A new characterization of perfect graphs
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Abstract

Let $D = (V(D), A(D))$ be a digraph; a kernel $N$ of $D$ is a set of vertices $N \subseteq V(D)$ such that $N$ is independent (for any $x, y \in N$ there is no arc between them) and $N$ is absorbent (for each $x \in V(D) - N$ there exists an $xN$-arc in $D$). A digraph $D$ is said to be kernel-perfect whenever each one of its induced subdigraphs has a kernel. A digraph $D$ is oriented by sinks when every semicomplete subdigraph of $D$ has at least one kernel.

Let $G$ be a graph and $\alpha = (\alpha_v)_{v \in V(G)}$ a family of mutually disjoint digraphs; a sum of $\alpha$ over $G$, denoted by $\sigma(\alpha, G)$ is a digraph defined as follows: Take $\bigcup_{v \in V(G)} \alpha_v$, and then for each $x \in V(\alpha_u)$ and $y \in V(\alpha_v)$ with $[u, v] \in E(G)$ we have at least one of the two arcs $(x, y)$ or $(y, x)$ in $\sigma(\alpha, G)$. The main result of this paper is the following theorem which provides a new characterization of Perfect Graphs.

**Theorem.** A graph $G$ is perfect if and only if for any family $\alpha = (\alpha_v)_{v \in V(G)}$ of mutually disjoint asymmetric kernel-perfect digraphs any sum of $\alpha$ over $G$, $\sigma(\alpha, G)$ oriented by sinks is kernel-perfect.

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**1 Introduction**

For general concepts we refer the reader to [2]. The concept of kernel of a digraph was introduced by J. von Neumann and O. Morgenstern in the context of Game Theory [8], the existence of a kernel in a digraph has been studied by several authors; a very complete survey on kernels can be found in [6].

Perfect Graphs have been deeply and widely studied by several authors. Since the statement of the Strong Perfect Graph Conjecture by Claude Berge around 1960 until its proof by M. Chudnovsky, P. Seymour et al. in 2006, [5], many authors have contributed to obtain nice properties and interesting characterizations of Perfect Graphs see for example [1], [9]. In particular Claude Berge
and Pierre Duchet conjectured in 1986 [3] that a graph $G$ is perfect if and only if any orientation by sinks of $G$ is a kernel-perfect digraph. This conjecture now proved (an implication was proved by E. Boros and V. Gurvich in [4] and the other implication is a consequence of the Strong Perfect Graph Theorem and some results by C. Berge and P. Duchet [3]) constructed an important bridge between two topics in Graph Theory (Colorings and Kernels)).

In this paper we give a new characterization of Perfect Graphs in terms of sums of digraphs. The main result of this paper is the following theorem.

**Theorem 1.1.** A graph $G$ is perfect if and only if for any family $\alpha = (\alpha_v)_{v \in V(G)}$ of mutually disjoint asymmetric kernel-perfect digraphs, any sum of $\alpha$ over $G$, $\sigma(\alpha, G)$ oriented by sinks is a kernel-perfect digraph.

2 Preliminaries

Let $D$ be a digraph; $V(G)$ and $A(D)$ will denote the sets of vertices and arcs of $D$ respectively. If $S_1, S_2 \subseteq V(D)$, the arc $(u_1, u_2)$ of $D$ will be called an $S_1$-$S_2$-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of $D$ induced by $S_1$. An arc $(u_1, u_2) \in A(D)$ is called asymmetric (resp. symmetric) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The directed cycle of length $n$ is denoted by $C_n$. The asymmetric part of $D$ (resp. symmetric part of $D$), which is denoted by $\text{Asym}(D)$ (resp. $\text{sym}(D)$) is the spanning subdigraph of $D$ whose arcs are the asymmetric (resp. symmetric) arcs of $D$. A digraph $D$ is called critical kernel-imperfect whenever $D$ has no kernel but every proper induced subdigraph has a kernel.

