TOWARD THE GENERAL THEORY OF AFFINE LINKING NUMBERS

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Abstract. Let N_1, N_2, M be smooth manifolds such that $\dim N_1 + \dim N_2 + 1 = \dim M$ and let ϕ_i, $i = 1, 2$, be the smooth mappings of N_i to M such that $\operatorname{Im} \phi_1 \cap \operatorname{Im} \phi_2 = \emptyset$. The classical linking number $\operatorname{lk}(\phi_1, \phi_2)$ is defined only when $\phi_{1*}[N_1] = \phi_{2*}[N_2] = 0 \in H_*(M)$.

Affine linking number alk is the generalization of the classical invariant to the case of nonzero-homologous $\phi_{1*}[N_1], \phi_{2*}[N_2]$. Recently in [5] we have constructed the first examples of alk-invariants of nonzero-homologous spheres in the spherical tangent bundle of a manifold and showed that alk is intimately related to the causality relation of wave fronts on manifolds.

In this paper we develop the general theory of alk-invariants in the case of nonzero-homologous $\phi_{1*}[N_1]$ and $\phi_{2*}[N_2]$. We show that alk is a universal Goussarov–Vassiliev invariant of order ≤ 1. In case of $\phi_{1*}[N_1] = \phi_{2*}[N_2] = 0 \in H_*(M)$ the alk-invar iant appears to be a splitting of the classical linking number into a collection of independent invariants.

To construct alk we introduce a new pairing on the bordism groups of space of mappings of N_1 and N_2 into M. For the case $N_1 = N_2 = S^1$ this pairing can be regarded as an analog of the string-homology pairing constructed by Chas and Sullivan.

Preliminaries

Throughout this paper M is a smooth connected oriented manifold (not necessarily compact), and N_1, N_2 are smooth oriented closed manifolds. The dimensions of M, N_1, N_2 are denoted by m, n_1, n_2, respectively. The one-point space is denoted by pt.

Given a space X, we denote by $\Omega_n(X)$ the n-dimensional oriented bordism group of X. Recall that $\Omega_n(X)$ is the set of the equivalence classes of (continuous) maps $f : V^n \to X$ where V is a closed oriented manifold. Here two maps $f_1 : V_1 \to X$ and $f_2 : V_2 \to X$ are equivalent if there exists a map $g : W^{n+1} \to X$ where W is a compact oriented manifold whose oriented boundary ∂W is diffeomorphic to $V_1 \cup (-V2)$ and $g|_{\partial W} = f$. Furthermore, the operation of disjoint sum converts $\Omega_n(X)$ into an abelian group. See [10, 12, 13] for details.

Let $[V] \in H_n(V)$ be the fundamental class of a closed oriented n-dimensional manifold V. Every map $f : V \to X$ gives us an element $f_*[V] \in H_n(X)$, and the correspondence $(V, f) \mapsto f_*[V]$ yields the

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Steenrod–Thom homomorphism

$$\Omega_n(X) \to H_n(X).$$

It turns out to be that this homomorphism is an isomorphism for $n \le 3$ and an epimorphism for $n \le 6$, [14], see [10, 12] for modern proofs.

Let $\mathcal{N}_i, i = 1, 2$, be a path-connected component of the spaces of smooth mappings of N_i to M. Let $\mathcal{B} = \mathcal{B}_{\mathcal{N}_1,\mathcal{N}_2}$ be the space of quadruples $(\phi_1, \phi_2, \rho_1, \rho_2)$ where $\phi_i : N_i \to M, i = 1, 2$, belong to $\mathcal{N}_i$ and $\rho_i : \mathrm{pt} \to N_i$ are such that $\phi_1\rho_1 = \phi_2\rho_2$.

1. The pairing $\mu : \Omega_i(\mathcal{N}_1) \otimes \Omega_j(\mathcal{N}_2) \to \Omega_{i+j+n_1+n_2-m}(\mathcal{B})$

Let $\alpha_1 : F_1 \to \mathcal{N}_1$ be a mapping representing $[\alpha_1] \in \Omega_i(\mathcal{N}_1)$ and let $\alpha_2 : F_2 \to \mathcal{N}_2$ be a mapping representing $[\alpha_2] \in \Omega_j(\mathcal{N}_2)$. Let $\widetilde{\alpha}_i : F_i \times N_i \to M$, $i = 1, 2$, be such that $\widetilde{\alpha}_i(f, n) = (\alpha_i(f))(n)$. Let $v_1 \in F_1 \times N_1$ and $v_2 \in F_2 \times N_2$ be such that $\widetilde{\alpha}_1(v_1) = \widetilde{\alpha}_2(v_2)$. We say that $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are transverse at (v_1, v_2) if $d\widetilde{\alpha}_2\big(T_{v_2}(F_2 \times N_2)\big)$ and $d\widetilde{\alpha}_1\big(T_{v_1}(F_1 \times N_1)\big)$ generate $T_{\widetilde{\alpha}_1(v_1)}M = T_{\widetilde{\alpha}_2(v_2)}M$. Following standard arguments we can assume that $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are *transverse,* i.e. they are transverse along all (v_1, v_2) such that $\widetilde{\alpha}_1(v_1) = \widetilde{\alpha}_2(v_2)$.

Consider the pull-back diagram

$$\begin{array}{ccc} V & \xrightarrow{j_1} & F_1 \times N_1 \\ \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle \widetilde{\alpha}_1} \\ F_2 \times N_2 & \xrightarrow{\widetilde{\alpha}_2} & M \end{array} \tag{1.1}$$

of the maps $\widetilde{\alpha}_i$, $i = 1, 2$.

