Improved Algorithms for Min Cuts and Max Flows in Undirected Planar Graphs

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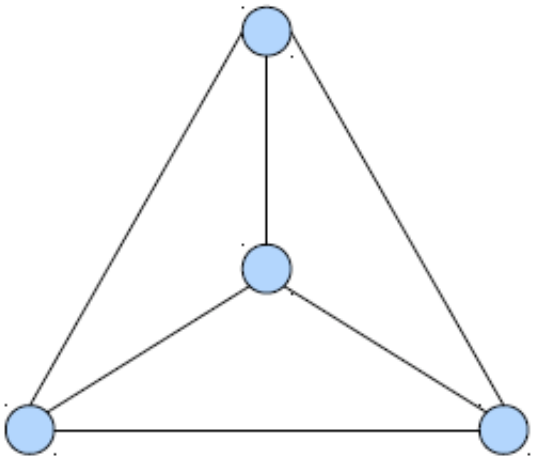
Joint work with Yahav Nussbaum, Piotr Sankowski and Christian Wulff-Nilsen

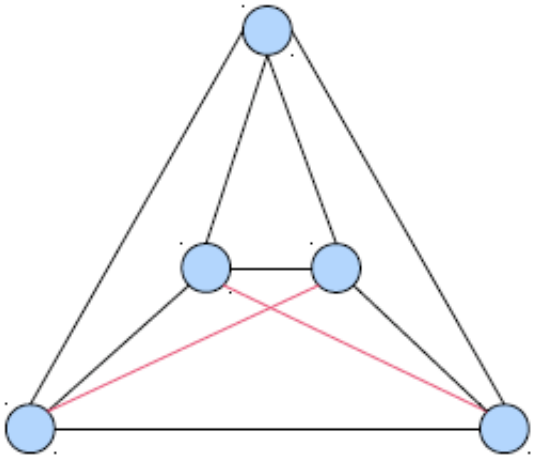
Outline

- Introduction
- History
- Dense Distance Graphs
- Minimum s-t cut in O(n log log n) time
- Applications, including max flow and dynamic min s-t cut

Planar Graphs

- A graph that has an embedding in the plane without crossing edges
- We assume that a combinatorial embedding is given
- Such embedding can be found in O(n) time – Hopcroft & Tarjan [1974]

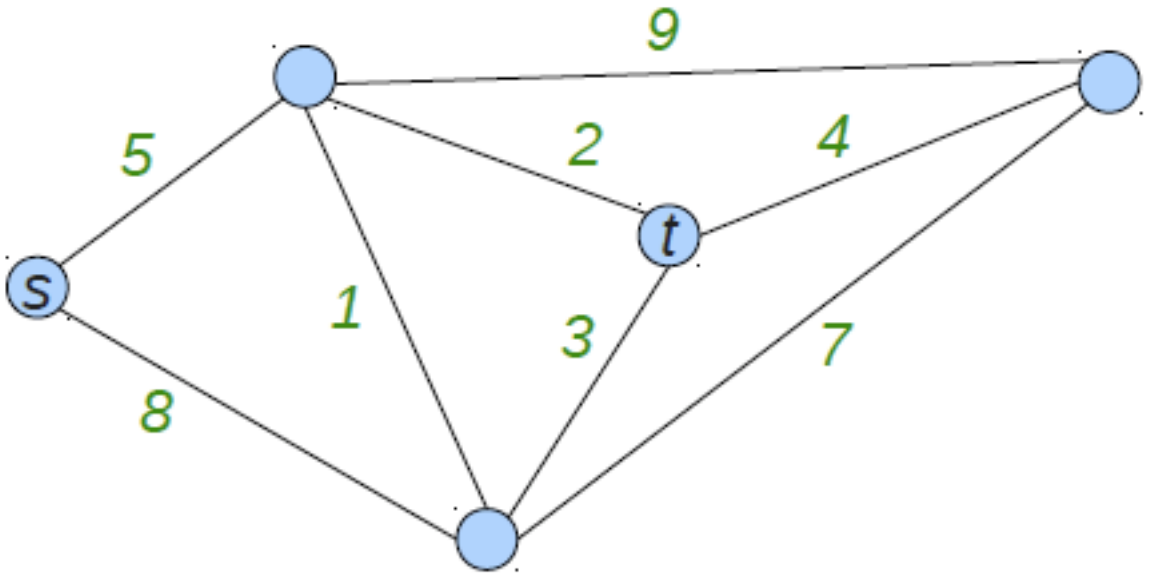




Why Planar Graphs?

- Special properties of planar graphs often allows for simpler and faster algorithms
- Arise in many applications – Transportation, VLSI, etc…

Min s-t Cut



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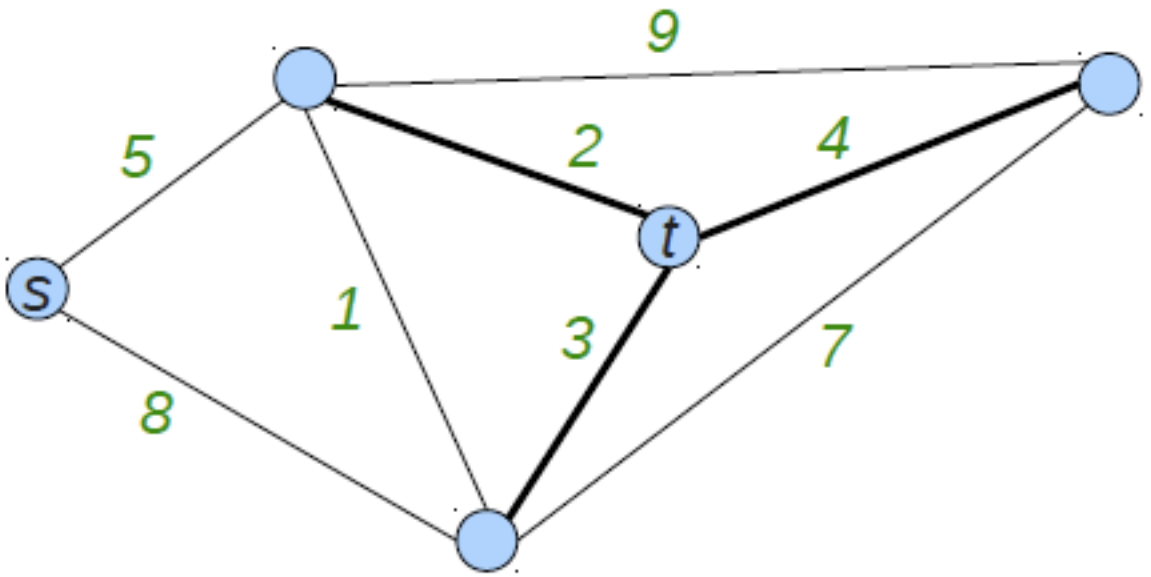


Among the Oldest Problems in CS

aka Warsaw Pact Rail Network

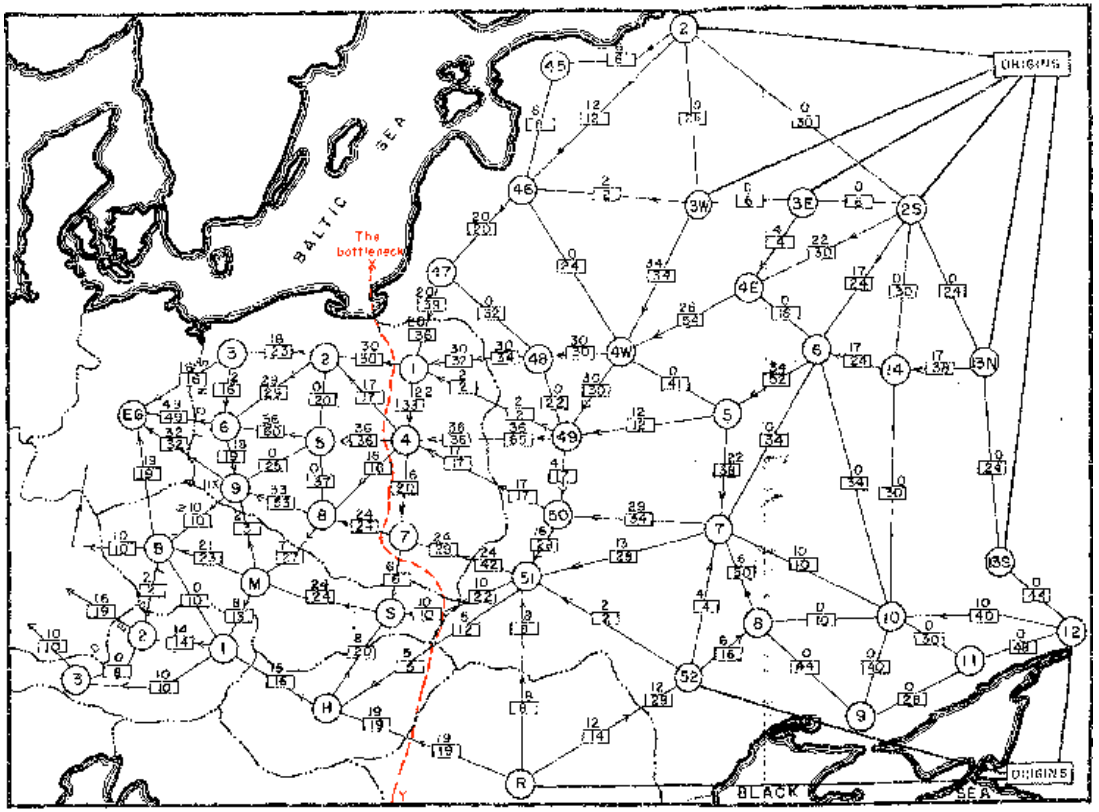
Harris & Ross [1955] studied rail network from Soviet Union to Eastern Europe

Edge weights = shipping rates

Computed by hand:

1. Max amount of stuff that could be moved from Soviet Union to Eastern Europe

2. "The bottleneck", i.e., cheapest way to disrupt network by removing links (blowing up train tracks)



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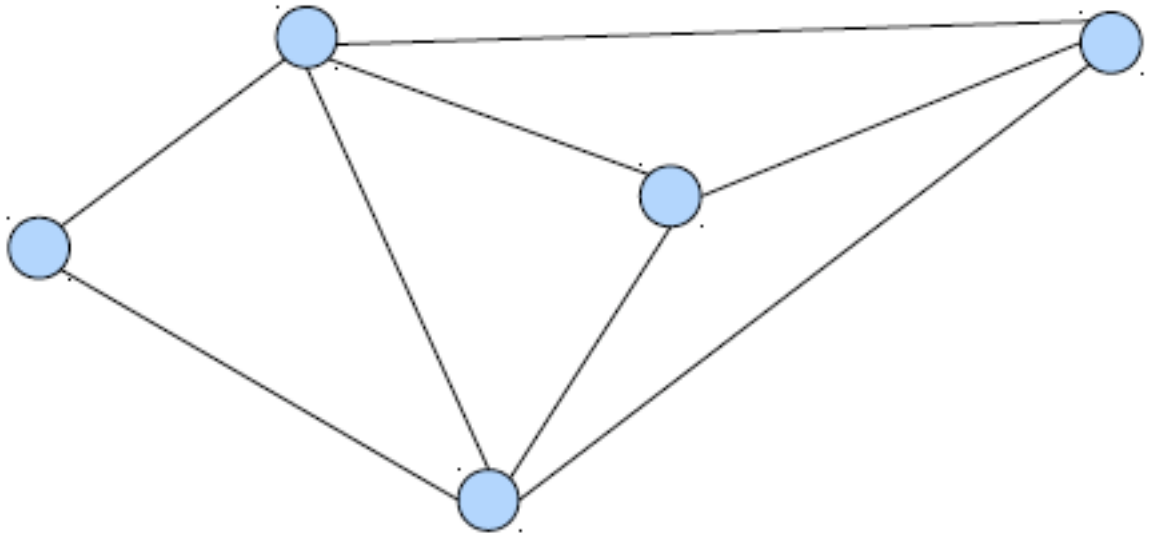
Dual Graph G^*

- Each face becomes a vertex



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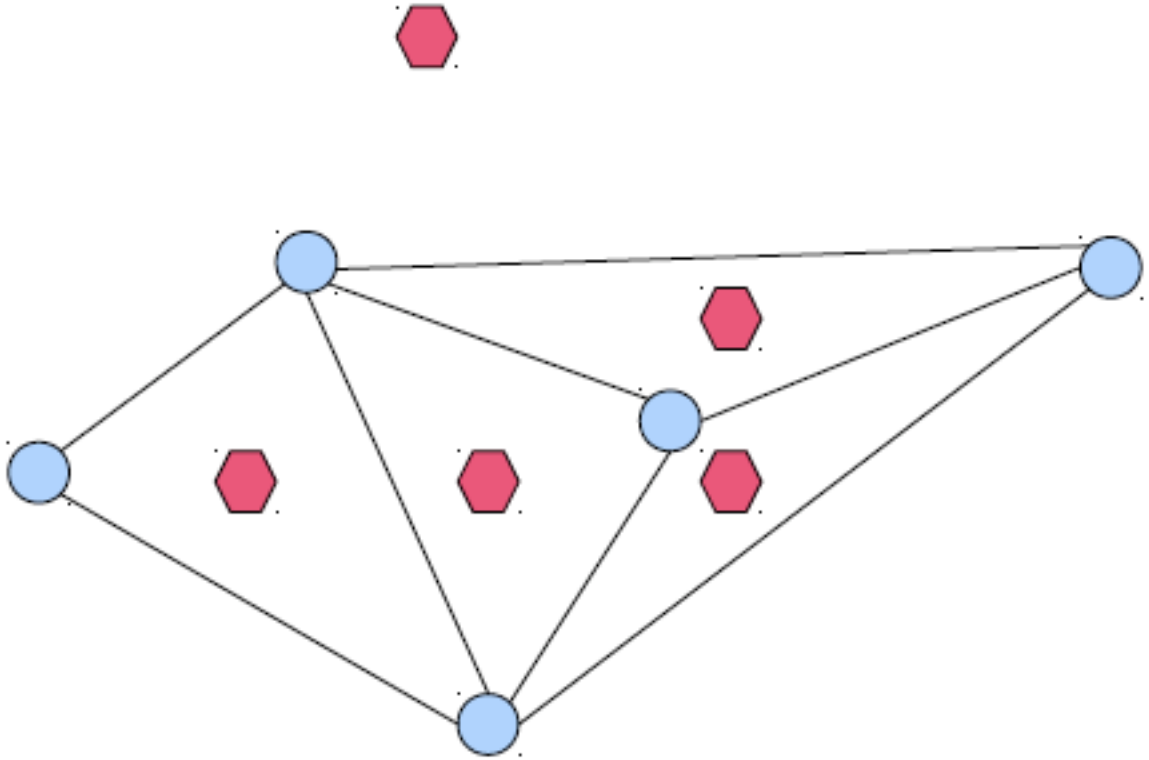
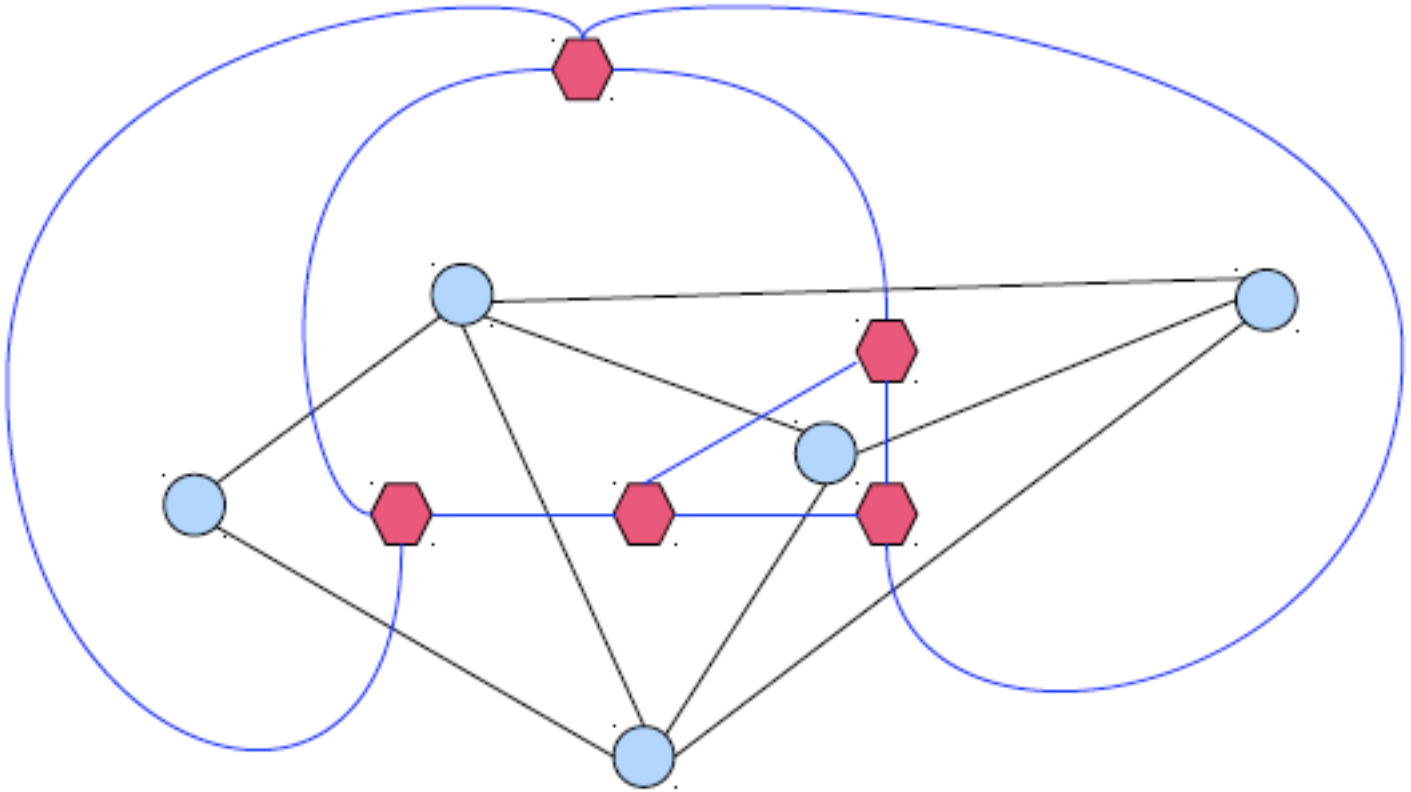