Define the digraph $C = C_n(j_1, j_2, \ldots, j_k)$ by $V(C) = \{0, 1, 2, \ldots, n-1\}$, $A(C) = \{(u, v) | v-u \equiv j_s (\text{mod} \ n) \text{ for } s = 1, \ldots, k\}$.

Let us recall that a graph $G$ is perfect if every induced subdigraph $H$ satisfies $\alpha(H) = \theta(H)$, where $\alpha(G)$ denotes the stability number of $G$ and $\theta(G)$ denotes the minimum number of cliques needed to cover the vertex-set of $G$. The Perfect Graph Conjecture is: $G$ is perfect if and only if there is no induced odd cycle $C_{2k+1}$ (with $k \geq 2$), and no induced $\overline{C}_{2k+1}$ (complement of a $C_{2k+1}$, $k \geq 2$).

An orientation of a graph $G$ is a digraph $D_G$ obtained from $G$ by replacing each edge either by an arc or by two parallel arcs in opposite directions; in a digraph the orientation is normal if every semicomplete subdigraph contains a kernel.

The following theorems will be useful along the proof of the main result.

**Theorem 2.1** ([5] The Perfect Graph Theorem). A graph $G$ is perfect if and only if there is no induced odd cycle $C_{2k+1}$ (with $k \geq 1$), and no induced $\overline{C}_{2k+1}$ (complement of a $C_{2k+1}$, $k \geq 2$).

**Theorem 2.2** ([7]). The digraphs $\overline{C}_3$ and $\overline{C}_n(1, \pm 2, \ldots, \pm [n/2])$ are critical kernel-imperfect digraphs.

**Theorem 2.3** ([4]). If $G$ is a perfect graph then any normal orientation of $G$ is a kernel-perfect digraph.
Theorem 2.4 ([3]). The graph $C_{2n+1}$, $n \geq 2$ (resp. $\overline{C}_{2n+1}$, $n \geq 2$) has a normal orientation which is a critical kernel-imperfect digraph.

Theorem 2.5 ([3]). If $G$ is a graph such that any normal orientation of $G$ is kernel-perfect then $G$ has no induced $C_{2k+1}$ and no induced $\overline{C}_{2k+1}$.

Theorem 2.6. A graph $G$ is perfect if and only if any normal orientation of $G$ is a kernel-perfect digraph.

Proof. It is a directed consequence of Theorems 2.1, 2.3, 2.4 and 2.5.

Theorem 2.7 ([3]). A semicomplete digraph has a normal orientation if and only if every directed cycle has at least one symmetric arc.

3 The main result

We start this section by proving the following lemma which allow us to prove the main result and also implies a characterization of critical kernel-imperfect digraphs which are a sum of two asymmetric digraphs.

Lemma 3.1. Let $D_1$ and $D_2$ be asymmetric digraphs; $N_1$ a kernel of $D_1$ and $N_2$ a kernel of $D_2$: $\alpha = (D_1, D_2)$, $G = K_2$; and $\sigma(\alpha, G)$ a sum of $\alpha$ over $G$. If $\sigma(\alpha, G)$ has no induced $\overline{C}_3$ and no induced $\overline{C}_4(1, \pm 2)$, then at least one of the following assertions holds: (i) For each $z \in V(D_2)$ there exists a $zN_1$-arc in $\sigma(\alpha, G)$, ($N_1$ absorbs $V(D_2)$ in $\sigma(\alpha, G)$). (ii) For each $z \in V(D_1)$ there exists a $zN_2$-arc in $\sigma(\alpha, G)$, ($N_2$ absorbs $V(D_1)$ in $\sigma(\alpha, G)$).

Proof. First we prove that at least one of the two following assertions holds: (a) For each $x \in (V(D_2) - N_2)$ there exists a $zN_1$-arc in $\sigma = \sigma(\alpha, G)$. (b) For each $z \in (V(D_1) - N_1)$ there exists a $zN_2$-arc in $\sigma$.