1.1. **Lemma.** *If $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are transverse, then V is a closed $(i + j + n_1 + n_2 - m)$-dimensional manifold.* □

Let $p_1 : F_i \times N_i \to F_i$ and $p_2 : F_i \times N_i \to N_i, i = 1, 2$, be the obvious projections. Consider the mapping

$$\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2) : V \to \mathcal{B}, \quad v \mapsto \big(\phi_1^v, \phi_2^v, \rho_1^v, \rho_2^v\big)$$

where $\phi_i^v(n) = \widetilde{\alpha}_i(p_1(j_i(v)), n)$ for $n \in N_i$ and $\rho_i(\mathrm{pt}) = p_2(j_i(v))$.

1.2. **Theorem.** *The bordism class $[V, \mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2)] \in \Omega_{i+j+n_1+n_2-m}(\mathcal{B})$ depends only on $[\alpha_1] \in \Omega_i(\mathcal{N}_1)$, $[\alpha_2] \in \Omega_j(\mathcal{N}_2)$ and defines a pairing*

$$\mu : \Omega_i(\mathcal{N}_1) \otimes \Omega_j(\mathcal{N}_2) \to \Omega_{i+j+n_1+n_2-m}(\mathcal{B}).$$

□

1.3. **Remark.** In the case of $N_1 = N_2 = S^1$ the pairing μ is intimately related to the bracket operation on the string homology introduced by Chas and Sullivan [4].

1.4. **Remark.** Although we defined the pairing μ for the connected components of spaces of maps $N_i \to M$, it is clear that the pairing extends and is well-defined for the whole mapping spaces.

2. Affine linking numbers

From here and till the end of the paper we assume that $n_1+n_2+1 = m$.

Put Σ to be the discriminant in $\mathcal{N}_1 \times \mathcal{N}_2$, i.e. the subspace that consists of pairs (f_1, f_2) such that there exist $y_1 \in N_1, y_2 \in N_2$ with $f_1(y_1) = f_2(y_2)$. (We do not include into Σ the maps that are singular in the common sense but do not involve double points between $f_1(N_1)$ and $f_2(N_2)$.)

2.1. **Definition.** We define Σ_0 to be the subset (stratum) of Σ consisting of all the pairs (f_1, f_2) such that there exists precisely one pair of points $y_1 \in N_1, y_2 \in N_2$ such that:

a: $f_1(y_1) = f_2(y_2)$, and, moreover this pair of points is such that:
b: y_i is a regular point of $f_i, i = 1, 2$;
c: $(df_1)(T_{y_1}N_1) \cap (df_2)(T_{y_2}N_2) = 0$.

2.2. **Construction.** Let $\gamma : (a, b) \to \mathcal{N}_1 \times \mathcal{N}_2$ be a path which intersects Σ_0 in a point $\rho(t_0)$. We also assume that

$$\gamma(t_0 - \delta, t_0 + \delta) \cap \Sigma = \gamma(t_0)$$

for δ small enough. We construct a vector $\mathbf{v} = \mathbf{v}(\gamma, t_0, \delta)$ as follows. We regard $\gamma(t_0)$ as a pair $(f_1, f_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ and consider the points y_1, y_2 as in 2.1. Set $z = f_1(y_1) = f_2(y_2)$. Choose a small $\delta > 0$ and regard $\gamma(t_0 + \delta)$ as a pair $(g_1, g_2) \in \mathcal{N}_1 \times \mathcal{N}_2$. Set $z_i = g_i(y_i), i = 1, 2$. Take a chart for M that contains z and z_i, $i = 1, 2$ and set

$$\mathbf{v}(\gamma, t_0, \delta) := \overrightarrow{zz_1} - \overrightarrow{zz_2} \in T_zM.$$

2.3. **Definition.** Let $\gamma : (a, b) \to \mathcal{N}_1 \times \mathcal{N}_2$ be a path as in 2.2. We say that γ *intersects* Σ_0 *transversally for* $t = t_0$ if there exists $\delta_0 > 0$ such that

$$\mathbf{v}(\gamma, t_0, \delta) \notin (df_1)(T_{y_1}N_1) \oplus (df_2)(T_{y_2}N_2) \subset T_zM$$

for all $\delta \in (0, \delta_0)$.

It is easy to see that the concept of transversal intersection does not depend on the choice of the chart.

2.4. **Definition.** A path $\gamma : (a, b) \to \mathcal{N}_1 \times \mathcal{N}_2, -\infty \leq a < b \leq \infty$ is said to be *generic* if

a: $\gamma(a, b) \cap \Sigma = \gamma(a, b) \cap \Sigma_0$;
b: the set $J = \{t | \gamma(t) \cap \Sigma_0 \neq \emptyset\} \subset (a, b)$ is an isolated subset of $\mathbb{R}$;
c: the path γ intersects Σ_0 transversally for all $t \in J$.

As one can expect, every path can be turned into a generic one by a small deformation. We leave a proof to the reader.

2.5. **Definition.** Let γ be a path in $\mathcal{N}_1 \times \mathcal{N}_2$ that intersects Σ transversally in one point $\gamma(t_0) \in \Sigma_0$. We associate a sign $\sigma(\gamma, t_0)$ to such a crossing as follows.