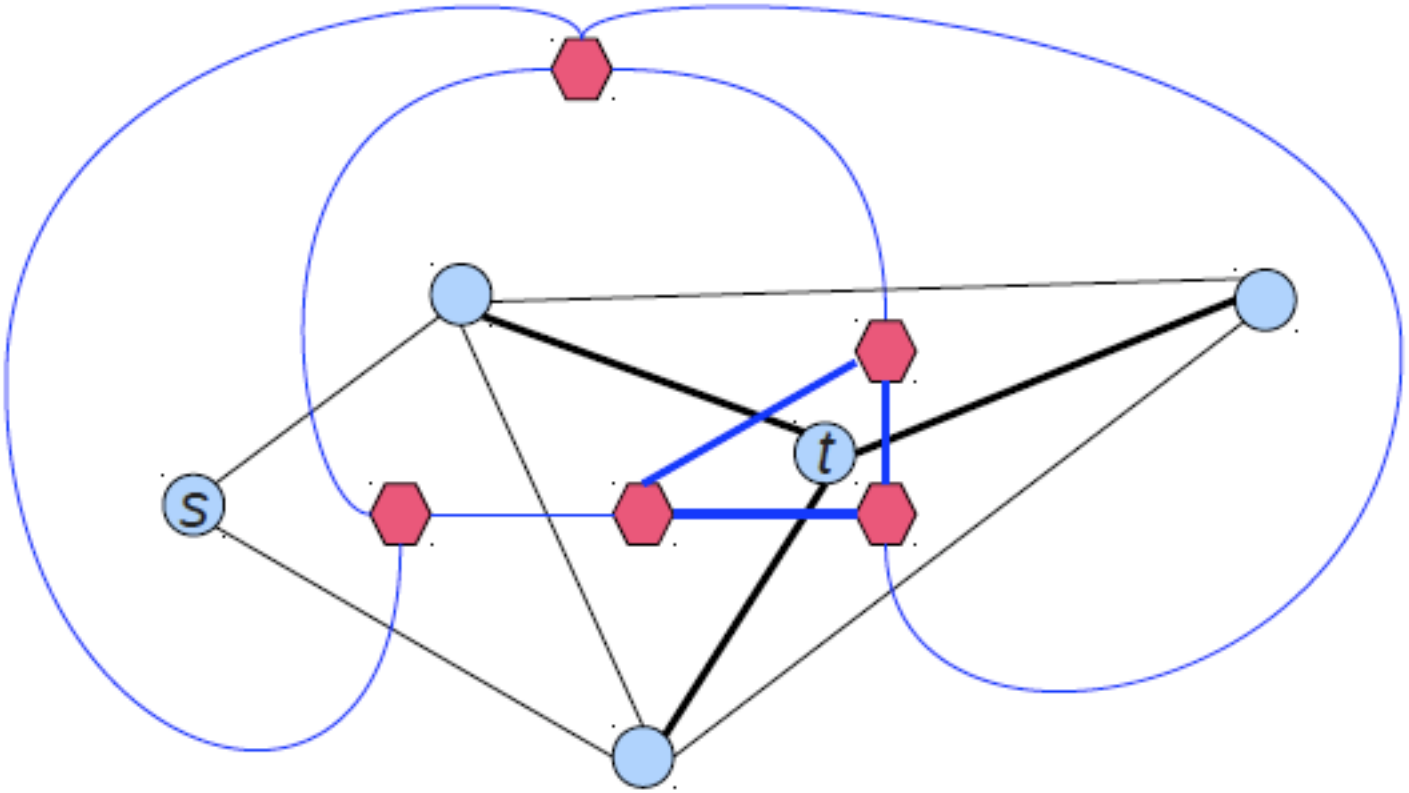
Dual Graph G^*

- Each face becomes a vertex
- Dual of e connects the faces adjacent to e
- The capacity of e is the length of its dual



Separating Cycles

- Dual of an s-t cut is is a cycle that separates s and t
- A min s-t separating cycle in G defines a min s-t cut in G^* – Itai & Shiloach [1979]



Itai & Shiloach [1979]

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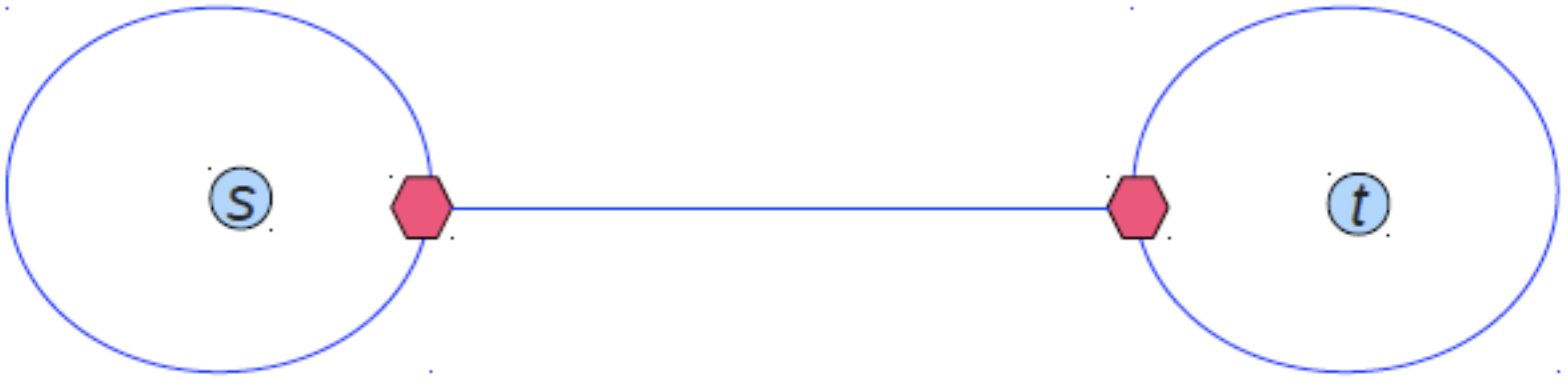

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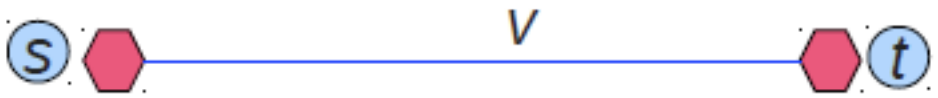
- Find the shortest path P from a face s* to a face t*
- The min s-t separating cycle that contains a specific vertex of P must cross P exactly once



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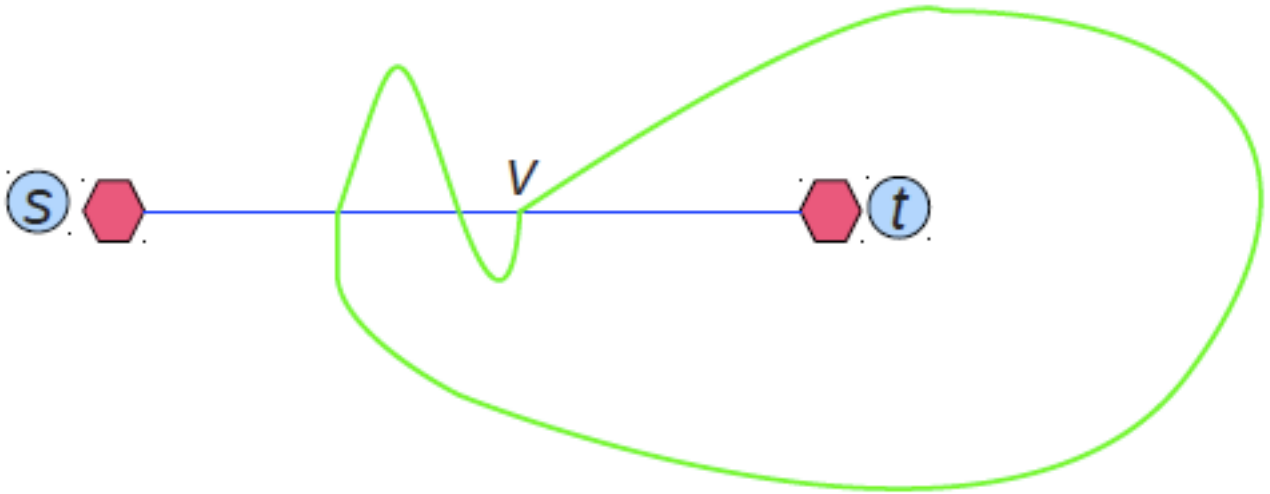

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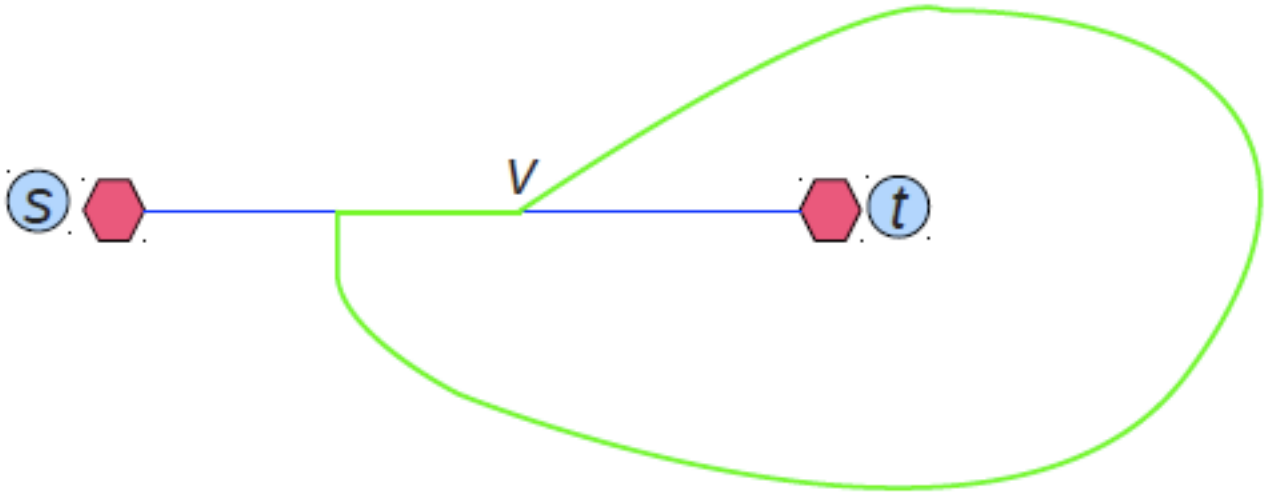

Itai & Shiloach [1979]

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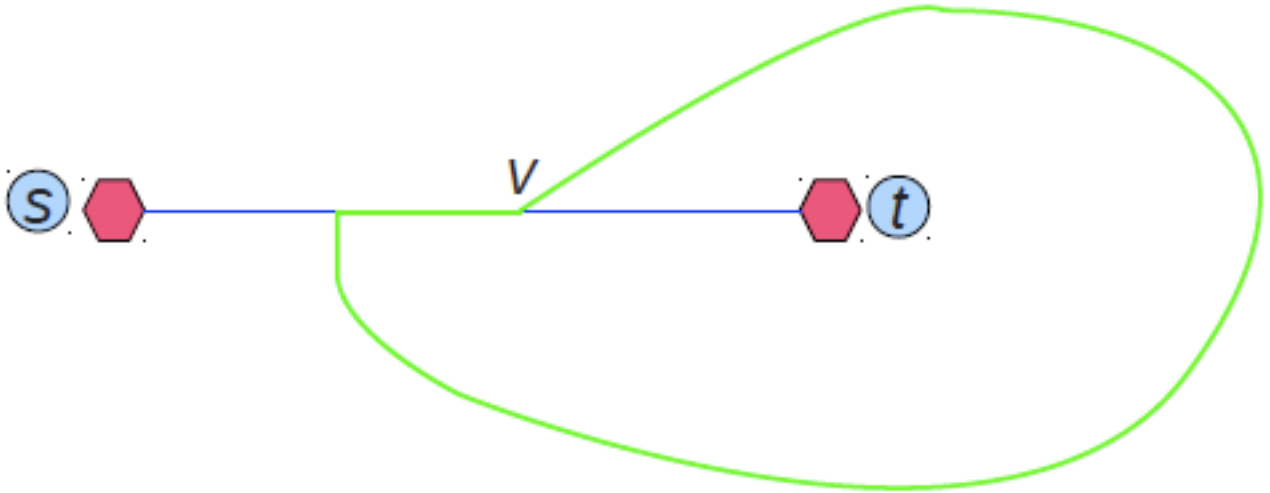

