Front-Running Dynamics

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October 9, 2006

1We thank Charlie Kahn and Burton Hollifield for helpful comments. We acknowledge financial support from the National Science Foundation, research grant #SES-0317706.
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Abstract

We integrate a monopolist dual trader who trades on his own account and processes orders from retail customers into a two-period dealership model of speculative trade. In static settings, Rochet and Vila establish a general irrelevance result—expected equilibrium outcomes are the same as if the monopolist speculator did not observe liquidity trade; and Roell shows that with multiple speculators, dual trading benefits liquidity traders. In dynamic settings, where the monopolist can front run—trade on knowledge of future liquidity trade—these results are reversed: the speculator profits from smoothing profits over time. Front running introduces positive serial correlation to order flow—as in the data. This causes market makers to discount past order flow in prices, but prices retain the martingale property.

Keywords: market microstructure finance, dual trading, informed speculators, informed traders, liquidity trade, private information.

JEL: G1, G14, D43
1 Introduction

This paper develops a dynamic model of a monopolist speculator who combines long-lived private information about asset values with knowledge of current and future liquidity trade. Specifically, the paper extends the Rochet and Vila (1994) model to dynamic settings, integrating a monopolist dual trader who both trades on his own account and processes all liquidity order flow from retail customers into a two-period dealership market (Kyle 1985). Such dual trading is endemic to many asset markets, including currency, commodity and futures markets. There is a widespread sense that dual trading must harm liquidity traders, as a speculator will exploit information about liquidity trade by taking partially-offsetting positions, recognizing that liquidity trade will drive prices away from true asset values.

While this intuition seems compelling, the literature has reached the opposite conclusion. Rochet and Vila (1994) prove that in static settings, equilibrium outcomes are exactly the same when a monopolist speculator sees all liquidity trade, as when the speculator sees no liquidity trade at all. This powerful irrelevance finding requires no structure on the distribution of private information and liquidity trade. What drives this result is that the benefits from seeing liquidity trade are just offset by the corresponding increased equilibrium price sensitivity to order flow. When the speculator offsets half of the liquidity trade, market makers respond by setting price schedules that are twice as sensitive to net order flow. The speculator scales down his trading intensity on private information about the asset proportionately, leaving the information content of prices and hence profits unaffected.

Roell (1990) reinforces this opposing conclusion, considering a single speculator who knows the asset’s (normally-distributed) value, and broker-dealers who see i.i.d. components of liquidity trade. She proves that liquidity trader losses are reduced by the presence of broker-dealers, as long as they do not handle all liquidity trade. Bernhardt and Taub (2006) extend the essence of Roell’s finding to settings with arbitrary numbers of speculators and general correlation structures on information. The key is that when speculators trade on liquidity trade with differing intensities, average trading intensity on liquidity trade rises, raising the signal-to-noise ratio in net order flow, reducing speculator profits.

We reconsider the impact of dual trading, using the simplest dynamic extension of the Rochet and Vila model—at the beginning of the two-date model, the monopolist speculator knows the asset’s value, as well as the liquidity order flow at each date. One might expect the neutrality result to extend. In fact, we show that the speculator now benefits at the expense of liquidity traders from the opportunity to trade on his knowledge of future liquidity trade. This result re-validates the lay wisdom that dual trading harms liquidity traders.

We find that the speculator trades against date-1 liquidity trade at both dates, because
at date 2, the market maker still confuses past liquidity order flow with speculative trade on fundamentals. In sharp contrast, trading on future liquidity trade takes a very different form—the speculator first trades in the same direction as the future liquidity trade, before taking a larger offsetting position at date 2. The speculator does this to offset the price impact of his larger date-2 trade on liquidity order flow. By doing so, the speculator adds liquidity trade at date 1, thereby smoothing profit extraction by transferring profits into the future. Thus, our model generates the front-running seen in practice, “Front-runners buy in front of large purchase orders and sell in front of large sell orders. Frontrunners [gain] by taking liquidity...[and] then offering this liquidity back at inferior prices (Harris (1997)).”

Standard market microstructure models predict that net order flow is serially uncorrelated: uncorrelated liquidity trade combined with the martingale property of prices imply that speculative trade has the martingale property relative to public information. Empirically, however, order flow exhibits strong autocorrelations. For example, using intraday data, Madhavan et al. (1997) find an autocorrelation coefficient on trades that exceeds .35. Our model delivers this autocorrelation structure. Even though liquidity order flow in our model is uncorrelated, the speculator’s front running introduces positive autocorrelation into net order flow. In turn, we prove that this positive autocorrelation implies that pricing does not take its standard form: in particular, the market maker partially discounts date-1 order flow in date-2 pricing. Despite this, pricing retains the martingale property.

Most fundamentally, we prove that the speculator uses his knowledge of liquidity trade to smooth expected profits over time. In standard contexts where liquidity trade is unobserved, the speculator extracts less profits as time passes, and trading intensities and information release are uneven. Front-running, which facilitates even profit extraction and even release of private information, leads to less aggregate information release through trade, and higher speculator profits, undoing the Rochet and Villa irrelevance result.