We regard $\gamma(t_0)$ as a pair $(f_1, f_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ and consider the points $y_1 \in N_1, y_2 \in N_2$ such that $f_1(y_1) = f_2(y_2)$. Set $z = f_1(y_1) = f_2(y_2)$. Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be frames which are tangent to N_1 at N_2 and y_2, respectively, and both are assumed to be positive. Consider the frame

$$\{df_1(\mathfrak{r}_1), \mathbf{v}, df_2(\mathfrak{r}_2)\}$$

at $z \in M$, where $\mathbf{v}$ is a vector described in 2.2. We put $\sigma(\gamma, t_0) = 1$ if this frame gives us the orientation of M, otherwise we put $\sigma(\gamma, t_0) = -1$. Because of the transversality and condition (c) from 2.1, the family $\{df_1(\mathfrak{r}_1), \mathbf{v}, df_2(\mathfrak{r}_2)\}$ is really a frame. Notice also that the vector $\mathbf{v}$ is not well-defined, but the above defined sign σ is.

Clearly if we traverse the path γ in the opposite direction then the sign of the crossing changes.

For every space X, the group $\Omega_0(X) = H_0(X)$ is the free abelian group with the base $\pi_0(X)$. So, every element of $\Omega_0(X)$ can be represented as a finite linear combination $\sum \lambda_k P_k$ with $\lambda_k \in \mathbb{Z}$ and $P_k \in X$, and every such linear combination gives us an element of $\Omega_0(X)$.

2.6. **Comment.** For N_1 and N_2 connected the set $\pi_0(\mathcal{B})$ can be described as follows. Given two pointed spaces X and Y, let $[X, Y]$ be the set of pointed homotopy classes of pointed maps $X \to Y$. Then $\pi_1(Y)$ acts on $[X, Y]$ in a usual way, see e.g. [11]. Furthermore, $\pi_1(X)$ acts on $[X, X]$, and therefore we get a *right* $\pi_1(X)$-action on $[X, Y]$ via the composition map $[X, X] \times [X, Y] \to [X, Y]$. Moreover, the left $\pi_1(Y)$-action commutes with the right $\pi_1(X)$-action.

Choose base points in N_1, N_2 and M and let $\mathcal{N}_i^* = [N_i, M], i = 1, 2$. Since $\pi_0(\mathcal{N}_1{}^* \times \mathcal{N}_2{}^*) = [N_1 \vee N_2, M]$, we get the right $\pi_1(M)$-action and the left $\pi_1(N_1 \vee N_2)$-action on $\pi_0(\mathcal{N}_1{}^* \times \mathcal{N}_2{}^*)$. Notice that $\pi_1(N_1 \vee N_2) =$

$\pi_1(N_1) * \pi_1(N_2)$. Now, one can verify that

$$\pi_0(\mathcal{B}) = (\pi_1(N_1) * \pi_1(N_2)) \Big\backslash \pi_0({\mathcal{N}_1}^* \times {\mathcal{N}_2}^*) \Big/ \pi_1(M).$$

Since $\pi_0({\mathcal{N}_1}^* \times {\mathcal{N}_2}^*) = \pi_0({\mathcal{N}_1}^*) \times \pi_0({\mathcal{N}_2}^*)$, the last formula facilitates calculations of $\pi_0(\mathcal{B})$, and hence of $\Omega_0(\mathcal{B})$.

2.7. **Definition.** Let γ be a path in $\mathcal{N}_1 \times \mathcal{N}_2$ that intersects Σ transversally in one point $\gamma(t_0) \in \Sigma_0$. We define $[\gamma(t_0)] \in \Omega_0(\mathcal{B})$ as $\sigma(t_0)\gamma(t_0)$.

Clearly,

$$\varepsilon([\gamma(t_0]) = \sigma(\gamma, t_0), \tag{2.1}$$

where $\varepsilon : \Omega_0(\mathcal{B}) \to \mathbb{Z}$ is the homomorphism induced by the map $\mathcal{B} \to$ pt.

2.8. **Definition.** We define the *indeterminacy subgroup* Indet to be the minimal subgroup of $\Omega_0(\mathcal{B})$ that contains both

$$\operatorname{Im}\big(\mu : \Omega_1(\mathcal{N}_1) \otimes \Omega_0(\mathcal{N}_2) \to \Omega_0(\mathcal{B})\big)$$

and

$$\operatorname{Im}\big(\mu : \Omega_0(\mathcal{N}_1) \otimes \Omega_1(\mathcal{N}_2) \to \Omega_0(\mathcal{B})\big).$$

We define the abelian group $\mathbb{A} = \mathbb{A}(\mathcal{N}_1, \mathcal{N}_2)$ to be the quotient group of $\Omega_0(\mathcal{B})/\operatorname{Indet}$. Let $q = q_{\mathcal{N}_1,\mathcal{N}_2} : \Omega_0(\mathcal{B}) \to \mathbb{A}$ be the corresponding quotient homomorphism.

2.9. **Theorem.** *Let $\mathbb{A}$ be as above. Then there exists an affine linking invariant* $\operatorname{alk} : \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma \to \mathbb{A}$ *such that:*

- **a:** alk *is constant on connected components of* $\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$*;*
- **b:** *if* $\gamma : [a, b] \to \mathcal{N}_1 \times \mathcal{N}_2$ *is a generic path such that* $\gamma(a), \gamma(b) \notin \Sigma$ *and* $t_i, i \in I$, *are the moments when* $\gamma(t_i) \in \Sigma$ (*and hence* $\gamma(t_i) \in \Sigma_0$ *by the definition of the generic path*), *then*

$$\operatorname{alk}(\gamma(b)) - \operatorname{alk}(\gamma(a)) = q\Big(\sum_{i \in I} [\gamma(t_i)]\Big) \in \mathbb{A}.$$

Moreover, these properties determine the invariant alk *uniquely up to an additive constant.*

We prove Theorem 2.9 in Section 3.