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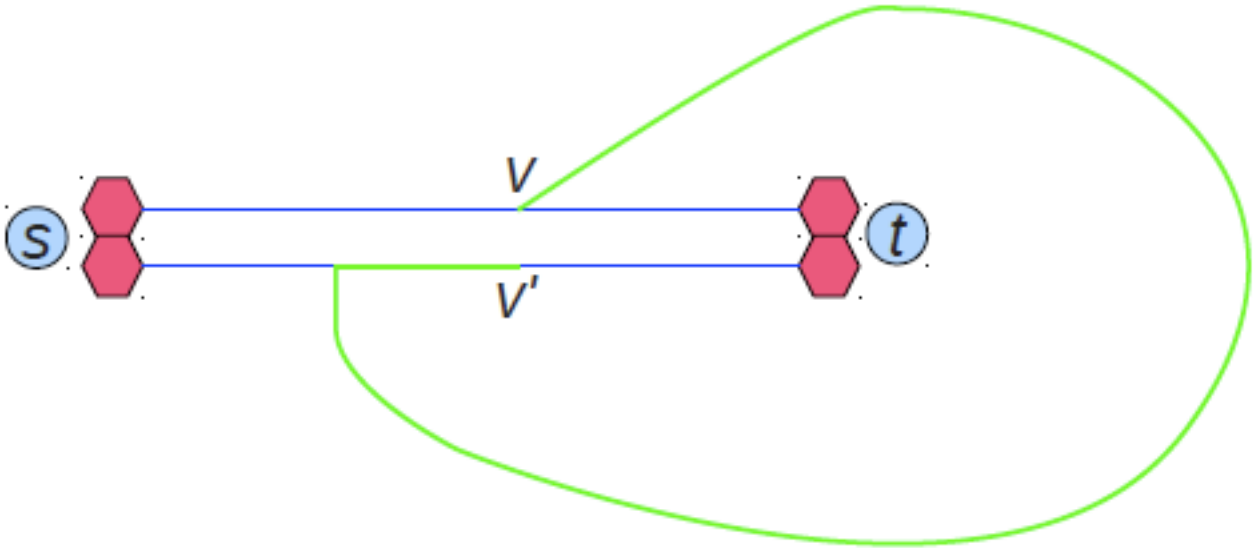
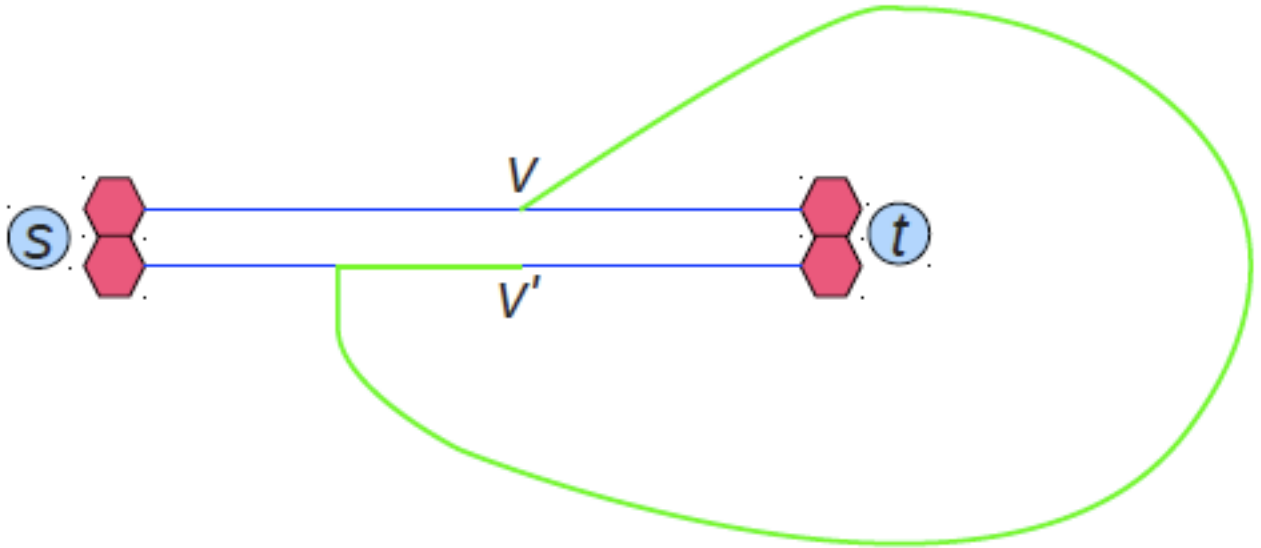


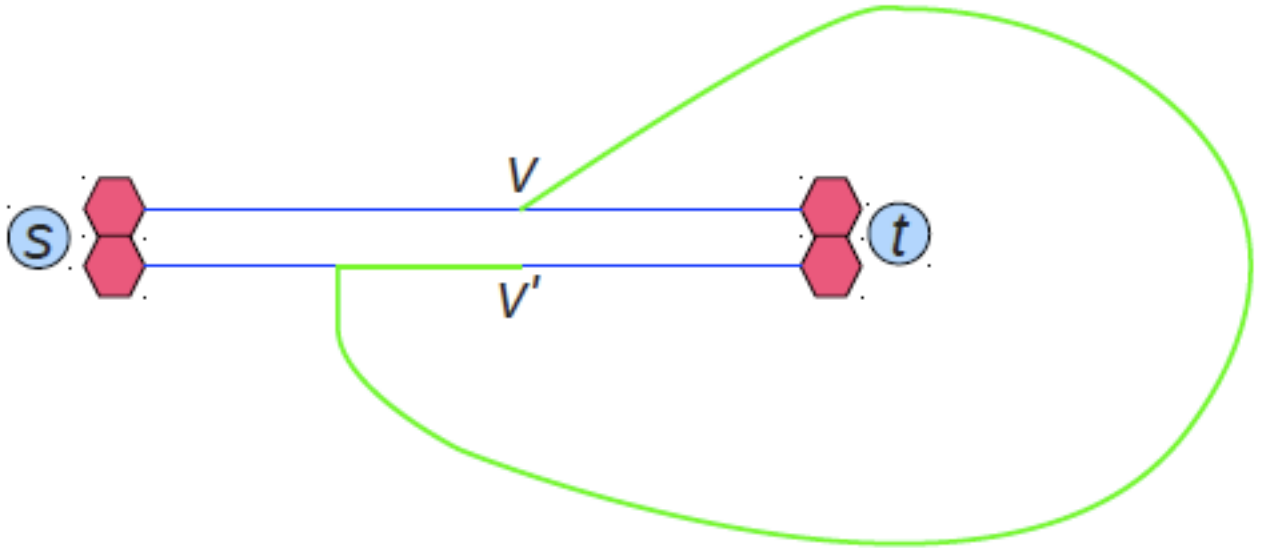
Itai & Shiloach [1979]

- Cut the graph along P
- Find a shortest path from each vertex of P to its copy
  - This gives us a min s-t separating cycle that contains each vertex of P



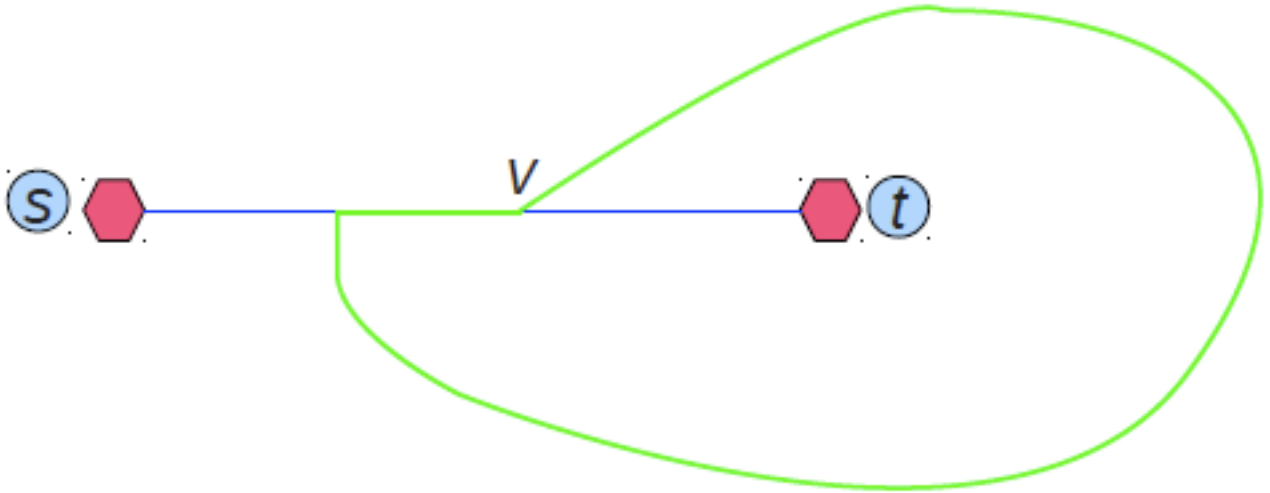
Itai & Shiloach [1979]

- Cut the graph along P
- Find a shortest path from each vertex of P to its copy
  - This gives us a min s-t separating cycle that contains each vertex of P
- O($n^2 \log n$) time
  - The length of P × time for running Dijkstra's algorithm



Reif [1983]

- Start with the middle vertex v of P



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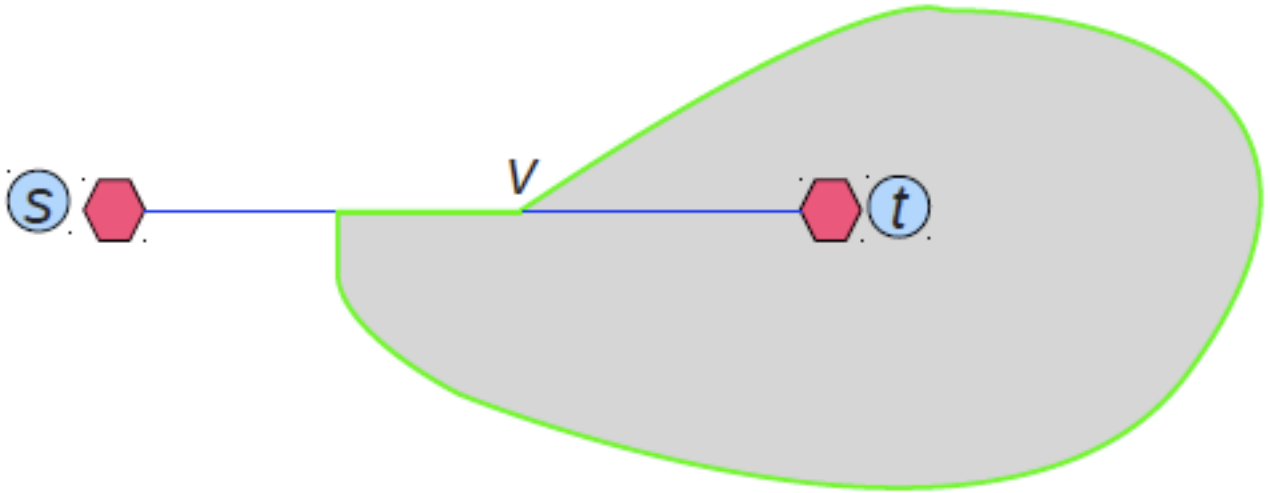
- Start with the middle vertex v of P
- Every min s-t separating cycle that contains a vertex of P is either inside, or outside the min s-t separating cycle that contains v



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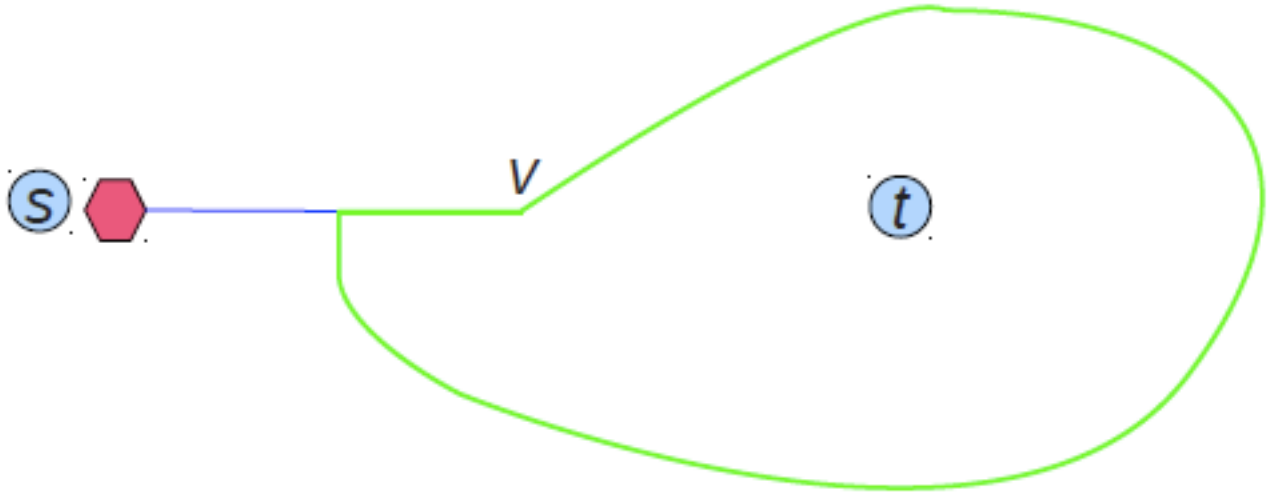