To emphasize that it is profit smoothing that drives this profit finding, and not the added opportunities to trade on knowledge of liquidity trade, we then consider a speculator who sees contemporaneous, but not future, liquidity trade. The speculator thus knows all of the components of net order flow affecting price when he submits his order. In this setting the analogue of the Rochet-Vila irrelevance result still does not hold. However, the nature of the result is the opposite of that when the speculator can front run: seeing contemporaneous, but not future, liquidity trade, exacerbates the speculator’s uneven extraction of profits, reducing total expected profits below those earned when liquidity trade is unobservable.

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1Cai (2002) documents that during the LTCM crisis there was “Extensive front-running by dealers who traded on their own accounts in the same direction as LTCM did, but just one or two minutes beforehand.
2 The model

There is a single risky asset whose date-2 liquidation value is given by

\[ v = v_0 + s, \]

where \( v_0 \) is public information at date 1 and \( s \) is drawn from a normal distribution with zero mean and variance \( \sigma_s^2 \). In addition to speculative trade, there is uninformed liquidity trade. Date-\( t \) liquidity trade is given by \( u_t \) where \( u_1, u_2 \sim N(0, \sigma_u^2) \) are independently and identically distributed. There is a single speculator who privately knows at the beginning of date 1 both the asset’s liquidation value and the liquidity trades at each date. In particular, the monopolist speculator can front-run by trading at date 1 on his knowledge of liquidity trade at date 2.

At the end of date 2, the asset’s liquidation value becomes public information. The speculator and liquidity traders trade in a market made by risk-neutral competitive, uninformed market makers who only see the net order flows, so that price equals the expected value of the asset given the net order flow history.

We conjecture that the equilibrium prices are linear functions of net order flows,

\[ p_1 = v_0 + \lambda_{11}(x_1 + u_1) \quad \text{and} \quad p_2 = v_0 + \lambda_{12}(x_1 + u_1) + \lambda_{22}(x_2 + u_2). \]

The conjecture allows the possibility that \( \lambda_{12} \neq \lambda_{11} \), i.e., date-2 price is not equal to date-1 price plus a martingale increment. We also conjecture that trading strategies take the form

\[ x_1 = a_0 s + a_1 u_1 + a_2 u_2 \quad \text{and} \quad x_2 = b_0 s + b_1 u_1 + b_2 u_2. \]

We organize and present the properties of the equilibrium in a series of characterizing propositions. The first proposition establishes the general form of pricing.

**Proposition 1** Prices evolve according to a martingale. However, date 2 price is not the sum of date 1 price plus a martingale increment. Rather, date-2 pricing discounts date-1 order flow, weighting it according to \( \lambda_{12} = \lambda_{11} - \lambda_{22}\rho \).

**Proof:** Since date \( t \) price equals the expected value of the asset given the net-order flow history, equilibrium pricing at each date takes the form of a projection. At date 1 we have

\[
\begin{align*}
p_1(x_1 + u_1) &= v_0 + P[s|x_1 + u_1] \\
&= v_0 + \frac{\text{cov}(s, a_0 s + a_1 u_1 + a_2 u_2 + u_1)}{\text{var}(a_0 s + a_1 u_1 + a_2 u_2 + u_1)}(x_1 + u_1) \\
&= v_0 + \frac{a_0 \sigma_s^2}{a_0^2 \sigma_s^2 + (1 + a_1)^2 \sigma_u^2 + a_2^2 \sigma_u^2}(x_1 + u_1) = v_0 + \lambda_{11}(x_1 + u_1). \quad (1)
\end{align*}
\]
At date 2, using the recursive projection formula (Sargent, 1987, p. 228), we have
\[
p_2(x_1 + u_1, x_2 + u_2) = v_0 + P[s|x_1 + u_1] + P \left[ s - P[s|x_1 + u_1] \right] x_2 + u_2 - P[x_2 + u_2|x_1 + u_1]
\]
\[
= v_0 + \lambda_{11}(x_1 + u_1) + P \left[ s - \lambda_{11}(x_1 + u_1) \right] x_2 + u_2 - P[x_2 + u_2|x_1 + u_1]
\]
\[
= v_0 + \lambda_{11}(x_1 + u_1) + P \left[ s \right] x_2 + u_2 - P[x_2 + u_2|x_1 + u_1].
\] (2)

The last equality follows because \( x_1 + u_1 \) is orthogonal to the forecast error \( x_2 + u_2 - P[x_2 + u_2|x_1 + u_1] \). This immediately establishes the martingale property.