2.10. **Remarks (Relations to front propagation).**

a) Let STM denote the total space of the sphere tangent bundle over M, $\dim STM = 2m-1$. In [5] we defined the affine linking invariant al for the mappings of $S^{m-1} \to STM$ that are homotopic to the inclusion of the fiber S^{m-1} to STM. Because of the orientability of the bundle $STM \to M$, the homotopy class of this inclusion is invariant under the $\pi_1(M)$-action on $[S^{m-1}, STM]$. Using 2.6 we get that in this case $\Omega_0(\mathcal{B}) = \mathbb{Z}$, and al is exactly alk for $N_1 = N_2 = S^{m-1}$ and the space $\mathcal{N}_1 = \mathcal{N}_2$ consisting of mappings $S^{m-1} \to STM$ as above.

In [5] we have shown that in this case $\varepsilon(\text{Indet}) = \text{Indet} = 0$ when m is even or when m is odd and M is not a rational homology sphere. This shows that $\varepsilon(\text{alk})$ can indeed be $\mathbb{Z}$-valued in many cases where the mappings are not zero-homologous.

This example is especially exciting since as it was shown in [5] this version of the alk invariant is intimately related to the causality relation invariant CR for wave fronts on M.

b) The classical winding number of a curve around a point in $\mathbb{R}^2$ is the linking number between the curve and the 0-cycle formed by the point and the point at infinity of $\mathbb{R}^2$. In [6] we considered the generalizations $\text{win}(F,p)$ and $\widetilde{\text{win}}(F,p)$ of the winding numbers of the mapping $F : N_1^{m-1} \to M$ around a point $p : \text{pt} = N_2 \to M$ to the case where $F_*([N_1]) \neq 0 \in H_*(M)$ and M does not have ends that could play the role of the infinity. (The invariants $\widetilde{\text{win}}$ and win are the generalizations of the winding number to the case where the observable point p moves and is fixed in M, respectively.) We showed that affine winding numbers can be effectively used to estimate from below the number of times a wave front has passed through a point between two moments of time.

One can verify that the generalized winding number $\widetilde{\text{win}}$ also is included into the theory explored in this work, if we consider affine linking number for the case $N_2 = \text{pt}$. (It is clear, see 2.6, that in this case $\Omega_0(\mathcal{B}) = \mathbb{Z}$.)

It is easy to construct the version $\overline{\text{alk}}$ of the alk invariant constructed in this paper, where $\overline{\text{alk}}$ will be a function on $\pi_0(\mathcal{N}_1 \times \{*\} \setminus \Sigma)$ for some fixed mapping $* \in \mathcal{N}_2$. A straightforward verification shows that $\overline{\text{alk}}$ is well-defined provided it takes values in the quotient group of $\Omega_0(\mathcal{B})$ by $\text{Im}\big(\mu : \Omega_1(\mathcal{N}_1) \otimes \Omega_0(\mathcal{N}_2) \to \Omega_0(\mathcal{B})\big)$. The win invariant constructed in [6] is a particular case of such $\overline{\text{alk}}$ where $N_2 = \text{pt}$.

2.11**. Remarks** (alk **is the universal Goussarov–Vassiliev invariant of order** ≤ 1**).**
Let $f = (f_1, f_2) \in \Sigma \subset \mathcal{N}_1 \times \mathcal{N}_2$ be such that $\text{Im}(f_1) \cap \text{Im}(f_2)$ consists of $(n+1)$ distinct double points of transverse intersection. Each double point can be resolved in two essentially different ways: positive and negative, where the sign is defined as in 2.5. Thus f with $(n+1)$ such double points admits $2^{(n+1)}$ possible resolutions of the double points. A sign of the resolution is put to be $+$ if the number of negatively resolved double points is even, and it is put to be $-$ otherwise. Let Γ

be an abelian group and let α be a Γ-valued invariant of nonsingular $f \in \mathcal{N}_1 \times \mathcal{N}_2$ i.e. α is a mapping $\alpha : \pi_0(\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma) \to \Gamma$. The invariant α is said to be of *finite order* (or *Vassiliev–Goussarov invariant,* [15], [7], [8]) if there exists a positive integer $(n+1)$ such that for any singular $f \in \Sigma$ with $(n+1)$ transverse double points the sum (with appropriate signs) of the values of α on the nonsingular mappings obtained by the 2^{n+1} resolutions of the double points is zero. An invariant is said to be of order not greater than n (of order $\leq n$) if n can be chosen as the integer in the above definition. The group of Γ-valued finite order invariants has an increasing filtration by the subgroups of the invariants of order $\leq n$.

It is easy to verify that alk is an $\mathbb{A}$-valued order ≤ 1 Vassiliev-Goussarov invariant of mappings belonging to $\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$. Furthermore, if $\alpha : \pi_0(\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma) \to \Gamma$ is a Γ-valued order ≤ 1 Vassiliev-Goussarov invariant, then the increment $\Delta_\alpha(\gamma(t_0))$ of α under the positive crossing of Σ at $\gamma(t_0) \in \Sigma_0$ depends only on the element of $\pi_0(\mathcal{B})$ that corresponds to $\gamma(t_0)$. In particular, we conclude that there exists the natural homomorphism $B : \Omega_0(\mathcal{B}) \to \Gamma$ that sends the bordism class of $(+1)\gamma(t_0) \in \Omega_0(\mathcal{B})$ to $\Delta_\alpha(\gamma(t_0))$. Moreover, this homomorphism B passes through the quotient homomorphism $q : \Omega_0(\mathcal{B}) \to \mathbb{A}$ as in 2.8, and therefore we get a homomorphism $A : \mathbb{A} \to \Gamma$ with $A \circ q = B$, cf. Theorem 4.1 below. One verifies that

$$A(\mathrm{alk}(f) - \mathrm{alk}(f')) = \alpha(f) - \alpha(f'),$$

for all $f, f' \in (\mathcal{N}_1, \times \mathcal{N}_2 \setminus \Sigma)$.