Proceeding by contradiction suppose that (a) does not hold and (b) does not hold. Then there exists $x_1 \in (V(D_1) - N_1)$ such that for each $y \in N_2$, $(y, x_1) \in \text{Asym}(\sigma)$ and there exists $x_2 \in (V(D_2) - N_2)$ such that for each $z \in N_1$, $(z, x_2) \in \text{Asym}(\sigma)$. Since $x_1 \in (V(D_1) - N_1)$ and $N_1$ is a kernel of $D_1$, there exists $y_1 \in N_1$ such that $(x_1, y_1) \in A(\sigma)$, moreover $(x_1, y_1) \in \text{Asym}(\sigma)$ (as $D_1$ is an asymmetric digraph); analogously there exists $y_2 \in N_2$ such that $(x_2, y_2) \in \text{Asym}(\sigma)$. Thus $(x_1, y_1, x_2, y_2, x_1)$ is an asymmetric directed cycle of length 4 in $\sigma$. Now, we have $(y_1, y_2) \in \text{Sym}(\sigma)$ (otherwise $(y_1, y_2) \in \text{Asym}(\sigma)$ or $(y_2, y_1) \in \text{Asym}(\sigma)$; when $(y_1, y_2) \in \text{Asym}(\sigma)$ (resp. $(y_2, y_1) \in \text{Asym}(\sigma)$), we have that $(y_2, x_1, y_1, y_2)$ (resp. $(y_1, x_2, y_2, y_1)$) is an induced subdigraph of $\sigma$ isomorphic to $C_3$, which is a contradiction). Analogously $(x_1, x_2) \in \text{Sym}(\sigma)$. We conclude that $D \{\{x_1, y_1, x_2, y_2\}\} \cong \overline{C}_4(1, \pm 2)$ which is a contradiction to our hypothesis.

Now assume without loss of generality that (a) holds. (i.e. $N_1$ absorbs to $V(D_2) - N_2$ in $\sigma$). We will prove that at least one of the two assertions (i) or (ii) in Lemma 3.1 holds. In fact we will prove that if assertion (i) does not hold, then assertion (ii) holds. Assume that assertion(i) does not hold; since we are assuming that assertion (a) holds; it follows that there exists $x_2 \in N_2$ such that for each $y \in N_1$, $(y, x_2) \in \text{Asym}(\sigma)$. Now we will show that $x_2$ absorbs $V(D_1)$ (i.e. for each
$z \in V(D_1)$, $(z, x_2) \in A(\sigma))$. We have observed that $\{x_2\}$ absorbs $N_1$. Let $z \in (V(D_1) - N_1)$ be; since $N_1$ is a kernel of $D_1$; there exists $y \in N_1$ such that $(z, y) \in A(D_1) \subseteq \text{Asym}(\sigma)$. Now we have $(z, x_2) \in A(\sigma)$ (otherwise $(x_2, z) \in \text{Asym}(\sigma)$ and $(x_2, z, y, x_2)$ is an induced subdigraph of $\sigma$ isomorphic to $\overrightarrow{C}_3$ which contradicts our hypothesis). Thus $N_2$ absorbs $V(D_1)$ (i.e. assertion (ii) in Lemma 3.1 holds).

The following result is a direct consequence of Lemma 3.1

**Theorem 3.2.** Let $D_1$ and $D_2$ be asymmetric kernel-perfect digraphs; $\alpha = (D_1, D_2)$, $G \cong K_2$.
Any sum of $\alpha$ over $G$, $\sigma(\alpha, G)$ is a kernel-perfect digraph if and only if $\sigma(\alpha, G)$ has no induced $\overrightarrow{C}_3$ and no induced $\overrightarrow{C}_4(1, \pm 2)$.

**Corollary 3.3.** Let $D_1$ and $D_2$ be asymmetric digraphs; $\alpha = (D_1, D_2)$, $G \cong K_2$; $\sigma(\alpha, G)$ is a critical kernel-imperfect digraph if and only if $\sigma(\alpha, G) \cong \overrightarrow{C}_3$ or $\sigma(\alpha, G) \cong \overrightarrow{C}_4(1, \pm 2)$.