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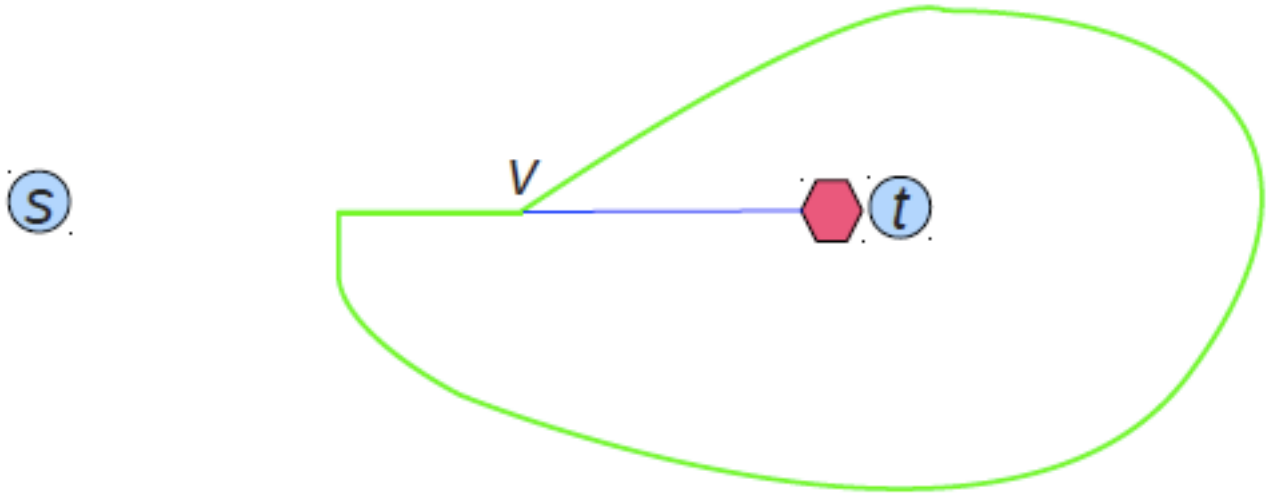
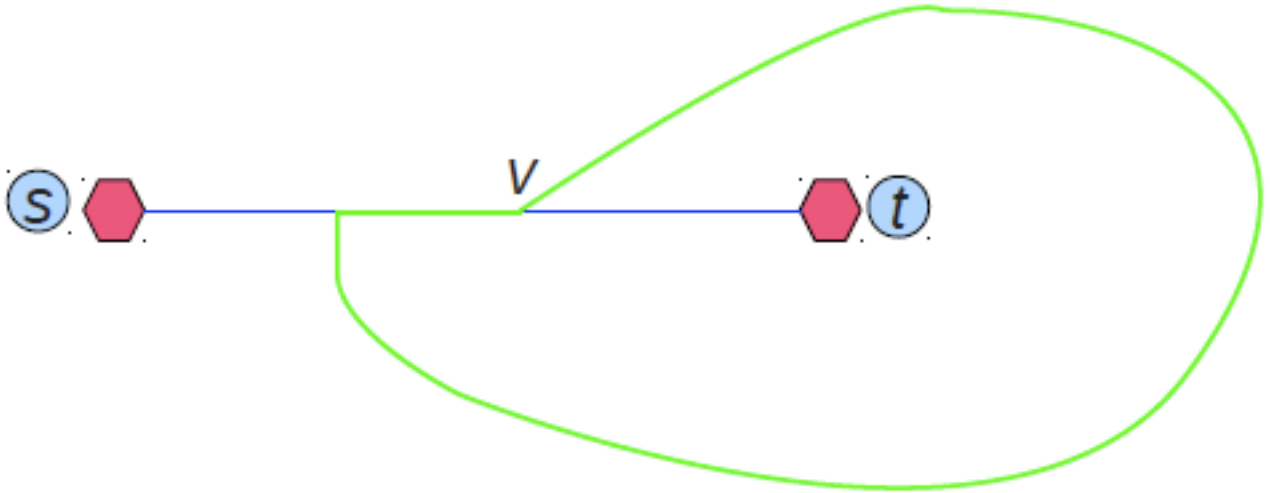
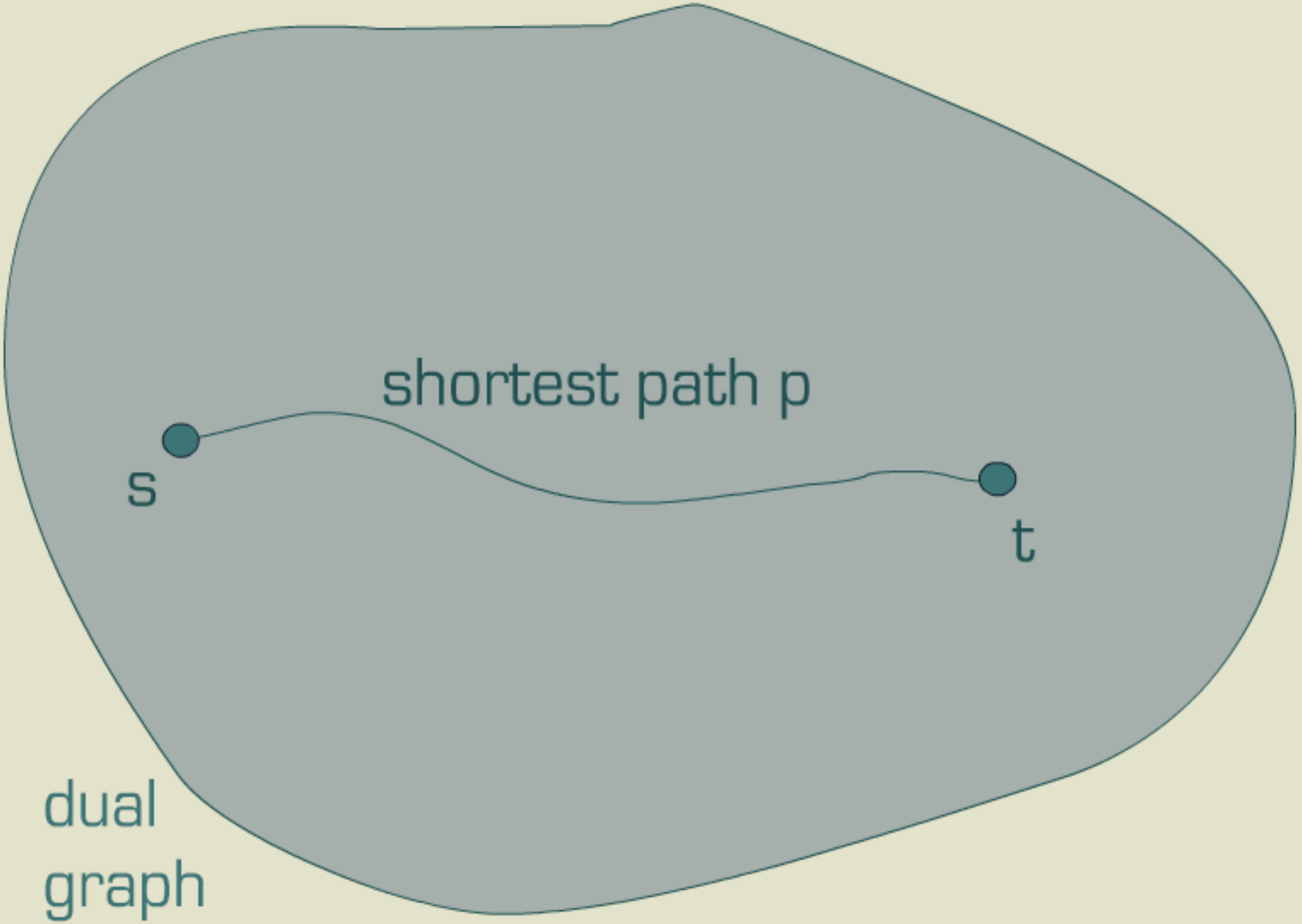


Reif [1983]

- Start with the middle vertex v of P
- Every min s-t cycle of a vertex of P is either inside, or outside the min s-t cycle that contains v
- Total time is $O(n \log^2 n)$
  - Recursion depth × time for running Dijkstra's algorithm



Reif's Algorithm



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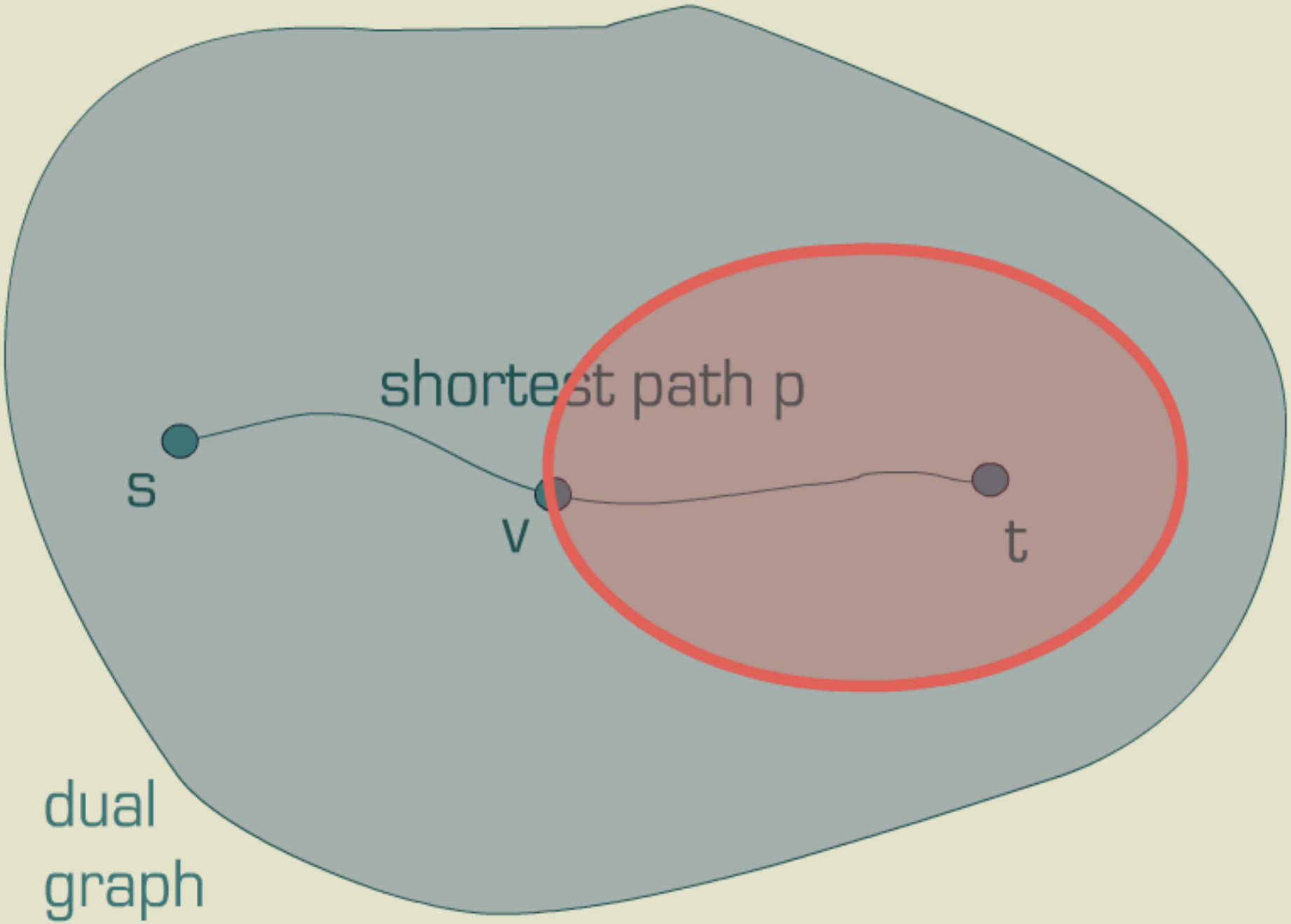

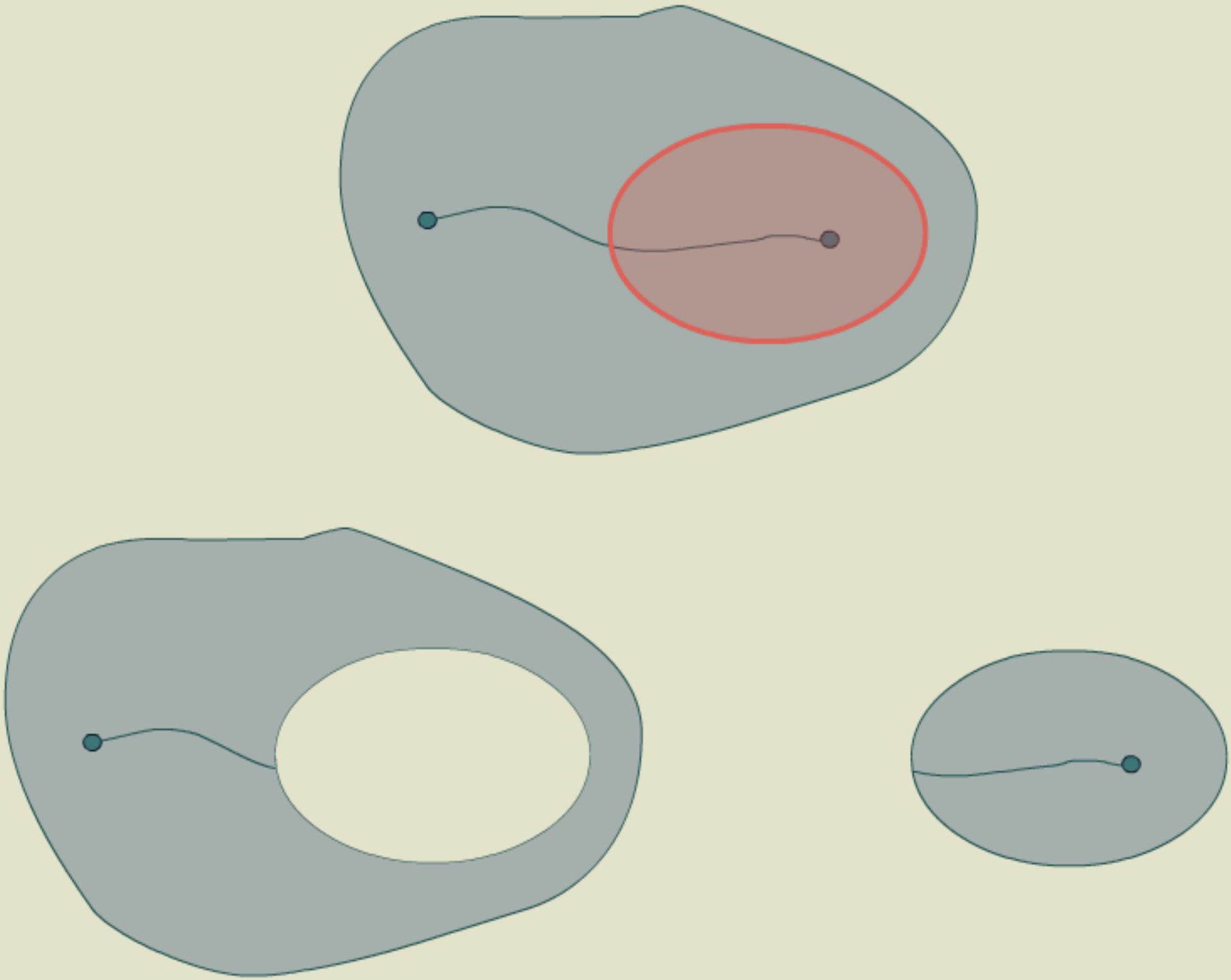

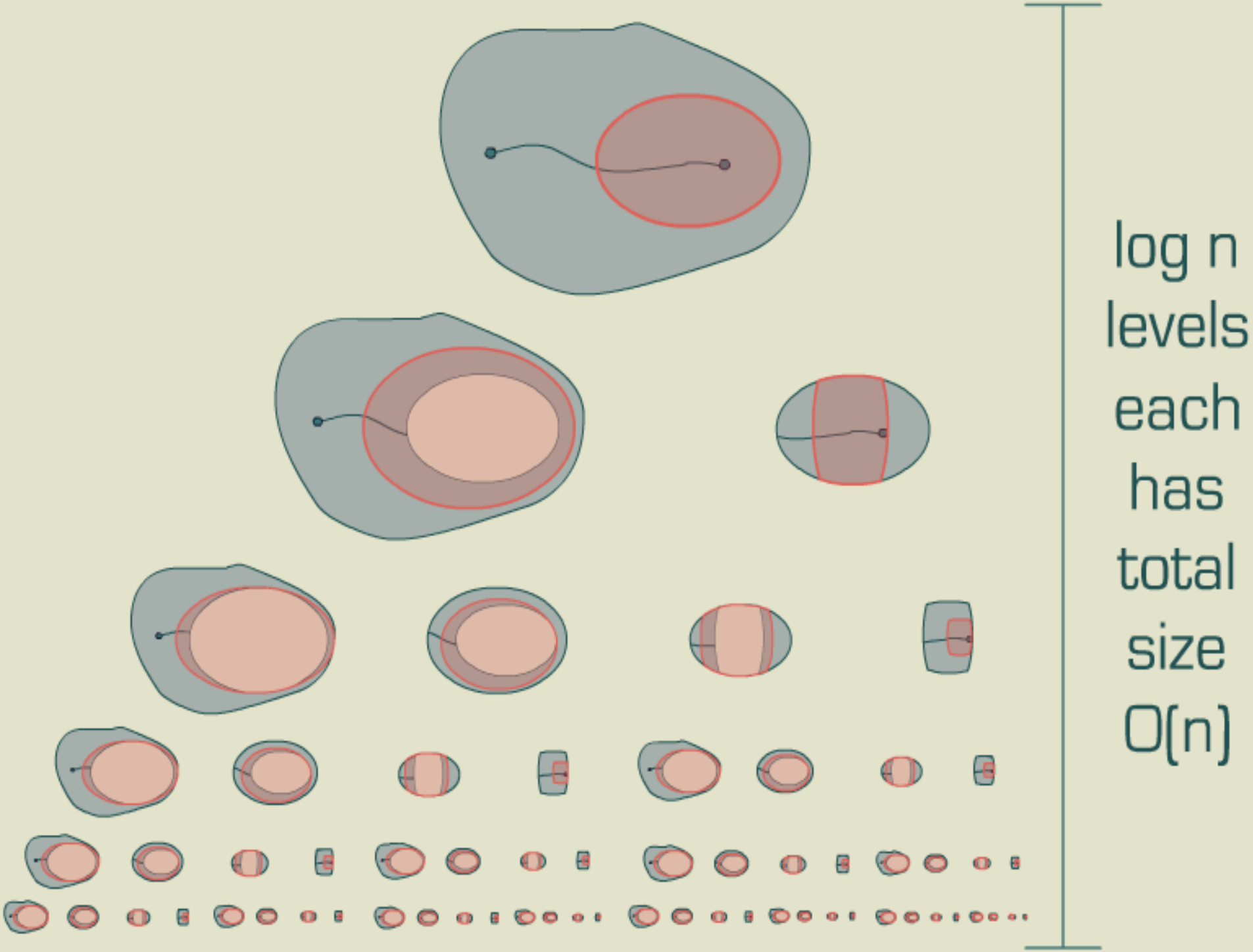
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State of the Art