Clearly if α and α' are Γ-valued Vassiliev-Goussarov invariants of order ≤ 1 such that $\alpha - \alpha'$ is a constant mapping, then the corresponding homomorphisms A and A' are equal. Let $f_0 \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ be a chosen preferred point. Then for every $f \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ we have $\alpha(f) = \alpha(f_0) + A(\mathrm{alk}(f)) - A(\mathrm{alk}(f_0))$. Thus alk completely determines the values of α on all f (modulo $\alpha(f_0)$), and hence alk is a universal Vassiliev-Goussarov invariant of order ≤ 1.

In particular, alk distinguishes all the elements $f, f' \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ that can be distinguished via Vassiliev-Goussarov invariants of order ≤ 1 with values in an arbitrary group Γ.

3. Proof of Theorem 2.9

3.1. **Definition.** We define Σ_1 to be the subset (stratum) of Σ consisting of all the pairs (f_1, f_2) such that there exists precisely two pairs of points $y_1 \in N_1, y_2 \in N_2$ as in 2.1. Here we assume that the two double points of the image are distinct.

Notice that, $\Sigma_i, i = 0, 1$, is the stratum of codimension i in Σ. In particular, a generic path in $\mathcal{N}_1 \times \mathcal{N}_2$ intersects Σ_0 in a finite number of points, and a generic disk in $\mathcal{N}_1 \times \mathcal{N}_2$ intersects Σ_1 in a finite number of points.

A generic path $\gamma : [0,1] \to \mathcal{N}_1 \times \mathcal{N}_2$ that connects two points in $\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ intersects Σ_0 in finitely many points $\gamma(t_j), j \in J$, and all the intersection points are of the types described in 2.5. Put

$$\Delta_{\text{alk}}(\gamma) = \sum_{j \in J} [\gamma(t_j)] \in \Omega_0(\mathcal{B}). \tag{3.1}$$

We let $A = \{(x,y) \in \mathbb{R}^2 \bigm| x^2 + y^2 \le 1\}$, $B_1 = \{(x,y) \in A \bigm| xy = 0\}$, $B_2 = \{(x,y) \in A \bigm| x = 0\}$, $B_3 = \{(x,y) \in A \bigm| y = 0\}$, $B_4 = \{(0,0)\}$, $B_5 = \emptyset$.

We define a regular disk in $\mathcal{N}_1 \times \mathcal{N}_2$ as a generically embedded disk D such that the triple $(D, D \cap \Sigma_0, D \cap \Sigma_1)$ is homeomorphic to a triple (A, B, C) where B is one of B_i's and $C \subset B_4$.

3.2. **Lemma.** *Let β be a generic loop that bounds a regular disk in $\mathcal{N}_1 \times \mathcal{N}_2$. Then $\Delta_{\text{alk}}(\beta) = 0$.*

Proof. It is easy to see that all the crossings of Σ_0 that happen along β can be subdivided into pairs, such that the elements of $\pi_0(\mathcal{B})$ corresponding to the two crossings in a pair are equal and the signs of the corresponding crossings of Σ_0 are opposite. Hence the inputs into $\Delta_{\text{alk}}(\beta)$ of the elements of $\Omega_0(\mathcal{B})$ corresponding to the two crossings in a pair cancel out and $\Delta_{\text{alk}}(\beta) = 0$. □

3.3. **Lemma.** *Let β be a generic loop that bounds a disk in $\mathcal{N}_1 \times \mathcal{N}_2$, then $\Delta_{\text{alk}}(\beta) = 0$.*

Proof. Without loss of generality we can (using a small deformation of the disk) assume that the disk is the union of regular ones, cf. Arnold [1], [2]. Now the proof follows from Lemma 3.2. □

3.4. **Corollary.** *Choose a base point $*$ of $\mathcal{N}_1 \times \mathcal{N}_2$ with $* \notin \Sigma$. The invariant Δ_{alk} induces a well-defined homomorphism*

$$\Delta_{\text{alk}} : \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *) \to \Omega_0(\mathcal{B}).$$

Proof. Since every element of $\pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$ can be represented by a generic loop, the proof follows from Lemma 3.3. □

3.5. **Lemma.**

$$\mathrm{Im}\{\Delta_{\mathrm{alk}} : \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *) \to \Omega_0(\mathcal{B})\} = \mathrm{Indet} \subset \Omega_0(\mathcal{B}).$$

Proof. First, we prove that $\mathrm{Im}\,\Delta_{\mathrm{alk}} \subset \mathrm{Indet}$. The base point $*$ in $\mathcal{N}_1 \times \mathcal{N}_2$ gives us the base points in both $\mathcal{N}_1$ and $\mathcal{N}_2$ which we also denote by $*$. Given a generic loop α in $(\mathcal{N}_1, *)$ and the constant loop e in $(\mathcal{N}_2, *)$, let (α, e) be the corresponding loop in $(\mathcal{N}_1 \times \mathcal{N}_2, *)$. The homotopy class of (α, e) gives us an element $[(\alpha, e)] \in \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$. Similarly, a generic loop β in $(\mathcal{N}_2, *)$ gives us an element $[(e, \beta)] \in \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$.

