorem 3.

Proof. of Theorem 3.

Corollary 20 proves the result when $n \ge N(71)$ and $\sum_{m_i \ge 72} m_i \le \frac{n}{8}(\epsilon(71)n - 4)$. Lemma 26 proves the result when $\sum_{m_i \in \{3,5\}} m_i \ge \frac{1}{3}\binom{n}{2} + c_2 n$ and $\sum_{m_i \ge 72} \ge c_3 n$. Lemma 27 proves the result when $\sum_{m_i \in \{3,5\}} m_i \le \frac{1}{2}\binom{n}{2} - c_4 n$ and $\sum_{m_i \ge 72} \ge c_5 n$. If we take n sufficiently large so that $n \ge N(71)$, $\frac{1}{8}(\epsilon(71)n - 4) \ge \max(c_3, c_5)$ and $(c_2 + c_4)n \le \frac{1}{6}\binom{n}{2}$ then in all cases at least one of these will prove the result. Indeed, we can take any $n \ge N_2 = \max(N(71), 1.2 \times 10^{10}/\epsilon(71))$. □

6 Conclusion

It is possible to reduce the lower bound of 72 on m_i in Theorem 4 (and hence Theorem 1) substantially by using the packings of Lemma 15 in Lemma 11 and Lemma 12. However Theorem 4 is false without some restriction on the m_i, since for example it is impossible to pack $O^{\cdot N}$ with a C_8 and $(3N-2)$ C_4's for any value of N. With considerably more effort, the following can however be proved.

Theorem 28 *There exists a constant c such that if $m_1, \dots, m_t$ are integers with $3 \leq m_i \leq L$, $\sum_{i=1}^{t} m_i = 12N$ and $\sum_{m_i \notin \{3,4,7,8\}} m_i \geq cL$ then cycles $C_{m_1}, \dots, C_{m_t}$ can be packed into $O^{\cdot N}$.*

Note that it is possible to avoid the use of $\widetilde{O}$ even when packing small cycles. All we need is that there are not too many C_3's, C_4's, C_7's or C_8's. Theorem 19 can be avoided, and hence the very large constants N_1 and N_2 in Theorem 2 and Theorem 3 can be reduced substantially if we avoid such cases.

The methods of Section 3 use cyclic Steiner Triple Systems. Theorem 14 is almost certainly not best possible, even for the Steiner triple system used in the proof. Similar arguments for other such systems (such as those constructed in [14]) should give a result similar to Theorem 14 for other values of n, and hence should give results similar to Theorem 1 and Theorem 2 at least for $n \equiv 2, 6 \bmod 12$ with some linear upper bound on the m_i's in terms of n.

More generally, we need $n \equiv 1, 3 \bmod 6$ in Theorem 14 to have any Steiner Triple System. Also if a k-self-avoiding trail of triangles exists then and consecutive sequence of k triangles must involve $2k+1$ vertices, so $2k+1 \leq n$. We do however make the following conjecture.

Conjecture 2 *There exists an absolute constant c such that for all k and n with $n \equiv 1, 3 \bmod 6$ and $n \geq 2k + c$ there exists a k-self-avoiding trail of triangles that pack K_n.*

Even if this conjecture were true, it would still only give an upper bound on m_i of $\frac{n}{2} - c$ in all three theorems and still require $n \equiv 2$ or $6 \bmod 12$ in Theorem 1 and Theorem 2. New ideas would still be needed to deal with cycles of length more than $\frac{n}{2}$.

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