To solve for date-2 pricing, we must calculate \( P[x_2 + u_2|x_1 + u_1] = \rho(x_1 + u_1) \):
\[
\rho(x_1 + u_1) = \frac{\text{cov}(b_0s + b_1u_1 + b_2u_2 + u_2, a_0s + a_1u_1 + a_2u_2 + u_1)}{\text{var}(a_0s + a_1u_1 + a_2u_2 + u_1)}(x_1 + u_1)
\]
\[
= \frac{b_0a_0\sigma_s^2 + b_1(1 + a_1)\sigma_u^2 + (1 + b_2)a_2\sigma_u^2}{a_0^2\sigma_s^2 + (1 + a_1)^2\sigma_u^2 + a_2^2\sigma_u^2}(x_1 + u_1).
\] (3)

Here, \( \rho > 0 \) would indicate that date-2 pricing discounts date-1 order flow. We substitute for
\[x_2 + u_2 - \rho(x_1 + u_1) = (b_0 - \rho a_0)s + (b_1 - \rho(1 + a_1))u_1 + ((b_2 + 1) - \rho a_2)u_2\]
to find the projection coefficient of \( P \left[ s \right] x_2 + u_2 - P[x_2 + u_2|x_1 + u_1] = P \left[ s \right] x_2 + u_2 - \rho(x_1 + u_1) \),
\[
\lambda_{22} = \frac{\text{cov}(s, (b_0 - \rho a_0)s)}{\text{var}((b_0 - \rho a_0)s) + \text{var}((b_1 - \rho(1 + a_1))u_1) + \text{var}(b_2 + 1 - \rho a_2)u_2)
\]
\[
= \frac{(b_0 - \rho a_0)^2\sigma_s^2 + (b_1 - \rho(1 + a_1))^2\sigma_u^2 + (b_2 + 1 - \rho a_2)^2\sigma_u^2}{(b_0 - \rho a_0)^2\sigma_s^2 + (b_1 - \rho(1 + a_1))^2\sigma_u^2 + (b_2 + 1 - \rho a_2)^2\sigma_u^2}.
\] (4)

Finally, grouping terms in (2) confirms that
\[
p_2(x_1 + u_1, x_2 + u_2) = (\lambda_{11} - \lambda_{22}\rho)(x_1 + u_1) + \lambda_{22}(x_2 + u_2). \quad \blacksquare
\]

Propositions 2 and 3 below detail how the correlation in order flow across periods induced by the speculator’s front running on liquidity trade complicates date-2 pricing. The market maker projects the asset’s value onto order flow at both dates, and due to the front running, which induces a positive autocorrelation in net order flows, he partially discounts date-1 order flow in date-2 pricing. To ease presentation we normalize \( v_0 \) to zero.

**Proposition 2** The weight in current price on current order flow does not vary over time: \( \lambda_{11} = \lambda_{22} = \lambda = \sqrt{\frac{2\sigma_s^2}{3\sigma_u^2}} \). Date 2 price discounts date 1 order flow by one-third. In particular, \( p_2(x_1 + u_1, x_2 + u_2) = \frac{4}{3}\lambda(x_1 + u_1) + \lambda(x_2 + u_2) \). The speculator first trades in the same direction as future liquidity trade, before taking a larger offsetting position at date 2:
\[
x_1 = \frac{3}{8\lambda} s - \frac{7}{16} u_1 + \frac{3}{16} u_2
\] (5)
\[
x_2 = \frac{3}{8\lambda} s - \frac{7}{16} u_1 - \frac{9}{16} u_2.
\] (6)
Proof: We begin by solving the speculator’s optimization problem:

\[
\max_{\{x_1, x_2\}} \{ (s - \lambda_{11}(x_1 + u_1))x_1 + (s - (\lambda_{11} - \lambda_{22}\rho))(x_1 + u_1) - \lambda_{22}(x_2 + u_2) \}.
\]

The associated first-order conditions are

\[
0 = s - 2\lambda_{11}x_1 - \lambda_{11}u_1 - (\lambda_{11} - \lambda_{22}\rho)x_2
\]
\[
0 = s - 2\lambda_{22}x_2 - (\lambda_{11} - \lambda_{22}\rho)x_1 - (\lambda_{11} - \lambda_{22}\rho)u_1 - \lambda_{22}u_2.
\]

Solving yields

\[
x_1 = \frac{(2\lambda_{22} - \lambda_{11} + \lambda_{22}\rho)s - (2\lambda_{22}\lambda_{11} - (\lambda_{11} - \lambda_{22}\rho)^2)u_1 + (\lambda_{11} - \lambda_{22}\rho)\lambda_{22}u_2}{4\lambda_{11}\lambda_{22} - (\lambda_{11} - \lambda_{22}\rho)^2}.
\]
\[
x_2 = \frac{(\lambda_{11} + \lambda_{22}\rho)s - (\lambda_{11} - \lambda_{22}\rho)\lambda_{11}u_1 - 2\lambda_{11}\lambda_{22}u_2}{4\lambda_{11}\lambda_{22} - (\lambda_{11} - \lambda_{22}\rho)^2}.
\]