**Corollary 3.4.** Let $D_1$ and $D_2$ be asymmetric digraphs; $\alpha = (D_1, D_2)$, $G \cong K_2$. For any asymmetric sum of $\alpha$ over $G$, $\sigma(\alpha, G)$ we have that $\sigma(\alpha, G)$ is a critical kernel-imperfect digraph if and only if $\sigma(\alpha, G) \cong \overrightarrow{C}_3$.

**Theorem 3.5.** Let $\alpha = (\alpha_v)_{v \in V(G)}$ any family of mutually disjoint asymmetric kernel-perfect digraphs, and $G$ any graph.
$G$ is a perfect graph if and only if for each sum of $\alpha$ over $G$, $\sigma(\alpha, G) = \sigma$ any normal orientation of $\sigma$ is a kernel-perfect digraph.

**Proof.** First we will suppose that for each family of mutually disjoint asymmetric kernel-perfect digraphs $\alpha = (\alpha_v)_{v \in V(G)}$ it holds that any normal orientation of $\sigma = \sigma(\alpha, G)$ is a kernel-perfect digraph. We will prove that $G$ is a perfect graph. From Theorem 2.6 we only need to prove that any normal orientation of $G$ is a kernel-perfect digraph. Let $D_G$ be any normal orientation of $G$. Now we consider the following sum of $\alpha$ over $G$, $\sigma = \sigma(\alpha, G)$: Take $\bigcup_{u \in V(G)} \alpha_u$ and $(x, y) \in A(\sigma)$ whenever $x \in \alpha_u$, $y \in \alpha_v$ and $(x, y) \in A(D_G)$. We will prove that $\sigma$ is normal oriented digraph (i.e. that each semicomplete subdigraph of $\sigma$ has a kernel). Let $Q$ be any semicomplete subdigraph of $\sigma$ (i.e. for any two vertices $x, y \in V(Q)$ there exists at least one arc of $\sigma$ between them); we will show that $Q$ has a kernel. When $Q \subseteq \alpha_v$ for some $v \in V(G)$, we have that $Q$ has a kernel because $\alpha_v$ is a kernel-perfect digraph. When $Q \not\subseteq \alpha_v$ for each $v \in V(G)$ we consider $K = \{v \in V(G) | Q \cap V(\alpha_v) \neq \emptyset\}$ since $Q$ is a semicomplete digraph we have that $D_G[K]$ is also a semicomplete digraph of $D_G$; since $D_G$ is a normal orientation of $G$ we have that $D_G[K]$ has a kernel; let $\{x_0\}$ be such a kernel. Clearly $\alpha_{x_0} \cap Q$ is a semicomplete subdigraph of $\alpha_{x_0}$ and thus it has a kernel; say $\{z_0\}$. Now from the definition of $\sigma$ we have that $\{z_0\}$ is a kernel of $Q$. Thus we conclude that $\sigma$ is normal oriented. Therefore it follows from our hypothesis that $\sigma$ is a kernel-perfect digraph. Clearly, from the definition of $\sigma$ we have that $D_G$ is isomorphic to an induced subdigraph of $\sigma$ and since $\sigma$ is a kernel-perfect digraph; we conclude that $D_G$ is also a kernel-perfect digraph.
Now we will suppose that $G$ is a perfect graph. Let $\alpha = (\alpha_v)_{v \in V(G)}$ any family of mutually disjoint asymmetric kernel-perfect digraphs and $\sigma = \sigma(\alpha, G)$ any normal oriented sum of $\alpha$ over $G$. We will prove that $\sigma$ has a kernel.

Let $D_G$ be the orientation of $G$ defined as follows: Suppose $u$ is adjacent to $v$ in $G$. Let $N_u$ be a kernel of $\alpha_u$ and $N_v$ a kernel of $\alpha_v$. It follows from Lemma 3.1 that at least one of the two following assertions holds: (i) $N_u$ absorbs $V(\alpha_u)$, (ii) $N_v$ absorbs $V(\alpha_u)$. We put $(u, v) \in A(D_G)$ (resp. $(v, u) \in A(D_G)$) whenever (i) holds (resp. (ii) holds). Notice that $D_G$ can have a pair of parallel arcs with opposite direction between $u$ and $v$. Now we will prove that $D_G$ is a normal orientation of $G$.