- Itai & Shiloach [1979] – $O(n^2 \log n)$ time for min cut
- Reif [1983] – $O(n \log^2 n)$ time for min cut
- Hassin & Johnson [1985] – Find also max flow
- Frederickson [1987] – $O(n \log n)$ time, with $O(n (\log n)^{1/2})$ shortest path algorithm
- Johnson [1987] – Also for directed graphs (parallel algorithm)
- Henzinger et al. [1997] – $O(n \log n)$ time, with faster $O(n)$ shortest path algorithm

Our Contribution

- Itai & Shiloach [1979] – $O(n^2 \log n)$ time for min cut
- Reif [1983] – $O(n \log^2 n)$ time for min cut
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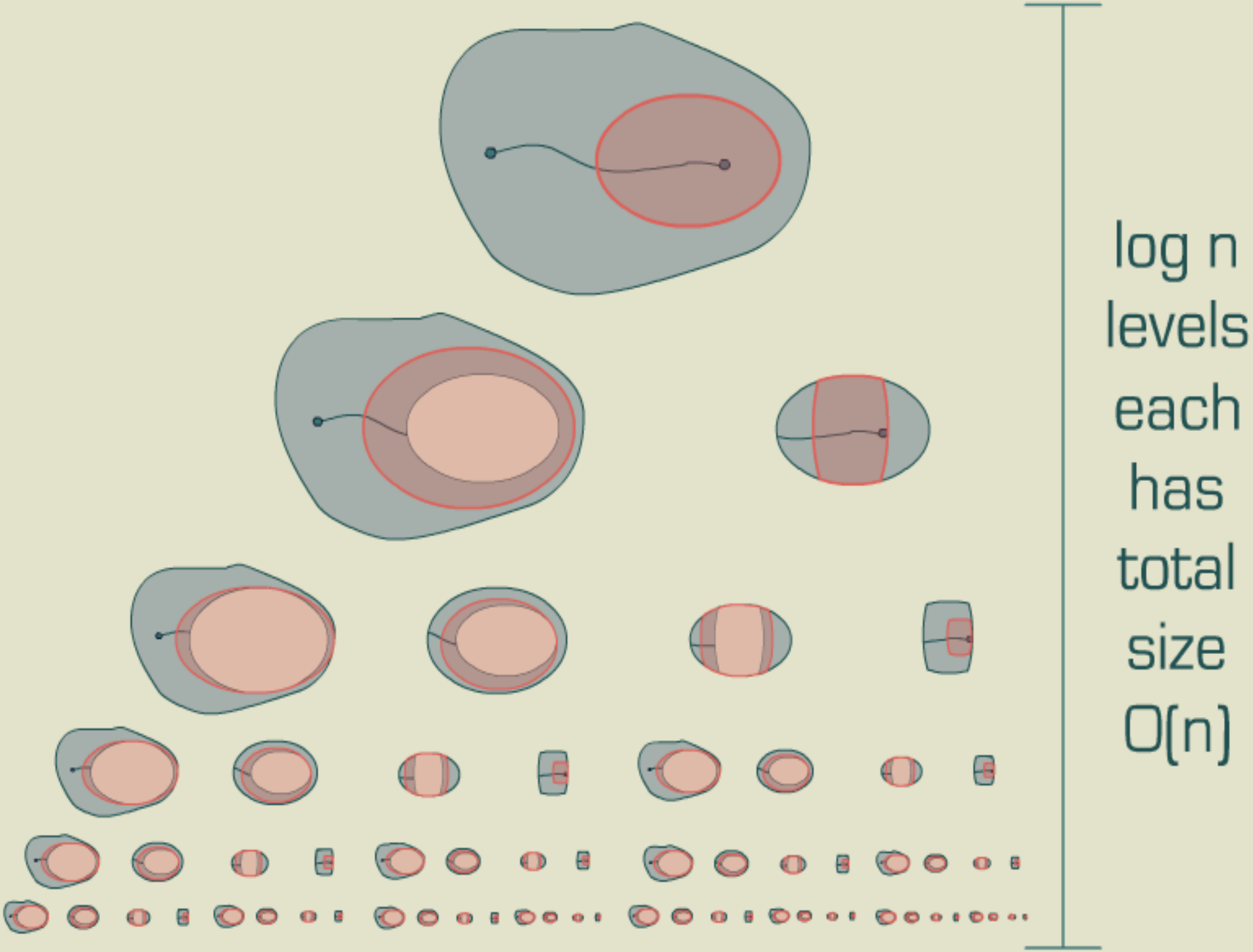
- Henzinger et al. [1997] – $O(n \log n)$ time, with faster $O(n)$ shortest path algorithm
- This talk – $O(n \log \log n)$ time

Our Contribution

- Min s-t cut in undirected planar graphs can be solved in $O(n \log \log n)$ time
- Max s-t flow in undirected planar graphs can be also solved in $O(n \log \log n)$ time
- Min s-t cut and max s-t flow can be maintained dynamically in $O(n^{2/3} \text{polylog}(n))$ time per update operation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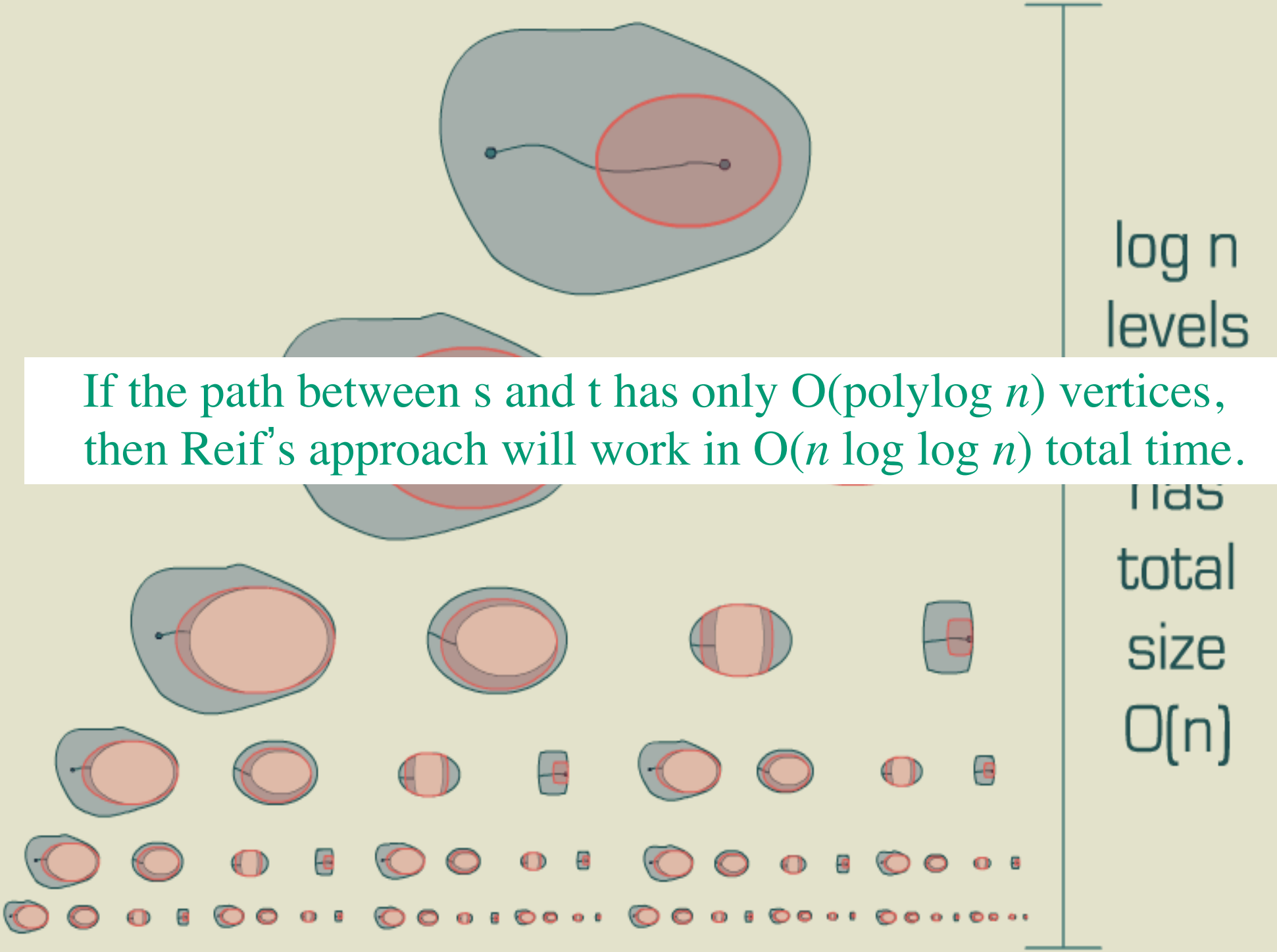

If the path between s and t has only O(polylog n) vertices, then Reif's approach will work in O(n log log n) total time.

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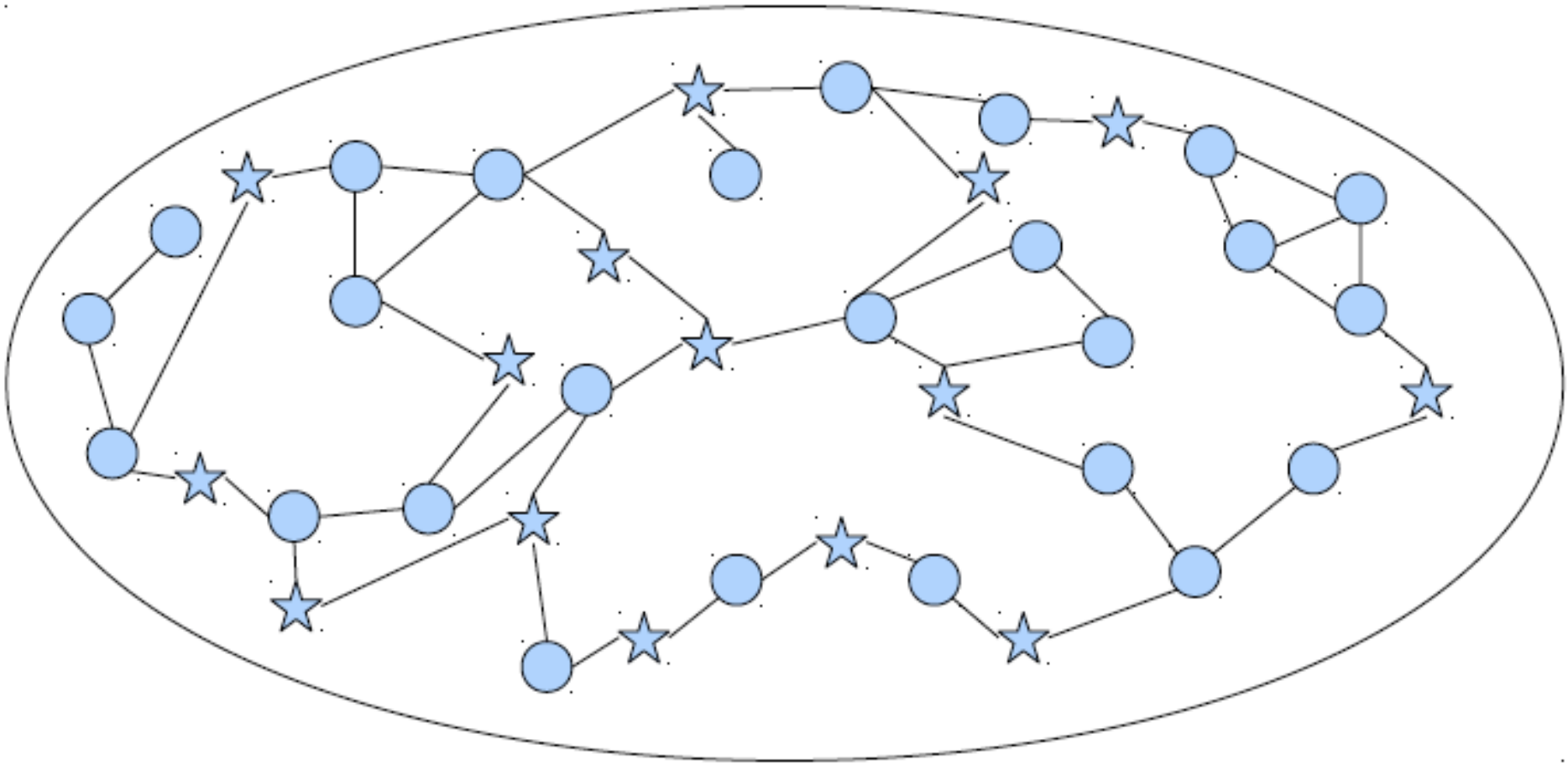
- Applications

$(\log^6 n)$-division

- We divide the graph into pieces
- $O(n / \log^6 n)$ pieces, each with $O(\log^6 n)$ vertices and edges, $O(\log^3 n)$ boundary vertices and **O(1)** holes