We next substitute the conjectures that \(\lambda_{11} = \lambda_{22} = \lambda\), and \(\rho = \frac{1}{3}\), and then verify these conjectures. Substituting these values into the speculator’s trading strategy yields

\[
x_1 = \frac{\frac{4}{3}s - (2 - \frac{4}{9})\lambda^2u_1 + \frac{2}{3}\lambda^2u_2}{4\lambda^2 - \frac{4}{9}\lambda^2} = \frac{\frac{4}{3}s - \frac{14}{9}\lambda^2u_1 + \frac{2}{3}\lambda^2u_2}{\frac{32\lambda^2}{9}} = \frac{3}{8\lambda}s - \frac{7}{16}u_1 + \frac{3}{16}u_2
\]
\[
x_2 = \frac{\frac{4}{3}s - \frac{23}{3}\lambda u_1 - 2\lambda u_2}{\frac{32\lambda^2}{9}} = \frac{3}{8\lambda}s - \frac{3}{16}u_1 - \frac{9}{16}u_2.
\]

We then solve for \(\lambda\) by substituting into the market maker’s date 1 zero-profit condition,

\[
0 = E[(s - \lambda(x_1 + u_1))(x_1 + u_1)] \Rightarrow E\left[\frac{5s}{8\lambda} \cdot \frac{3s}{8\lambda}\right] = E\left[\lambda \left(\frac{9u_1}{16} + \frac{3u_2}{16}\right)^2\right]
\]
\[
\Rightarrow \frac{15\sigma^2_s}{64\lambda} = \lambda\sigma^2_u = \frac{90}{256} \Rightarrow \lambda = \sqrt{\frac{2\sigma^2_s}{3\sigma^2_u}}.
\]

To confirm our conjecture that \(\lambda_{22} = \lambda_{11}\), we substitute for \(\lambda\) into the equation (4) for \(\lambda_{22}\),

\[
\lambda_{22} = \frac{\left(\frac{3}{8\lambda} - \frac{1}{3\lambda}\right)^2\sigma^2_s}{\lambda^2} = \frac{\left(-\frac{3}{16} - \frac{9}{32}\right)^2\sigma^2_u}{\lambda^2} + \left(1 - \frac{9}{16} - \frac{1}{16}\right)^2\sigma^2_u
\]
\[
= \frac{\frac{1}{16}\lambda\sigma^2_s + \frac{9}{32}\sigma^2_u}{\frac{32\sigma^2_u}{2}} = \sqrt{\frac{3\sigma^2_s}{2\sigma^2_u}} = \sqrt{\frac{3\sigma^2_s}{2\sigma^2_u}} = \lambda.
\]

Finally, substituting for \(\lambda\) into (3) verifies that \(\rho = \frac{1}{3}\):

\[
\rho = \frac{3\sigma^2_s}{\frac{3}{8\lambda}\sigma^2_s - \frac{3}{16}(1 - \frac{7}{16})\sigma^2_u + (1 - \frac{9}{16})\frac{3}{16}\sigma^2_u}{(\frac{3}{8\lambda})^2\sigma^2_s + (1 - \frac{7}{16})^2\sigma^2_u + (\frac{3}{16})^2\sigma^2_u}
\]

5
There are three key features to note in the solution for the trading strategy. Most importantly, at date 1, the speculator trades against his information about date-2 liquidity trade, i.e., \( a_2 = \frac{3}{16} > 0 \), trading in the same direction as future liquidity trade. This is because the speculator recognizes that at date 2, he will take a position on the opposite side (as he offsets the impact of date-2 liquidity trade), and he wants to reduce the price impact of his (larger) date-2 trade on liquidity order flow. The central impact of trading against information about date-2 liquidity trade at date 1 is to reduce date-1 profit (by providing added date-1 liquidity trade of \( a_2 u_2 \)), and to increase date-2 profit, thereby smoothing profit extraction intertemporally. Next note that in contrast to his trade on date-2 liquidity, the speculator trades in the same direction at both dates on his information about date-1 liquidity trade. This is because, at date 1, the speculator only partially offsets the date-1 liquidity trade, offsetting fraction \( \frac{7}{16} \), so that the market maker still confuses high liquidity trade with a high \( s \), leading the speculator to continue to take a position opposite the date-1 liquidity trade at date 2. Finally, at each date, the speculator trades with the same intensity on his private information about the value of the asset.

We now derive the implications of Proposition 3 for the correlation structure in order flow.