Because of the isomorphism $\pi_1(\mathcal{N}_1 \times \mathcal{N}_2) = \pi_1(\mathcal{N}_1) \times \pi_1(\mathcal{N}_2)$, the classes $[(\alpha, e)]$ and $[(e, \beta)]$ generate the group $\pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$. So, it suffices to prove that $\Delta_{\mathrm{alk}}[(\alpha, e)] \subset \mathrm{Indet}$ and $\Delta_{\mathrm{alk}}[(e, \beta)] \subset \mathrm{Indet}$ for all loops α and β as above.

We calculate $\Delta_{\mathrm{alk}}[(\alpha, e)] \in \Omega_0(\mathcal{B})$. Fix a mapping $\overline{e} : N_2 \to M$ in $\mathcal{N}_2$ and consider the mapping

$$\overline{\alpha} : S^1 \times (N_1 \sqcup N_2) \to M$$

such that $\overline{\alpha}\big|_{S^1 \times N_1} = \alpha$ and $\overline{\alpha}\Big|_{S^1 \times N_2}$ coincides with the composition

$$S^1 \times N_2 \xrightarrow{\text{projection}} N_2 \xrightarrow{\ \overline{e}\ } M$$

Without loss of generality we may assume that $\overline{\alpha}\Big|_{S^1 \times N_1}$ is transverse to $\overline{e}$. Now it is easy to see that

$$\Delta_{\mathrm{alk}}[(\alpha, e)] = \mu\Big(\Big[\overline{\alpha}\big|_{S^1 \times N_1}\Big] \otimes \Big[\overline{e}\Big]\Big) \in \Omega_0(\mathcal{B}). \tag{3.2}$$

Hence

$$\Delta_{\mathrm{alk}}([\alpha, e]) \in \mathrm{Im}\Big(\mu : \Omega_1(\mathcal{N}_1) \otimes \Omega_0(\mathcal{N}_2) \to \Omega_0(\mathcal{B})\Big) \subset \mathrm{Indet}\,.$$

Similarly we show that

$$\Delta_{\mathrm{alk}}([e, \beta]) \in \mathrm{Im}\big(\mu : \Omega_0(\mathcal{N}_1) \otimes \Omega_1(\mathcal{N}_2) \to \Omega_0(\mathcal{B})\big) \subset \mathrm{Indet}\,.$$

Conversely, since $\Omega_0(\mathcal{N}_2) = \mathbb{Z}$ is generated by $[e]$, it follows from (3.2) (and the symmetric formula for (e, β)) that $\mathrm{Indet} \subset \mathrm{Im}\,\Delta_{\mathrm{alk}}$. □

3.6. **Definition.** Choose $k \in \mathbb{A}$. Take an arbitrary point $f = (f_1^1, f_2^1) \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ and choose a generic path γ going from $*$ to f. We set

$$\mathrm{alk}(f) = k + q\big(\Delta_{\mathrm{alk}}(\gamma)\big) \in \mathbb{A}$$

and call alk the *affine linking invariant*. Here q is the epimorphism from Definition 2.8.

3.7. **Theorem.** *The function* alk $: \pi_0(\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma) \to \mathbb{A}$ *is well-defined and increases by* $q([\gamma(\bar{t})]) \in \mathbb{A}$ *under a transverse passage by a path* γ *through the stratum* Σ_0 *at the point* $\gamma(\bar{t})$*.*

Furthermore, the above property determines the function alk *uniquely up to an additive constant.*

Proof. To show that alk is well-defined we must verify that the definition of alk is independent on the choice of the path γ that goes from $*$ to f. This is the same as to show that $q(\Delta_{\text{alk}}(\varphi)) = 0$ for every closed generic loop φ at $*$. But this follows from Lemma 3.5 directly.

All the other claims are obvious. □

Clearly, Theorem 2.9 is a direct consequence of Theorem 3.7.

4. Relations between alk and the classical linking invariant lk

Given a closed oriented manifold N^n with the fundamental class $[N] \in H_n(M)$, we say that a map $f : N \to M$ is zero-homologous if $f_*([M]) = 0 \in H_n(M)$.

Let $\varepsilon : \Omega_0(\mathcal{B}) \to \mathbb{Z}$ be the homomorphism from (2.1).

4.1. **Theorem.** *Suppose that* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *consist of zero-homologous mappings. Then* $\varepsilon(\text{Indet}) = 0$*. Furthermore, for all* $f = (f_1, f_2), f' = (f_1', f_2') \in \mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$*, we have*

$$\varepsilon(\text{alk}(f)) - \varepsilon(\text{alk}(f')) = \text{lk}(f_1, f_2) - \text{lk}(f_1', f_2') \in \mathbb{Z}.$$

Proof. Since $\mathcal{N}_1$ and $\mathcal{N}_2$ consist of zero-homologous maps, the classical linking invariant lk $: \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma \to \mathbb{Z}$ is well-defined. Now, similarly to Δ_{alk}, we define

$$\Delta_{\text{lk}} : \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *) \to \mathbb{Z}, \quad \Delta_{\text{lk}}(\gamma) = \sum_{i=1}^{k} \sigma(\gamma, t_i),$$

where the generic loop γ in $(\mathcal{N}_1 \times \mathcal{N}_2, *)$ intersects $\Sigma_0 \subset \Sigma \subset \mathcal{N}_1 \times \mathcal{N}_2$ in certain points $\gamma(t_1), \cdots, \gamma(t_k)$. (Here we use the notation γ for the loop as well as for its homotopy class.) Since lk is well-defined, we conclude that $\Delta_{\text{lk}}(\gamma) = 0$ for all γ, i.e. $\sum \sigma(\gamma, t_i) = 0$.