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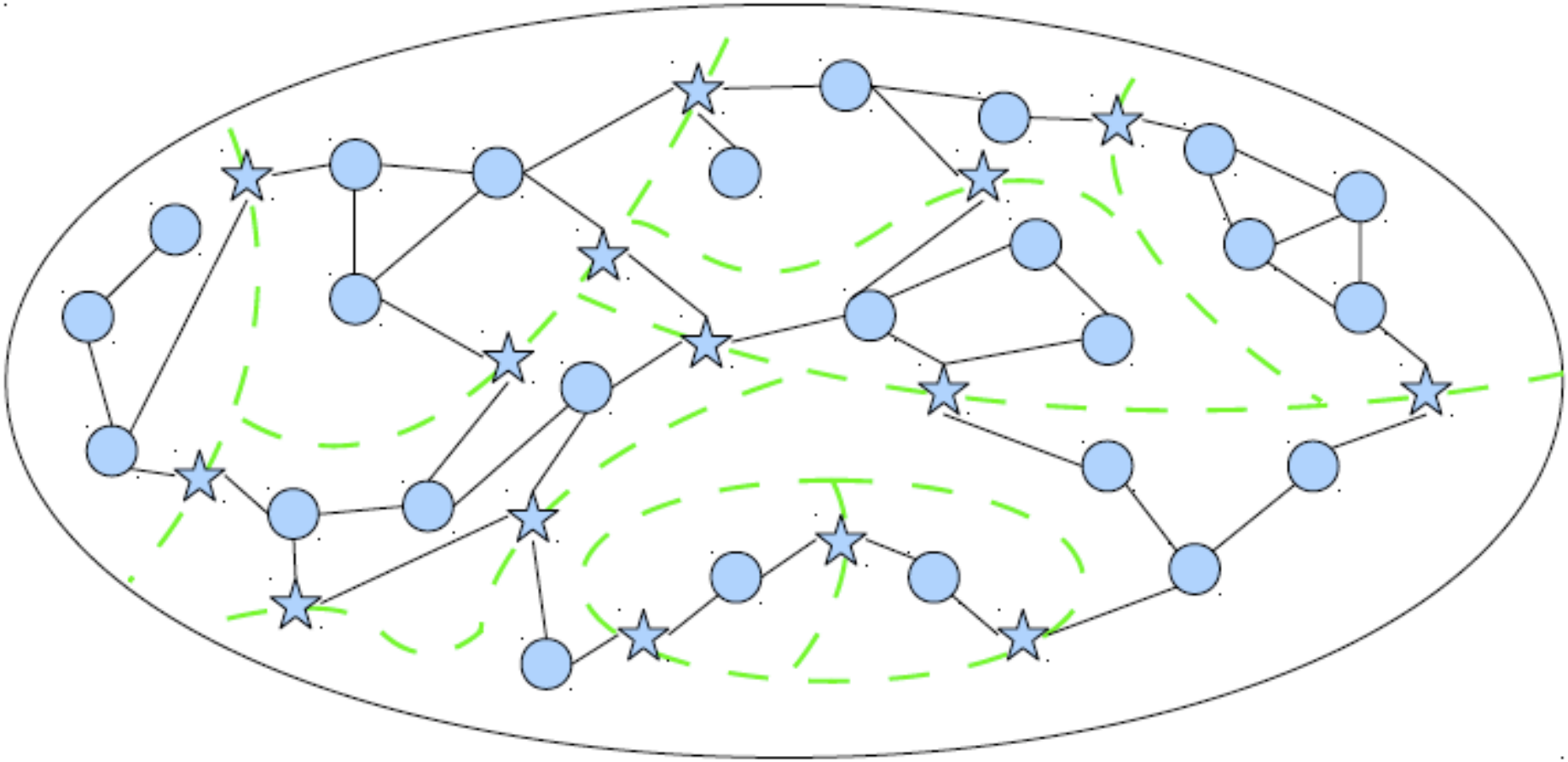
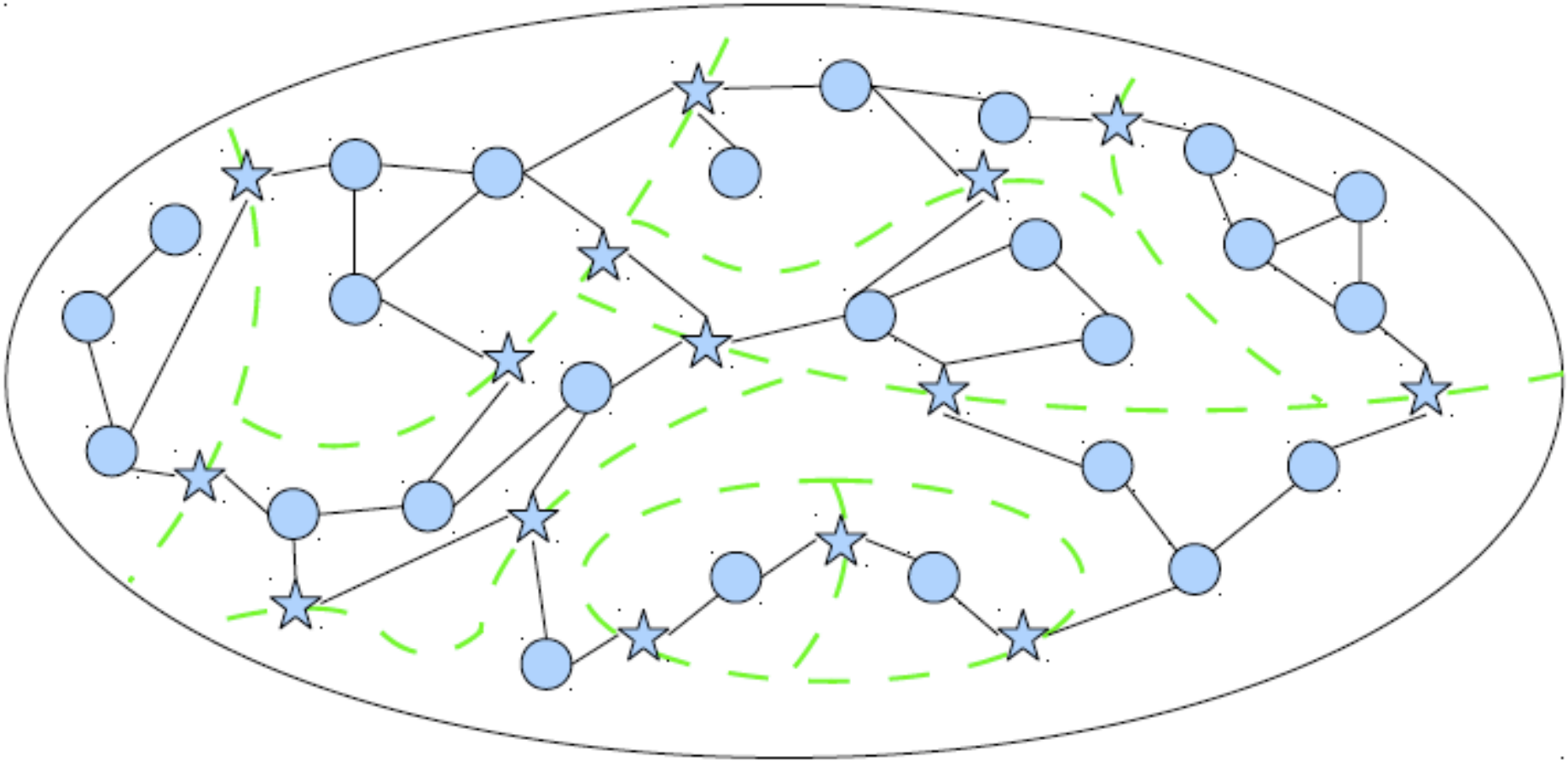


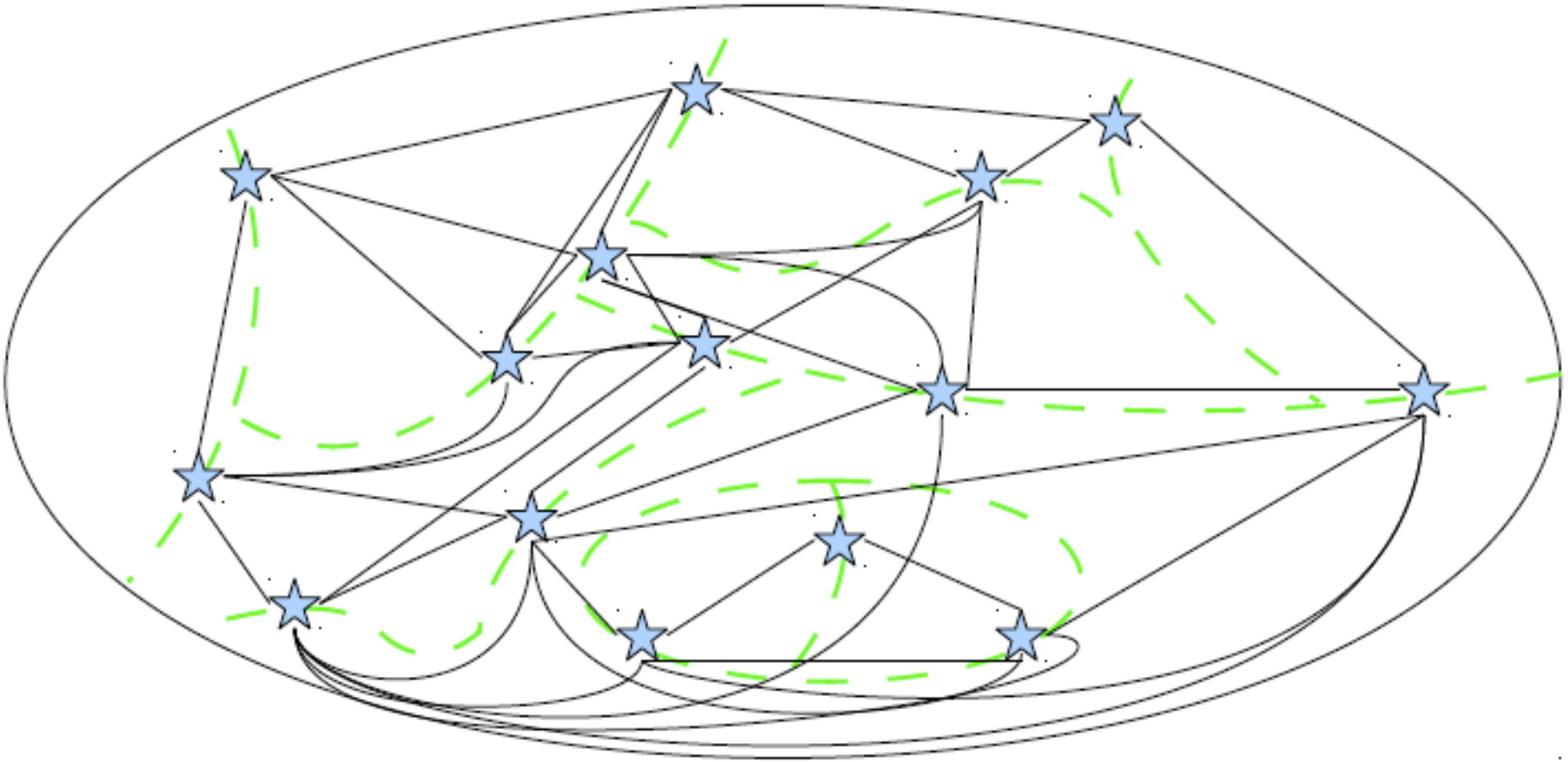
Dense Distance Graph

- Fakcharoenphol and Rao [2006]
- We are interested in distances among boundary vertices



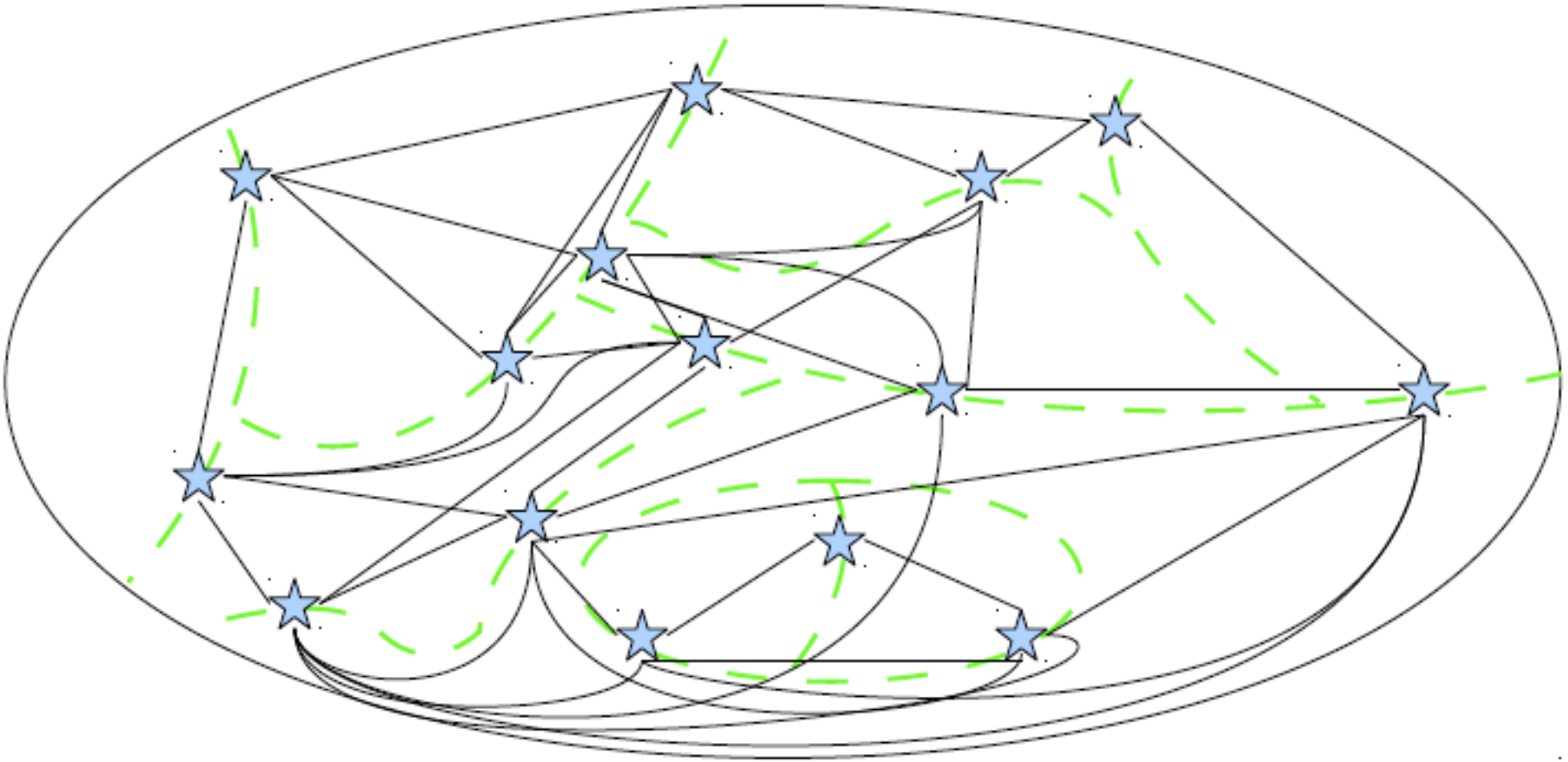
Dense Distance Graph

- Replace internal vertices with cliques on the boundary vertices
- The length of an edge is the distance between its endpoints