**Corollary 1** *Net order flow is positively autocorrelated.*

**Proof:** Adding liquidity trade to the speculator orders in (6) yields net order flow:

\[
\begin{align*}
x_1 + u_1 &= \frac{3}{8\lambda} s + \frac{9}{16} u_1 + \frac{3}{16} u_2 \\
x_2 + u_2 &= \frac{3}{8\lambda} s - \frac{3}{16} u_1 + \frac{7}{16} u_2.
\end{align*}
\]

The covariance between date-1 and date-2 order flow is

\[
\text{cov}(x_1 + u_1, x_2 + u_2) = \frac{9\sigma_s^2}{64\lambda^2} - \frac{3\sigma_u^2}{128} = \frac{9\sigma_s^2}{64\frac{27}{16}\sigma_s^2} - \frac{3\sigma_u^2}{16} = \frac{3\sigma_u^2}{16} > 0.
\]

This corollary is important. In standard models of speculation, if correlation is not built into the liquidity trade process, net order flow is serially uncorrelated. This is most easily
observed by noting that period \( t \) price is typically a martingale increment on period \( t - 1 \) price, and if liquidity trade is serially uncorrelated, then net order flow must be as well. In our model, liquidity trade is uncorrelated, but the speculator’s front running introduces serial correlation into the order flow. Thus, this corollary indicates that front running can rationalize Madhavan et al.’s (1997) finding that order flow is strongly autocorrelated.

We next establish that the speculator’s front running allows him to smooth profits over time, so that ex-ante, the speculator expects to earn the same profit in each period.

**Proposition 3** The speculator uses his advance knowledge of future liquidity trade to perfectly smooth expected profits over time.

**Proof:** We compute the speculator’s expected period profits via the market maker’s zero-profit condition, as they must equal the expected losses of the liquidity traders. Hence,

\[
E[\pi_1] = -E[(s - \lambda (x_1 + u_1))u_1] = E[\lambda \frac{9}{16} u_1^2] = \frac{3\sqrt{3}}{8\sqrt{2}} \sigma_s \sigma_u. \tag{12}
\]

\[
E[\pi_2] = -E[(s - \frac{2}{3} \lambda (x_1 + u_1) - \lambda (x_2 + u_2))u_2] = E[(\frac{2}{3} \lambda \frac{3}{16} u_2 + \lambda \frac{7}{16} u_2)u_2] = E[\lambda \frac{9}{16} u_2^2] = \frac{3\sqrt{3}}{8\sqrt{2}} \sigma_s \sigma_u. \tag{13}
\]

We contrast this profit result with its counterpart in standard settings where the speculator does not see liquidity trade. In this setting, the speculator’s profits fall over time: as time passes, the market maker combines information in each period’s order flow to extract more information about \( s \). This inability to smooth profits hurts the speculator.

**Proposition 4** At date 1, the speculator expects higher profits when liquidity trade is unobservable than when it is observable; but at date 2, the speculator expects lower profits when liquidity trade is unobservable. The speculator’s total expected profit is higher when he can front run on knowledge of liquidity trade than when the speculator does not observe liquidity trade.

**Proof:** When the speculator does not observe liquidity trade, prices are linear functions of net order flows and date-2 price equals date-1 price plus a martingale increment,

\[
p_1(x_1 + u_1) = \lambda_1(x_1 + u_1) \quad \text{and} \quad p_2(x_1 + u_1, x_2 + u_2) = p_1(x_1 + u_1) + \lambda_2(x_2 + u_2).
\]

At date 2, the speculator maximizes

\[
E[(s - p_1 - \lambda_2(x_2 + u_2))x_2] \Rightarrow x_2 = \frac{s - \lambda_1(x_1 + u_1)}{2\lambda_2}.
\]
Substituting the functional form for \( x_2 \), we see that the speculator chooses \( x_1 \) to solve

\[
\max_{x_1} E \left[ (s - \lambda_1(x_1 + u_1))x_1 \right] + E \left[ \frac{(s - \lambda_1(s_1 + u_1))^2}{4\lambda_2} \right].
\]

The first-order condition for \( x_1 \) is

\[
s - 2\lambda_1 x_1 + \frac{(s - \lambda_1 x_1)(-\lambda_1)}{2\lambda_2} = 0 \quad \Rightarrow \quad x_1 = \frac{s(2\lambda_2 - \lambda_1)}{\lambda_1(4\lambda_2 - \lambda_1)} \quad \text{and} \quad x_2 = \frac{s}{4\lambda_2 - \lambda_1} - \frac{\lambda_1 u_1}{2\lambda_2}.
\]

We next substitute for \( x_1 \) and \( x_2 \) into the projection formulas for \( \lambda_1 \) and \( \lambda_2 \):

\[
\lambda_1 = \frac{\text{cov}(x_1 + u_1, s)}{\text{var}(x_1 + u_1)} = \frac{\sigma_s^2 (2\lambda_2 - \lambda_1)}{\sigma_s^2 \left( \frac{(2\lambda_2 - \lambda_1)}{4\lambda_2 - \lambda_1} \right)^2 + \sigma_u^2} = \frac{2\lambda_2 - \lambda_1}{4\lambda_2 - \lambda_1} \lambda_1 \frac{(4\lambda_2 - \lambda_1)}{2\lambda_2 - \lambda_1}
\]

\[
\lambda_2 = \frac{\text{cov}(x_2 + u_2, s)}{\text{var}(x_2 + u_2)} = \frac{\sigma_s^2 \left( \frac{(2\lambda_2 - \lambda_1)}{4\lambda_2 - \lambda_1} \right)^2 + \sigma_u^2}{(4\lambda_2 - \lambda_1)^2 + \sigma_u^2} = \frac{2\lambda_2 - \lambda_1}{4\lambda_2 - \lambda_1} \lambda_2 \frac{(4\lambda_2 - \lambda_1)}{2\lambda_2 - \lambda_1}
\]