Now, we have

$$\Delta_{\text{alk}}([\gamma]) = \sum [\gamma(t_i)] \in \Omega_0(\mathcal{B}).$$

So, in view of (2.1)

$$\varepsilon(\Delta_{\text{alk}}([\gamma])) = \sum \varepsilon([\gamma(t_i)]) = \sum \sigma(\gamma, t_i) = 0.$$

Thus, by Lemma 3.5,

$$\varepsilon\,(\text{Indet}) = \varepsilon(\text{Im}(\Delta_{\text{alk}})) = 0.$$

To prove the last claim of the Theorem, take a generic path γ which connects f and f'. Then

$$\begin{aligned}\varepsilon(\text{alk}(f)) - \varepsilon(\text{alk}(f')) &= \varepsilon\left(\sum[\gamma(t_i)]\right) = \sum\sigma(\gamma,t_i)\\ &= \text{lk}(f_1,f_2) - \text{lk}(f_1',f_2').\end{aligned}$$

□

4.2**. Remark.** Theorem 4.1 demonstrates that, up to an additive constant, $\varepsilon\circ\text{alk}$ is equal to the classical linking number lk whenever $\mathcal{N}_1$ and $\mathcal{N}_2$ consist of zero-homologous mappings.

So, alk is an extension of the classical lk-invariant of zero-homologous submanifolds. In general there is no way to choose this additive constant canonically, and for this reason we call alk the *affine* linking invariant.

Since the homomorphism ε is the summation over the components of $\mathcal{B}$, we conclude that, for zero-homologous mappings, alk **can be regarded as a splitting of the classical linking invariant** into a collection of independent invariants. On the other hand, as it was explained in this paper, the alk invariant exists regardless of whether the mappings are zero-homologous or not.

Also, as it was pointed in 2.10, $\varepsilon(\text{Indet}) = \text{Indet} = 0$ and thus $\varepsilon\circ\text{alk}$ is a $\mathbb{Z}$-valued invariant in many cases where $\mathcal{N}_1, \mathcal{N}_2$ do not consist of zero-homologous mappings.

5. Examples where the indeterminacy subgroup vanishes

Given the manifolds M, N_1, N_2 as in Section 2, we assume in addition that $N_i, i = 1,2$, are connected and that $n_1n_2 > 0$.

5.1**. Theorem** (Preissman [9])**.** *Let M be a closed manifold that admits a Riemannian metric of negative sectional curvature. Then the following holds:*

(i) *every nontrivial abelian subgroup of π is an infinite cyclic group;*

(ii) *for every nontrivial abelian subgroup A of π there exists a unique abelian subgroup B_A of π which contains A and is maximal with respect to this property.*

□

5.2**. Comment.** The claim (i) is proved in [9, Ch. 3, Theorem 9] explicitly. The claim (ii) is not formulated in [9] but follows from the results of the paper. Indeed, it is proved in [9, Ch. 3, Theorems 5* and 6*] that, for every simply connected complete Riemannian manifold X of negative sectional curvature, an isometry of X can transform at most one geodesic in itself and that every other isometry commuting with it leaves the same geodesic invariant. Now, considering the universal cover $\widetilde{M} \to M$ and applying this result to $\widetilde{M}$ with the Riemannian metric induced from M, we easily get (i) and (ii).

5.3**. Definition.** A finitely generated group π is called a *Preissman group* if it satisfies the properties (i) and (ii) from Theorem 5.1.

5.4**. Proposition.** *Let π be a Preissman group. Let $\alpha, \beta \in \pi$ be such that $\alpha\beta \neq \beta\alpha$, and let $\gamma \in \pi$ be such that $\alpha\gamma = \gamma\alpha$ and $\beta\gamma = \gamma\beta$, then $\gamma = e$.*

Proof. Let $G = \{x\}$ be the (unique) maximal cyclic subgroup of π which contains $\gamma \neq e$. Since $\alpha\gamma = \gamma\alpha$, the subgroup $\{\alpha, \gamma\}$ of π is contained in G, and so $\alpha = x^m$ for some m. Similarly, $\beta = x^k$, and thus $\alpha\beta = \beta\alpha$. This is a contradiction. □

5.5**. Theorem.** *Given M, N_1, N_2 as above, assume that $\pi_1(M)$ is either a Preissman group or finite. Assume also that $\pi_i(M) = 0$ for $2 \leq i \leq 1 + \max\{n_1, n_2\}$. Then the indeterminacy subgroup* Indet $\subset \Omega_0(\mathcal{B})$ *is the zero subgroup,* Indet $= \{0\} \subset \Omega_0(\mathcal{B})$.

Proof. Throughout the proof we denote $\pi_1(M)$ by π. We must prove that, for every $\alpha \in \pi_1(\mathcal{N}_1)$ and $\beta \in \pi_1(\mathcal{N}_2)$,

$$\Delta_{\text{alk}}[(\alpha, e)] = 0 = \Delta_{\text{alk}}[(e, \beta)]. \tag{5.1}$$

cf. Lemma 3.5.

We prove the first equality from (5.1) only, the second equality is proved in the similar way. Fix $\phi_i \in \mathcal{N}_i, i = 1, 2$, and consider a loop α in $(\mathcal{N}_1, \phi_1)$. Let $\widetilde{\alpha} : S^1 \times N_1 \to M$ be the adjoint map $\widetilde{\alpha}(t, n) = \alpha(t)(n)$. Since $\pi_i(M) = 0$ for $2 \leq i \leq 1 + n_1$, it follows from the elementary obstruction theory that the homomorphism

$$\widetilde{\alpha}_* : \pi_1(S^1 \times N) \to \pi$$

completely determines the homotopy class of $\widetilde{\alpha}$. We use the isomorphism $\pi_1(S^1 \times N) \simeq \pi_1(S^1) \times \pi_1(N)$ and set

$$\gamma = \widetilde{\alpha}_*(\iota, e), \tag{5.2}$$

where $\iota \in \pi_1(S^1)$ is the generator.