Let $K$ be any semicomplete subdigraph of $D_G$. Assume by contradiction that $K$ is not normal oriented; it follows from Theorem 2.7 that $K$ contains an asymmetric directed cycle. Let $\gamma$ be an asymmetric directed cycle of minimum length contained in $K$; $\gamma = (z_0, z_1, \ldots, z_{n-1}, z_0)$. Thus $D_G[V(\alpha)] \cong \vec{C}_n \left(1, \pm 2, \pm 3, \ldots, \pm \left[\frac{n}{2}\right]\right)$. Let $N_{z_i}$ a kernel of $\alpha_{z_i}$ for each $i \in \{0, 1, \ldots, n-1\}$. Since $(z_i, z_{i+1}) \in \text{Asym}(D_G)$, it follows from the definition of the orientation of $D_G$ that there exists $w_{i+1} \in V(\alpha_{z_{i+1}})$ such that there is no $w_{i+1}N_{z_i}$-arc in $\sigma$ and then for each $x \in V(N_{z_i})$, $(x, w_{i+1}) \in \text{Asym}(\sigma)$. Now for each $i \in \{0, 1, \ldots, n-1\}$ we define $x_{i+1}$ and $y_{i+1} \equiv i \pmod{n}$ as follows: When $w_{i+1} \in N_{z_{i+1}}$ we denote $x_{i+1} = y_{i+1} = w_{i+1}$; when $w_{i+1} \notin N_{z_{i+1}}$ we denote $x_{i+1} = w_{i+1}$ and $y_{i+1} = \text{any fixed vertex in } N_{z_{i+1}}$ such that $(w_{i+1}, z_{i+1}) \in A(\alpha_{z_{i+1}})$. Clearly $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1}, x_0, y_0)$ is an asymmetric directed closed walk; so it contains an asymmetric directed cycle which is contained in $\sigma(\langle \alpha_v \rangle_{v \in K}, K)$ which is a semicomplete subdigraph of $\sigma(\alpha, G)$, contradicting that $\sigma(\alpha, G)$ is normally oriented. We conclude that $D_G$ is normal oriented. Since $G$ is a perfect graph and $D_G$ is a normal orientation of $G$, it follows from Theorem 2.6 that $D_G$ is a kernel-perfect digraph. Let $N$ be a kernel of $D_G$. Clearly $\bigcup_{u \in N} \{N_u \text{ kernel of } \alpha_u\}$ is a kernel of $\sigma(\alpha, G)$.

Now let $H$ be any proper induced subdigraph of $\sigma(\alpha, G)$.

Clearly $H = \sigma(\alpha', G')$ where $G'$ is an induced subgraph of $G$ and $\alpha' = (\alpha'_v)_{v \in V(G')}$. Moreover $\alpha'$ is a family of mutually disjoint asymmetric kernel-perfect digraphs and $H = \sigma(\alpha', G')$ is a normal oriented sum of $\alpha'$ over $G'$. Since $G$ is a perfect graph and $G'$ is an induced subgraph of $G$; it follows that $G'$ is also a perfect graph. So preceeding as before we can prove that $\sigma(\alpha', G')$ has a kernel. Thus $\sigma(\alpha, G)$ is a kernel perfect digraph, and Theorem 3.2 is proved.

Corollary 3.6. If $D$ is a critical kernel-imperfect digraph such that $D \cong \sigma(\langle \alpha_v \rangle_{v \in V(G)}, G)$ where $\alpha_v$ is asymmetric for each $v \in V(G)$ and $G$ is a perfect graph. Then $D \cong \vec{C}_3$ or $D \cong \vec{C}_n \left(1, \pm 2, \ldots, \pm \left[\frac{n}{2}\right]\right)$.

Corollary 3.7. If $D$ is an asymmetric critical kernel-imperfect digraph such that $D \cong \sigma(\langle \alpha_v \rangle_{v \in V(G)}, G)$ where $G$ is a perfect graph. Then $D \cong \vec{C}_3$.

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