Writing \( \lambda_2 = a \lambda_1 \), we rearrange the two pricing relationships to obtain

\[
(4a\lambda_1 - \lambda_1)^2 \sigma_u^2 = \sigma_s^2 (2a - 1)2a \quad \Rightarrow \quad \lambda_1 = \sqrt{\frac{(2a - 1)2a \sigma_s}{(4a - 1)^2 \sigma_u}} \tag{14}
\]

\[
(4a^2 + 1)(4a\lambda_1 - \lambda_1)^2 \sigma_u^2 = \sigma_s^2 (3a - 1)4a \quad \Rightarrow \quad \lambda_1 = \sqrt{\frac{(3a - 1)4a \sigma_s}{(4a^2 + 1)(4a - 1)^2 \sigma_u}}. \tag{15}
\]

Dividing the first pricing relationship, equation (14) by the second, equation (15), yields a single equation in \( a \), which we manipulate to obtain a cubic equation for \( a \),

\[
8a^3 - 4a^2 - 4a + 1 = 0,
\]

and then solve to obtain \( a \sim 0.901 \). Substituting for \( a \) into (14) yields \( \lambda_1 = 0.464 \frac{\sigma_s}{\sigma_u} \), and hence \( \lambda_2 = 0.418 \frac{\sigma_s}{\sigma_u} \). Date-1 profits are then \( 0.464 \sigma_s \sigma_u \), which exceed the date-1 profits when the speculator sees liquidity trade and front runs, \( \frac{3\sqrt{3}}{8\sqrt{2}} \sigma_s \sigma_u \sim 0.459 \sigma_s \sigma_u \), and those profits, in turn exceed date-2 profits when the speculator does not see liquidity trade of \( 0.418 \sigma_s \sigma_u \).

Summing over time reveals that expected profits are 4% higher when the speculator processes liquidity trade, and can use that information to smooth profits intertemporally, than they are when liquidity trade is unobservable.

Thus, we prove that the general irrelevance result identified by Rochet and Villa in static contexts does not hold in dynamic contexts. More concretely, introducing longer-lived private information to dynamic settings provides a countervailing influence on the static competitive forces that we identify in multi-speculator contexts in Bernhardt and Taub (2006).
2.1 Contemporaneously-observable liquidity trade

In this section, we show that it is the ability to front run on advance knowledge of future liquidity trade that allows the speculator to increase lifetime profits by smoothing profit extraction intertemporally. We now consider a speculator who only sees contemporaneous liquidity trade, so that he does not observe $u_2$ until date 2. We prove that the analogue of the Rochet-Vila irrelevance result still does not hold. However, the nature of the result is the opposite of what happens when the speculator can front run: when the speculator only sees contemporaneous liquidity trade, this exacerbates his already uneven profit extraction, reducing his profits below those that he would receive were liquidity trade unobservable.

Mimicking our earlier analysis, it is easy to show that

$$x_1 = a_0 s + a_1 u_1 \quad \text{and} \quad x_2 = b_0 s + b_1 u_1 + b_2 u_2$$

$$p_1(x_1 + u_1) = \lambda_{11}(x_1 + u_1) \quad \text{and} \quad p_2(x_1 + u_1, x_2 + u_2) = (\lambda_{11} - \lambda_{22} \rho)(x_1 + u_1) + \lambda_{22}(x_2 + u_2),$$

where $\lambda_{11}$ and $\lambda_{22}$ take their forms in (1) and (4), after substituting $a_2 = 0$:

$$\lambda_{11} = \frac{a_0 \sigma_s^2}{a_0^2 \sigma_s^2 + (1 + a_1)^2 \sigma_u^2} \quad (16)$$

$$\lambda_{22} = \frac{b_0 - \rho a_0}{(b_0 - \rho a_0)^2 \sigma_s^2 + (b_1 - \rho (1 + a_1))^2 \sigma_u^2 + (b_2 + 1)^2 \sigma_u^2}. \quad (17)$$

The speculator’s date-2 optimization problem is

$$\max_{\{x_2\}} \left\{ (s - \lambda_{11}(x_1 + u_1))x_1 + (s - (\lambda_{11} - \lambda_{22} \rho)(x_1 + u_1) - \lambda_{22}(x_2 + u_2))x_2 \right\}.$$

The associated first-order condition is

$$0 = s - 2 \lambda_{22} x_2 - (\lambda_{11} - \lambda_{22} \rho)x_1 - (\lambda_{11} - \lambda_{22} \rho)u_1 - \lambda_{22} u_2.$$  