5.6. **Lemma.** *If $\gamma = e$, then $\Delta_{al}[(\alpha, e)] = 0$.*

Proof. Indeed, in this case $\widetilde{\alpha} : S^1 \times N_1 \to M$ is homotopic to the map

$$S^1 \times N_1 \xrightarrow{\text{proj}} N_1 \xrightarrow{\phi_1} M$$

because both maps induce the same homomorphism of fundamental groups. So, since $n_1 + n_2 = m - 1 < m$, there is a generic map $\widehat{\alpha}$ homotopic to $\widetilde{\alpha}$ such that $\widehat{\alpha}(S^1 \times N_1)$ does not meet $\phi_2(N_2)$. □

5.7. **Lemma.** *If $\operatorname{Im} \widetilde{\alpha}_* = \mathbb{Z} \subset \pi$, then $\Delta_{\text{alk}}[(\alpha, e)] = 0$.*

Proof. If $\operatorname{Im} \widetilde{\alpha}_* = \mathbb{Z}$, then $\widetilde{\alpha}_*$ can be decomposed as

$$\pi_1(S^1 \times N_1) \to \mathbb{Z} \subset \pi. \tag{5.3}$$

Since $S^1 = K(\mathbb{Z}, 1)$, the map $\pi_1(S^1 \times N_1) \to \mathbb{Z}$ in (5.3) can be induced by a map $\varphi : S^1 \times N_1 \to S^1$. Furthermore, the inclusion $\mathbb{Z} \subset \pi$ in (5.3) can be induced by a map (inclusion) $\psi : S^1 \to M$, and we can assume that $\psi(S^1)$ does not meet $\phi_2(N_2)$ since $m - n_2 > 1$. Now, the map $\widetilde{\alpha} : S^1 \times N_1 \to M$ is homotopic to a map

$$\widehat{\alpha} : S^1 \times N_1 \xrightarrow{\varphi} S^1 \xrightarrow{\psi} M.$$

Clearly, $\widehat{\alpha}$ does not meet $\phi_2(N_2)$ and, moreover, any small perturbation of $\widehat{\alpha}$ does not. Thus, $\Delta_{\text{alk}}[(\alpha, e)] = 0$. □

Now, assume that π is a Preissman group and consider the homomorphism

$$(\phi_1)_* : \pi_1(N_1) \to \pi.$$

If $\operatorname{Im}(\phi_1)_*$ is non-abelian, then $\gamma = e \in \pi$ by Proposition 5.4, because γ commutes with $\operatorname{Im}(\phi_1)_*$. So, $\Delta_{\text{alk}}[(\alpha, e)] = 0$ by Lemma 5.6.

Furthermore, assume that $\operatorname{Im}(\phi_1)_*$ is abelian. Since γ commutes with $\operatorname{Im}(\phi_1)_*$, we conclude that $\operatorname{Im}(\widetilde{\alpha}_*)$ is an abelian subgroup of π. So, $\operatorname{Im}(\widetilde{\alpha}_*) = \mathbb{Z}$ or 0 because π is a Preissman group. If it is $\mathbb{Z}$ then $\Delta_{\text{alk}}[(\alpha, e)] = 0$ by Lemma 5.7. If it is 0, then $\gamma = e \in \pi$ and $\Delta_{\text{alk}}[(\alpha, e)] = 0$ by Lemma 5.6.

Finally, assume that π is finite. Let k be the order of γ in π. Then, by Lemma 5.6,

$$0 = \Delta_{\text{alk}}[(\alpha^k, e)] = k\Delta_{\text{alk}}[(\alpha, e)] \in \Omega_0(\mathcal{B}).$$

Thus, $\Delta_{\text{alk}}[(\alpha, e)] = 0$ because the abelian group $\Omega_0(\mathcal{B})$ is torsion free. □

5.8. **Examples.** It is well-known that every closed manifold M^m that admits a complete Riemannian metric of negative sectional curvature has a universal cover homeomorphic to $\mathbb{R}^m$, and thus $\pi_i(M) = 0$, $i > 1$. Combining this with Theorems 5.1 and 5.5 we get that for such M the group Indet $= 0$ for all $N_1, N_2, \mathcal{N}_1, \mathcal{N}_2$, and hence $\mathbb{A}$ is a free abelian group. Actually using 2.6 it is possible to show that in most such cases $\mathbb{A}$ is infinitely generated.

Consider an orientation-preserving action of a (finite) group π on a homotopy sphere Σ^m (in fact, m must be odd). Then $\pi_k(\Sigma^m/\pi) = 0$ for $k = 2, \dots, m-1$, and using Theorem 5.5 we get that Indet $= 0$ for all $N_1, N_2, \mathcal{N}_1, \mathcal{N}_2$.

Theorem 4.1 implies that in all these cases, if $\mathcal{N}_1, \mathcal{N}_2$ consist of zero-homologous mappings, then alk is a splitting of the classical linking invariant lk into a direct sum of independent $\mathbb{Z}$-valued invariants. One uses 2.6 to show that this direct sum is infinite in most cases where M is closed and admits a complete metric of negative sectional curvature and the images of $\pi_1(N_1), \pi_1(N_2)$ in $\pi_1(M)$ under the homomorphisms induced by the mappings from $\mathcal{N}_1, \mathcal{N}_2$ are nontrivial.

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