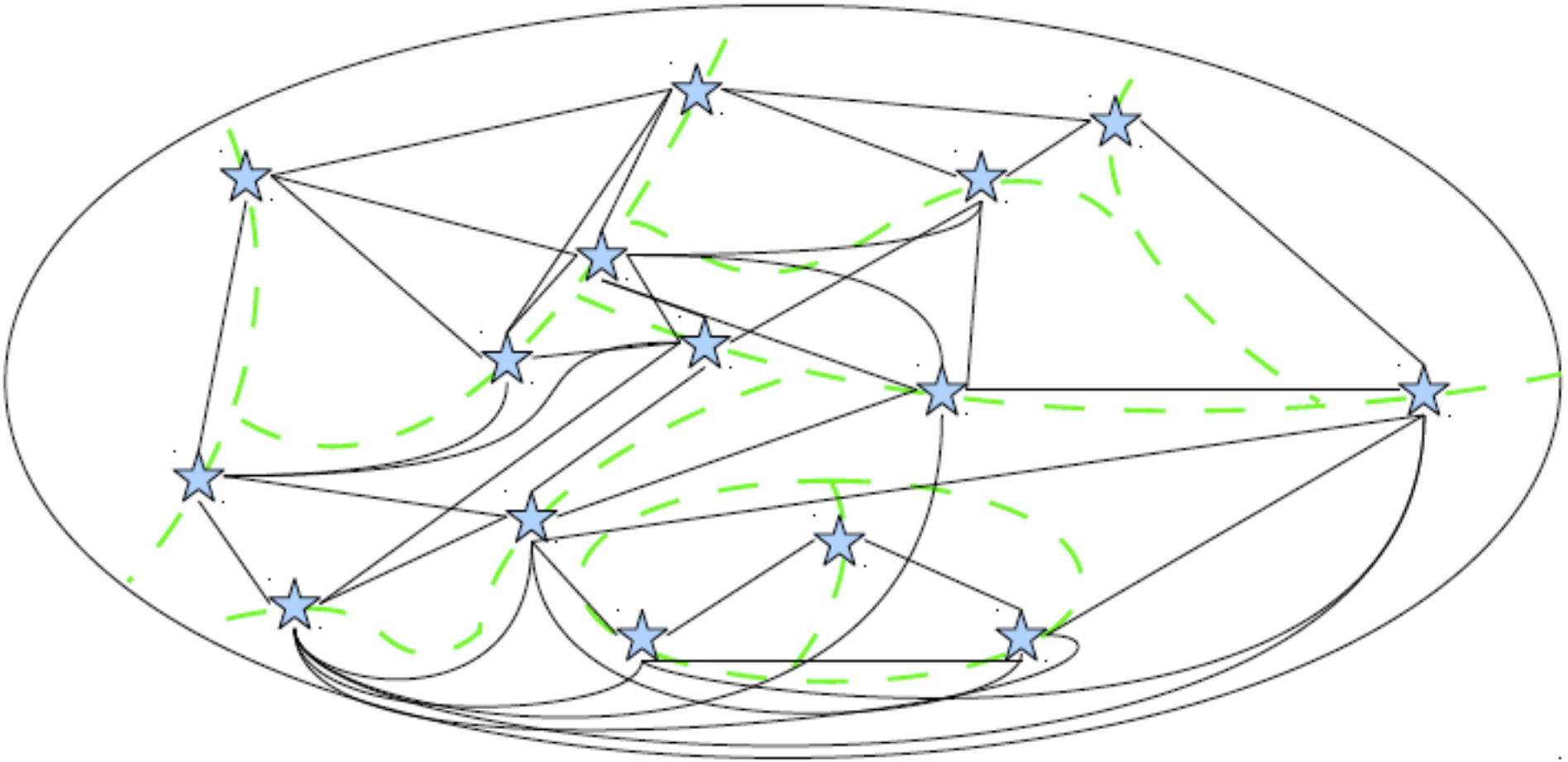
Dense Distance Graph

- Encode distances between boundary vertices
- Graph is no longer planar
- Much more edges but less vertices overall – $O(n / \log^3 n)$



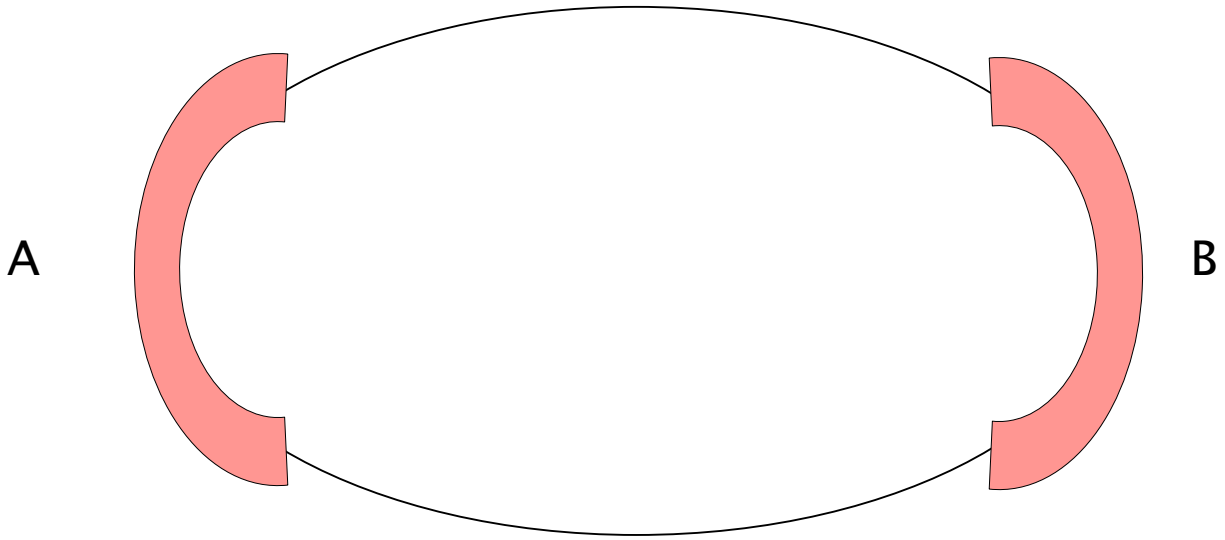
Dense Distance Graph

- Can compute the dense distance graph for a single piece in time $O(\log^6 n \log\log n)$ with the algorithm of Klein [2006] for multiple source shortest paths on a single face
- Total time is $O(n / \log^6 n) \times O(\log^6 n \log\log n) = O(n \log\log n)$



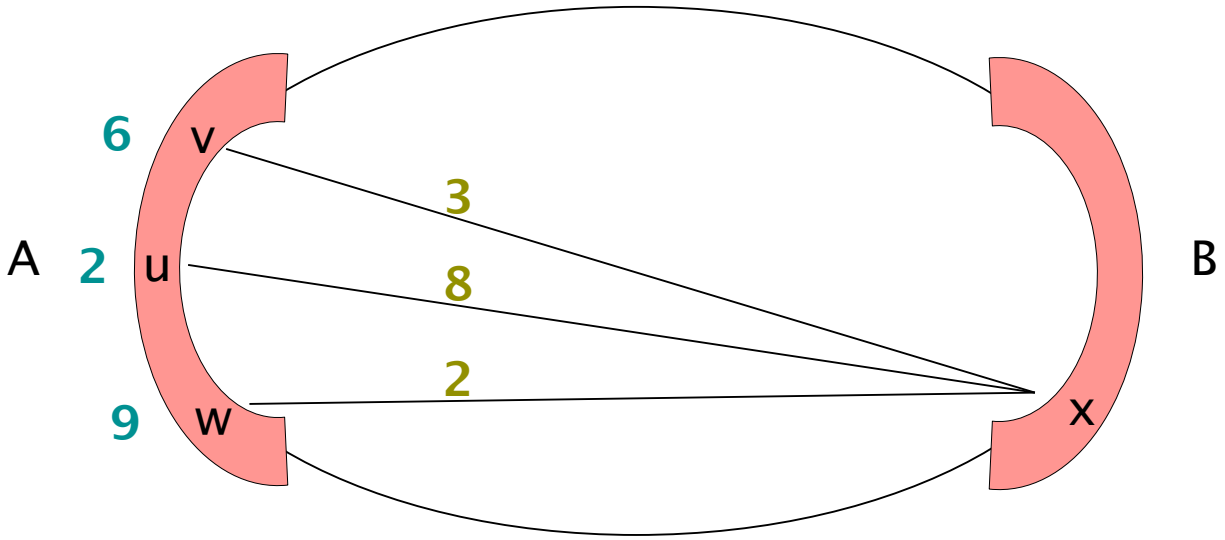
Distance between boundary vertices

- Consider the distances between two disjoint sets of consecutive boundary vertices A and B
- We have a complete bipartite graph between A and B



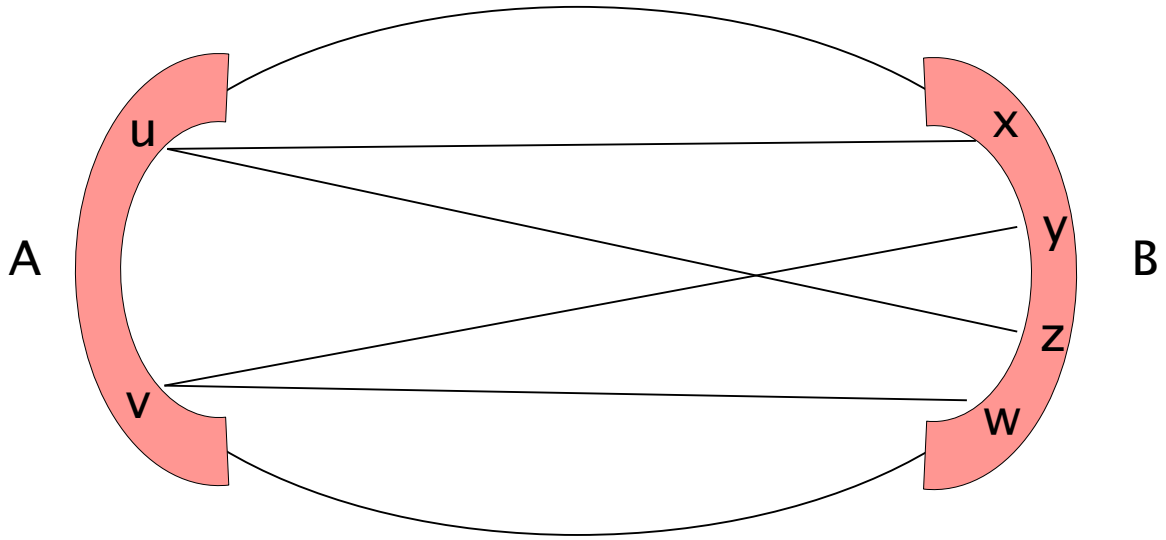
Distance between boundary vertices

- Fast implementation of Dijkstra after Fakcharoenphol & Rao [2006]
- Assume we have some distance labeling
- $v \in A$ is the parent of $x \in B$ if v minimizes $d(v) + d(v,x)$



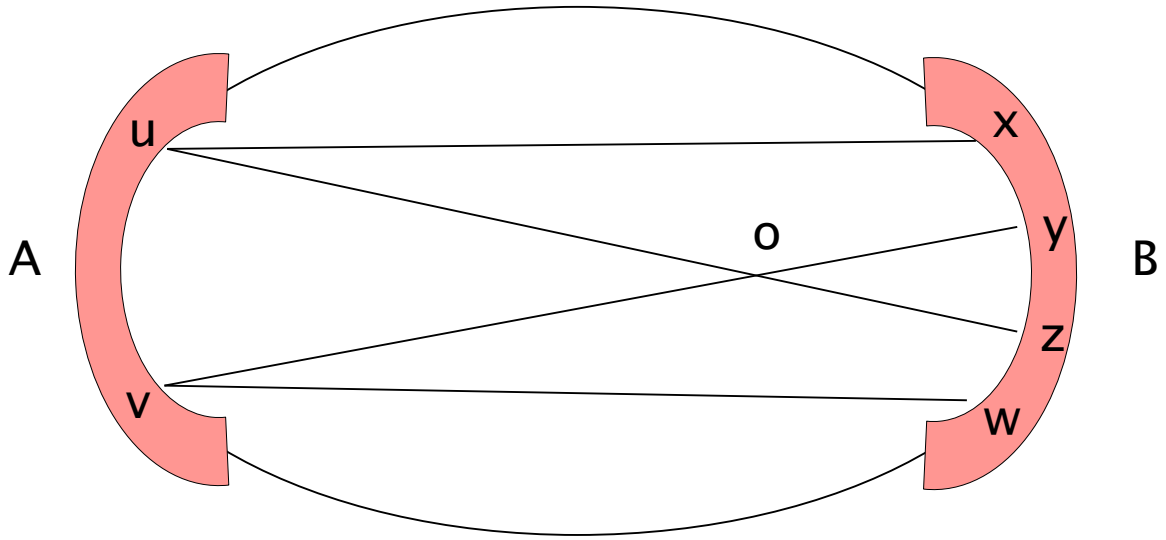
The non crossing property

- The children of a member of A are consecutive in B



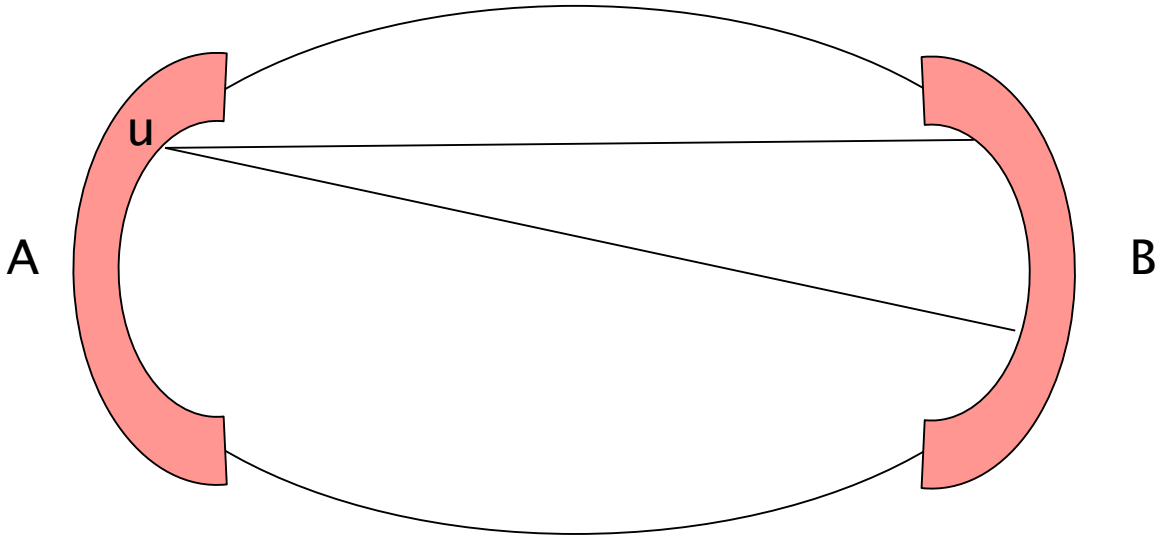
The non crossing property

- The children of a member of A are consecutive in B
- $d(u, z) < d(v, z) \Rightarrow d(u, o) < d(v, o) \Rightarrow$
  $d(u, y) < d(v, y)$



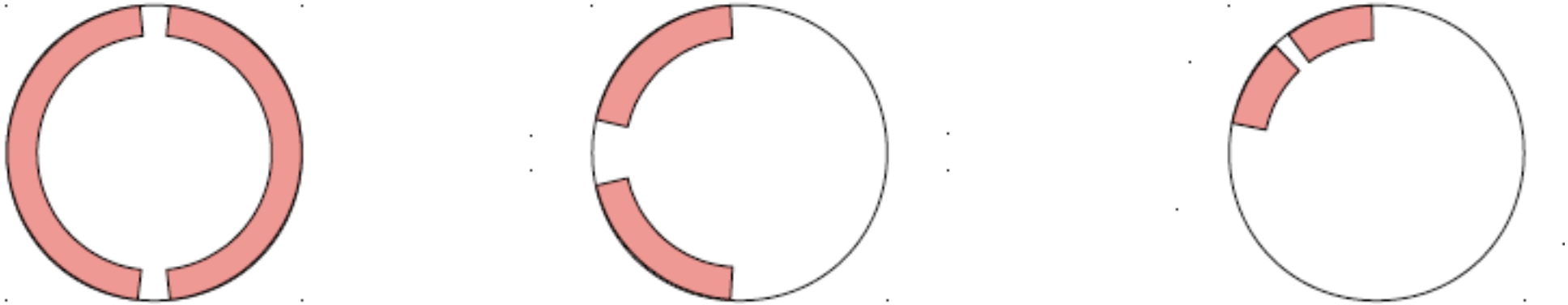
Fast Dijkstra with Global Heap

- Due to the non crossing property, we can run Dijkstra's shortest path algorithm more efficiently on the dense distance graph
- When a vertex in A gets a new label, we update the labels of all of its children in B at once
- This takes O(log n) time per vertex