Solving yields

$$x_2 = \frac{s}{2 \lambda_{22}} - \frac{(\lambda_{11} - \lambda_{22} \rho)(x_1 + u_1)}{2 \lambda_{22}} - \frac{u_2}{2}.$$  

At date 1, the speculator solves

$$\max_{\{x_1\}} \left\{ (s - \lambda_{11}(x_1 + u_1))x_1 + E[(s - (\lambda_{11} - \lambda_{22} \rho)(x_1 + u_1) - \lambda_{22}(x_2 + u_2))x_2|s, u_1] \right\}.$$

The associated first-order condition is

$$0 = s - 2 \lambda_{11} x_1 - \lambda_{11} u_1 - (\lambda_{11} - \lambda_{22} \rho)E[x_2|s, u_1].$$
Substituting for \( E[x_2|s, u_1] = \frac{s}{2\lambda_{22}} - \frac{(\lambda_{11} - \lambda_{22})u_1}{2\lambda_{22}} \), we solve for
\[
x_1 = \frac{(2\lambda_{22} - \lambda_{11} + \lambda_{22})s - (2\lambda_{22}\lambda_{11} - (\lambda_{11} - \lambda_{22})^2)u_1}{4\lambda_{11}\lambda_{22} - (\lambda_{11} - \lambda_{22})^2} \tag{18}
\]
\[
E[x_2|s, u_1] = \frac{(\lambda_{11} + \lambda_{22})s - (\lambda_{11} - \lambda_{22})\lambda_{11}u_1}{4\lambda_{11}\lambda_{22} - (\lambda_{11} - \lambda_{22})^2} - \frac{u_2}{2} = E[x_2|s, u_1] - \frac{u_2}{2}. \tag{19}
\]
Comparing these solutions with those when the speculator sees \( u_2 \) at date 1, equations (7) and (8) reveals that the coefficients on \( s \) and \( u_1 \) are unchanged. However, the coefficient on \( u_2 \) at date 1 is zero; and at date 2, the speculator does not have to account for date-1 trading on \( u_2 \), leading him to offset half of date-2 liquidity order flow.

Proposition 5 characterizes equilibrium outcomes. The proposition shows that when future liquidity trade is unobservable, net order flow becomes serially uncorrelated, so that pricing takes it standard form at date 2—that is, the sum of date-1 price and a linear weighting of date-2 net order flow. Crucially, Proposition 5 shows that observing contemporaneous liquidity trade causes the speculator to increase date-1 trading intensities, leading to more uneven profit extraction and lower expected total profits than when liquidity trade is unobservable.

**Proposition 5** In equilibrium, with contemporaneously-observable liquidity trade, net order flow is serially uncorrelated so that \( \rho = 0 \). We have
\[
\lambda_{11} = \sqrt{\frac{9 - \sqrt{33}}{6}} \frac{\sigma_s}{\sigma_u} \quad \text{and} \quad \lambda_{22} = \frac{\lambda_{11}}{2(1 - \frac{\lambda_{11}^2}{\sigma_s^2})}.
\]
Profit extraction is less even than when liquidity trade is unobservable, with date-1 expected profit higher and date-2 expected profit lower than their unobservable liquidity trade counterparts. In turn, total expected profits when liquidity trade is observable contemporaneously are even lower than when liquidity trade is unobservable.

**Proof:** It is straightforward to verify that, as is typical, pricing is proportional to the signal-to-noise ratio, \( \frac{\sigma_s}{\sigma_u} \), and to simplify presentation, we set \( \sigma_s^2 = \sigma_u^2 = 1 \). We conjecture and verify that \( \rho = 0 \). Under this conjecture, we have
\[
a_0 = \frac{2\lambda_{22} - \lambda_{11}}{4\lambda_{11}\lambda_{22} - \lambda_{11}^2}, \quad a_1 = -\lambda_{11}a_0, \quad b_0 = \frac{\lambda_{11}}{4\lambda_{11}\lambda_{22} - \lambda_{11}^2}, \quad b_1 = -\lambda_{11}b_0, \quad b_2 = -\frac{1}{2}.
\]
Using \( 1 + a_1 = 1 - \lambda_{11} \frac{2\lambda_{22} - \lambda_{11}}{4\lambda_{11}\lambda_{22} - \lambda_{11}^2} \), we substitute for \( a_0 \) and \( a_1 \) into (16) to obtain
\[
\lambda_{11} = \frac{(2\lambda_{22} - \lambda_{11})(4\lambda_{11}\lambda_{22} - \lambda_{11}^2)}{(2\lambda_{22} - \lambda_{11})^2 + 4\lambda_{11}^2\lambda_{22}^2}.
\]
Dividing both sides by $\lambda_{11}$ and then multiplying through yields
\[(2\lambda_{22} - \lambda_{11})^2 + 4\lambda_{11}^2 \lambda_{22}^2 = (2\lambda_{22} - \lambda_{11})(4\lambda_{22} - \lambda_{11})\]
Subtracting the common $(2\lambda_{22} - \lambda_{11})^2$ from both sides, and then dividing by $2\lambda_{22}$ yields
\[2\lambda_{11}^2 \lambda_{22} = 2\lambda_{22} - \lambda_{11} \Rightarrow \lambda_{22} = \frac{\lambda_{11}}{2(1 - \lambda_{11}^2)}. \tag{20}\]

We now turn to the $\lambda_{22}$ equation (17), substituting for $b_0$, $b_1$ to obtain
\[\lambda_{22} = \frac{4(4\lambda_{11} \lambda_{22} - \lambda_{11}^2)\lambda_{11}}{4\lambda_{11}^2 + 4\lambda_{11}^4 + (4\lambda_{11} \lambda_{22} - \lambda_{11}^2)^2}\]
Substituting (20) for $\lambda_{22}$ above, and then dividing both sides by $\lambda_{11}$ yields
\[\frac{1}{2(1 - \lambda_{11}^2)} = \frac{4(\frac{4z}{2(1-z)} - z)}{4z + 4z^2 + (\frac{4z}{2(1-z)} - z)^2} = \frac{4(z + z^2)(1 - z)}{4(z + z^2)(1 - z)^2 + (z + z^2)^2}\]
Noting that $\lambda_{11}$ only enters quadratically, we substitute for $z = \lambda_{11}^2$ to obtain
\[\frac{1}{2(1 - z)} = \frac{4(\frac{4z}{2(1-z)} - z)}{4z + 4z^2 + (\frac{4z}{2(1-z)} - z)^2} = \frac{4(z + z^2)(1 - z)}{4(z + z^2)(1 - z)^2 + (z + z^2)^2}\]
Cross-multiplying yields
\[4(1 - z)^2 + (z + z^2) = 8(1 - z)^2 \Rightarrow 3z^2 - 9z + 4 = 0 \Rightarrow z = \frac{9 - \sqrt{33}}{6}\]
With the solutions for $\lambda_{11}$ and $\lambda_{22}$ in hand, it remains to confirm that $\rho = 0$, by showing that net order flow is serially uncorrelated. Direct substitution yields
\[\text{cov}(x_1 + u_1, x_2 + u_2) = \text{cov}(a_0 s + a_1 u_1 + u_1, b_0 s + b_1 u_1 + b_2 u_2 + u_2)\]
\[= a_0 b_0 \sigma_s^2 + (1 + a_1) b_1 \sigma_u^2\]
\[= a_0 b_0 \sigma_s^2 - \lambda_{11} (a_0 + b_0) \lambda_{11} b_0 \sigma_u^2\]
Dividing out $b_0$ and substituting for the remaining $a_0$ and $b_0$ terms,
\[a_0 \sigma_s^2 - (a_0 + b_0) \lambda_{11} \sigma_u^2 = (2\lambda_{22} - \lambda_{11}) \sigma_s^2 - (2\lambda_{22}) \lambda_{11} \sigma_u^2\]
\[= 2\lambda_{22} (1 - \lambda_{11} \frac{\sigma_u^2}{\sigma_s^2}) - \lambda_{11} \sigma_s^2 = 0.\]
Finally, to calculate profit, we exploit the fact that it is proportional to $\sigma_s \sigma_u$, and set $\sigma_s = \sigma_u = 1$. Expected date-1 profit equals expected date-1 liquidity trader losses, which equal $\lambda_{11} (1 - a_1) \sigma_s^2 \sim .477511$. Expected date-2 profit is $\lambda_{11} (1 - b_1) \sigma_u^2 \sim .402575$. Comparisons of these profits with those in the other settings completes the argument. \[\blacksquare\]
3 Conclusion

One’s economic intuition may be that speculators should profit from private information about liquidity trade just as they profit from private information about asset values. Rochet and Vila demonstrated that this intuition is misplaced in static settings: when the monopolist speculator trades against his knowledge of liquidity trade, effectively taking a fraction of liquidity trade out of the market, market makers respond by proportionately increasing the sensitivity of pricing to net order flow. This causes speculators to scale down their trading on information about asset values, leaving aggregate outcomes unaffected.

Extending the model dynamically destroys this irrelevance result. When front-running on future liquidity trade, a monopolist speculator reverses his static dual trading strategy: he first trades in the same direction as future liquidity trade, before taking a larger offsetting position. With this sequence of trades, the speculator smoothes profits intertemporally, thereby increasing expected lifetime trading profits beyond those obtained when liquidity trade is unobserved. To emphasize how it is the intertemporal profit smoothing that drives this result, we then contrast outcomes with those that obtain when the speculator sees contemporaneous liquidity order flow, so that he is fully informed about the components of date-t order flow when he trades, but he does not know future liquidity trade. We show that this causes the speculator to exacerbate the uneven extraction of profits intertemporally, lowering total expected profits below those when liquidity trade is unobserved.

4 Bibliography