Fast Dijkstra

- With O(log n) bipartite graphs we can cover all pairs of boundary vertices



- This gives O(b log^2 n) implementation of Dijkstra's algorithm for a graph composed of dense distance graphs with a total of b boundary vertices

Shortest paths in a ($\log^6 n$)-division

- We can divide the graph in $O(n \log \log n)$ time
  - Based on planar separators
  - Requires a new, more efficient algorithm (O(1) holes)
- We build the dense distance graphs on this division in $O(n \log \log n)$ time
- Using the fast Dijkstra implementation, we can find the shortest path between any two boundary vertices in $O(n / \log n)$ time:
  - $O(n / \log^6 n)$ pieces × $O(\log^3 n)$ boundary vertices per piece × extra $O(\log^2 n)$ per boundary vertex

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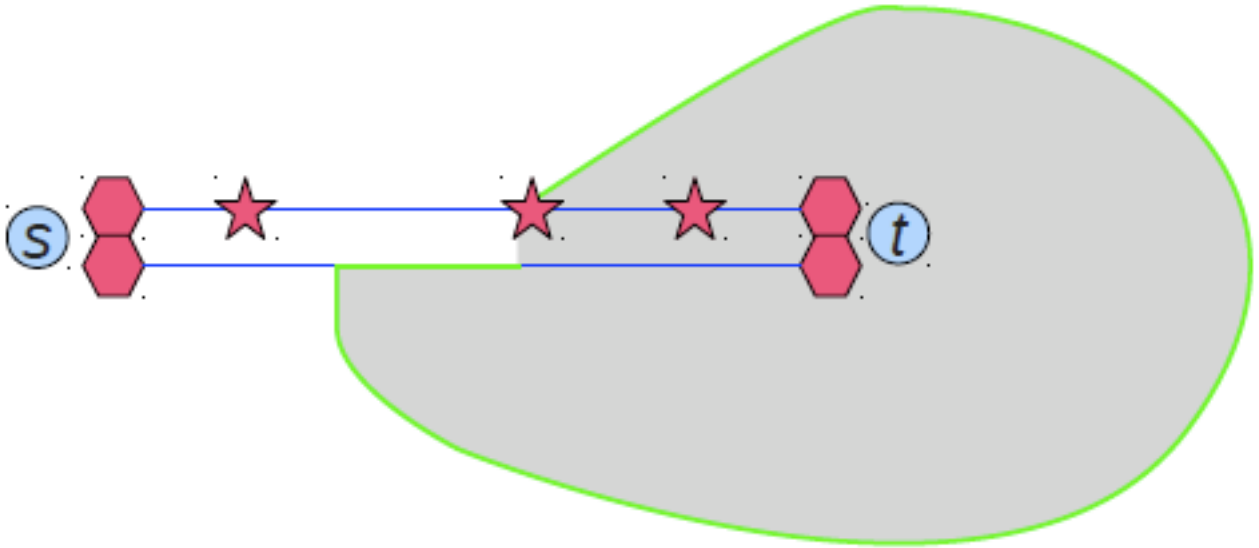
Our min s-t cut algorithm

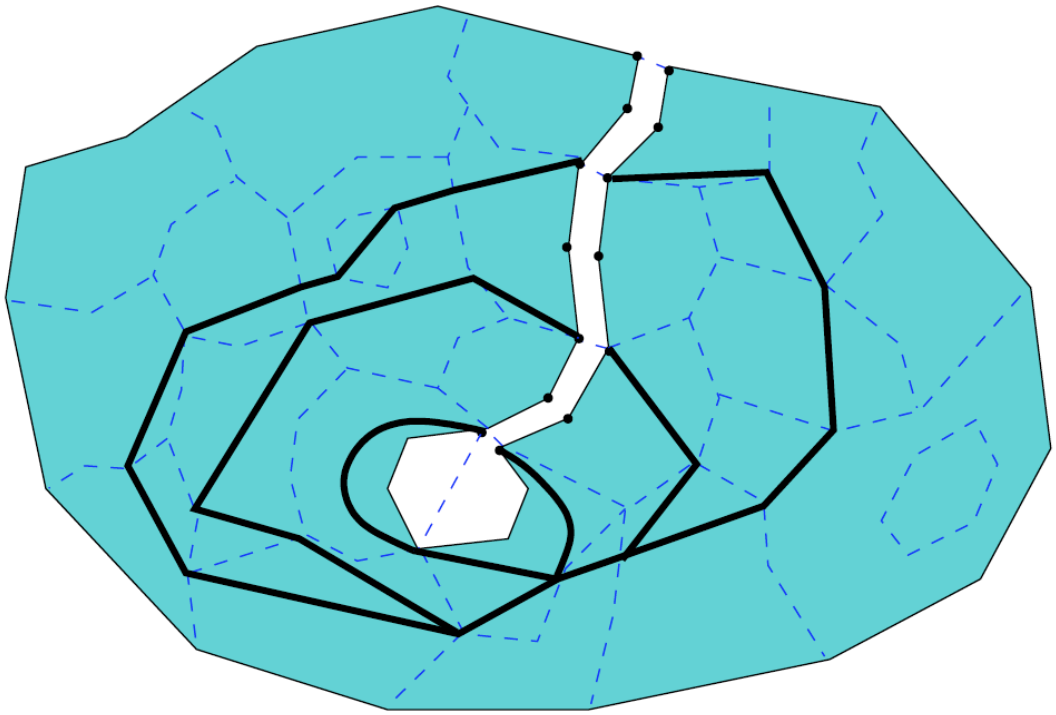
- Compute ($\log^6 n$)-division and dense distance graphs in $O(n \log \log n)$ time
- Based on Reif's min separating cycle algorithm
- Two levels of Reif's algorithm:
  - "Coarse Reif" on the dense distance graph
  - "Refined Reif" for the rest of the s* to t* shortest path P

“Coarse Reif”

- Find path P as in Reif's algorithm and cut open graph along it
- Run Reif's algorithm, but only on the boundary vertices in P
  - This takes $O(n / \log n) \times O(\log n)$ (fast Dijkstra × recursion depth) = $O(n)$ time
- Now have min s-t separating cycle containing each boundary vertex in P



“Coarse Reif”



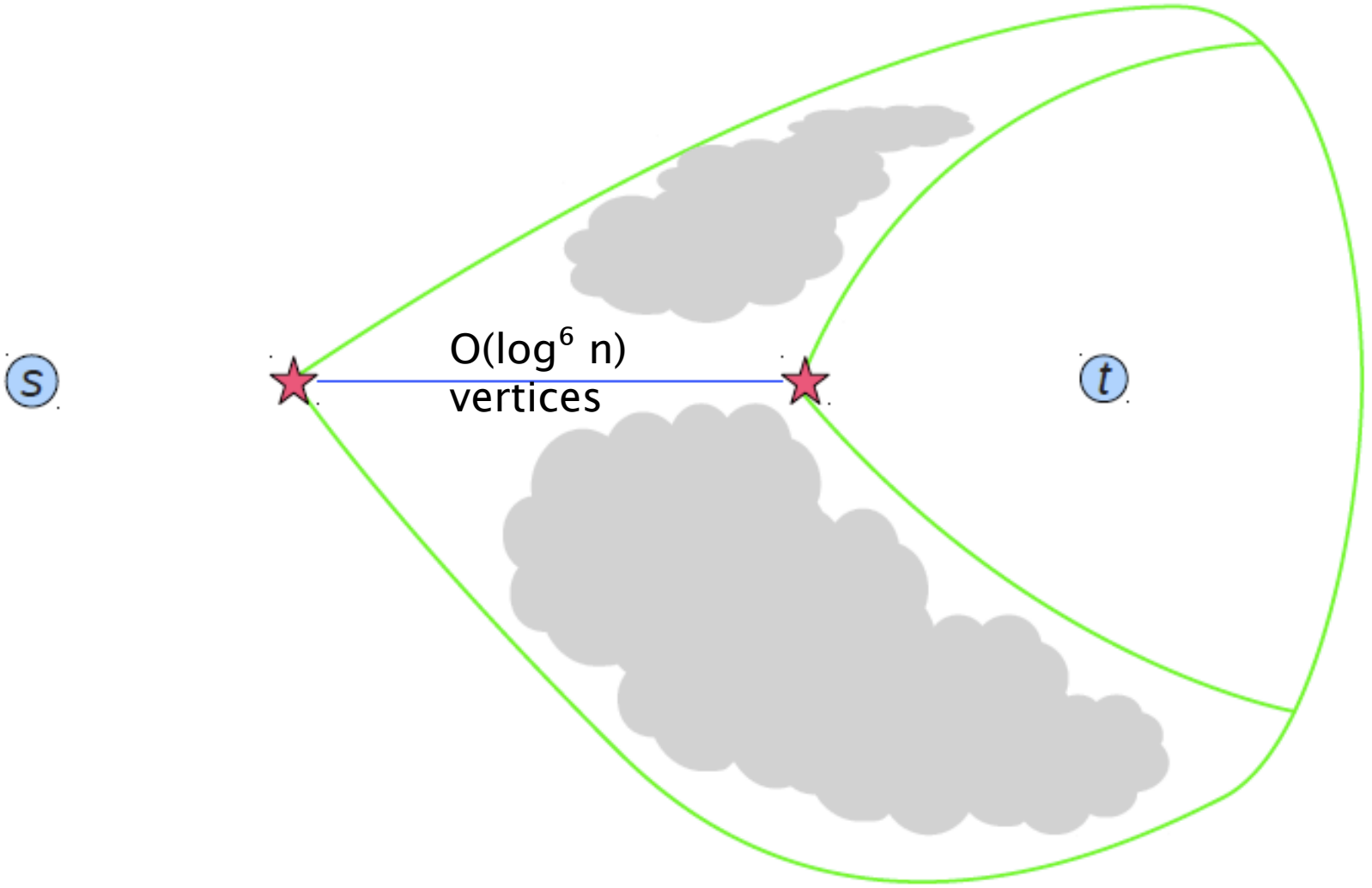
- Obtain min s-t separating cycles for boundary vertex in P
- The s-t separating cycles obtained partition G into subgraphs: each subgraph has subpath of P fully contained inside a piece
- Each such subpath has length $O(\log^6 n)$ [# edges in a piece]

“Refined Reif”

- Between two boundary vertices u, v continue with normal Reif algorithm

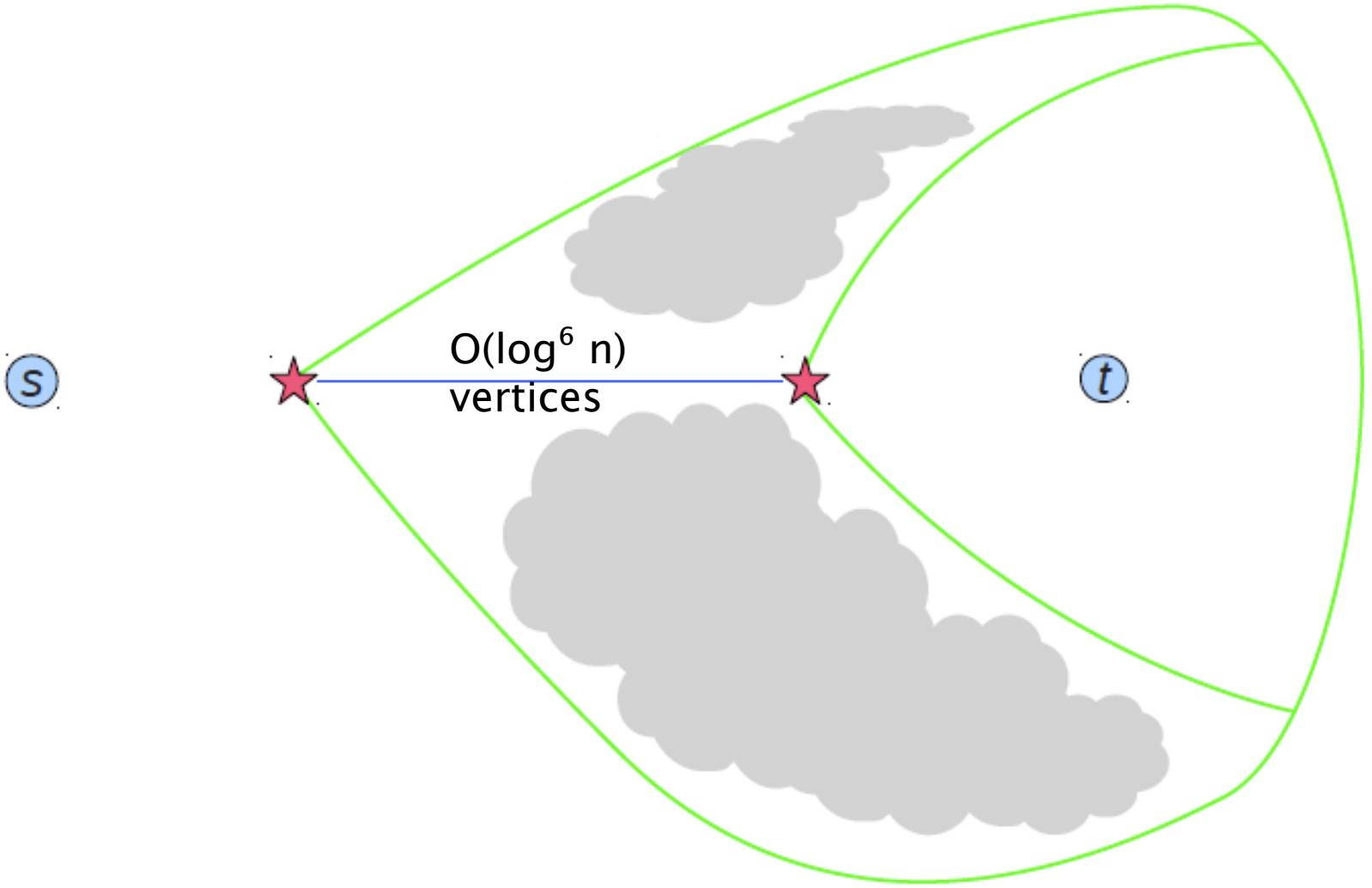
"Refined Reif"

- There are $O(\log^6 n)$ vertices of P between u and v



"Refined Reif"

- Depth of recursion is O(log log n)
- Make sure total size of all parts still O(n) (non-trivial)



Time Analysis

- O(n log log n) for constructing the (log^6 n)-division and the dense distance graph
- O(n) for "Coarse Reif" step
- O(n log log n) for "Refined Reif" step
- Obtain min s-t separating cycle for each vertex of P, exactly as Itai & Shiloach (but faster!)

Outline

- ✔ Introduction
- ✔ History
- ✔ Dense Distance Graphs
- ✔ Minimum s-t cut in O(n log log n) time
- Applications

Applications

- Computing min separating cycle in planar graphs is the bottleneck for the following problems:
  - Maximum flow in undirected planar graphs – Hassin & Johnson [1985], improved to $O(n \log \log n)$
  - Global minimum cut / girth of planar graphs – Charlermoosk et al. [2004], improved to $O(n \log n \log \log n)$
  - Shortest non-trivial cycle in undirected graphs on a surface with a bounded genus – Kutz [2006], improved to $O(g^{O(g)} n \log \log n)$
  - Minimum s-t cut in undirected graphs on a surface with a bounded genus – Chambers et al. [2009], improved to $O(g^{O(g)} n \log \log n)$

Dynamic data structure for min cut queries

- Extend efficient Dijkstra implementation of Fakcharoenphol and Rao [2006] to a dynamic shortest path queries data structure
  - Need to recompute pieces affected by edge insertion / deletion / capacity change
- This allows also a dynamic data structure for min cut algorithm, using our ideas
- $O(n^{2/3} \log^{8/3} n)$ time per operation