Backstepping observer using weighted spatial average for 1-dimensional parabolic distributed parameter systems

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Abstract: This paper is concerned with designing a backstepping-based observer for a class of 1-dimensional parabolic distributed parameter systems whose output is a weighted spatial average of the state. We first show that an integral transformation converts the original system into another parabolic system with boundary observation if the weighting function satisfies a parameterized ordinary differential equation. Then the backstepping observer for the transformed system is available and an estimate of the original state is obtained through the inverse transformation. We also show that the estimation error exponentially converges to 0 in terms of the $L^2$ norm with arbitrary decay rate. Furthermore, a closed-form expression of the observer is provided for systems described by linear reaction-diffusion equations.

Keywords: Distributed parameter systems; Partial differential equations; Observers; Average values

1. INTRODUCTION

A difficulty in measurement for distributed parameter systems arises from the fact that the state is infinitely distributed in the spatial domain. Perfect point measurement is impossible since there are no infinitesimal sensors. Furthermore, although some distributed sensors such as fiber optic sensors [Hotate, 2006] have been developed, distributed sensing with ultimately high resolution is not available. What we can measure is some kind of spatially averaged value of the state on a region. Therefore, we must estimate the state on the basis of such coarse information to obtain the detailed distribution.

Extension of the observer to infinite dimensional systems can be found in standard textbooks, e.g. [Curtain and Zwart, 1995, Lasiecka and Triggiani, 2000], and one of recent results is found in [Vries et al., 2010]. Many researches in this field are based on the theory of abstract systems. Although such an approach gives quite general results similar to those of lumped parameter systems, we often encounter feasibility problem in practical applications. Recently, Smyshlyaev and Krstic proposed the backstepping observer for distributed parameter systems with boundary observation [Smyshlyaev and Krstic, 2005]. The backstepping observer is a Luenberger-type observer, and a key feature is that backstepping for distributed parameter systems [Balogh and Krstic, 2002, Liu, 2003, Smyshlyaev and Krstic, 2004] is utilized to stabilize the error system. The backstepping method provides a systematic procedure to design an exponentially stabilizing observer gain and requires a solution of a simple linear partial differential equation (PDE) only. Hence, backstepping has an advantage in practice, since numerical computation of such a PDE is relatively easy in comparison to that of operator Riccati equations. However, in contrast to these advantages, backstepping is applicable only to the system with triangular or spatially causal structure.

As mentioned in the first paragraph, one of realistic choices of the output is a spatial average of the state. It is beneficial if backstepping is applicable to the systems with such output. Hence, in this paper, we consider an observer design for the systems whose output is a spatially average based on the backstepping method. As a first step, we focus on the case in which the output is a weighted spatial average on overall domain. Clarification of admissible weighting functions is also our objective. Unfortunately, backstepping is not directly applicable in our case, since the error feedback term destroys the triangular structure. To overcome this difficulty, we employ an integral transformation which converts the system into a system with boundary observation. We show that if the weighting function satisfies an ordinary differential equation (ODE) containing a design parameter, then a backstepping observer for the converted system is available. An estimate of the original state is obtained through inverse transformation. The estimation error exponentially converges to 0 with respect to the $L^2$ norm. Furthermore, a closed-form expression for observer gain is available in a special case. Although this facts limit practical applications, we can obtain a number of weighting functions by changing the design parameter.

This paper is organized as follows. In Section 2, we define a class of systems considered in this paper and mention the lack of triangular structure due to the error feedback. Section 3 is the key section of this paper, where we intro-
duce an integral transformation to overcome the lack of triangular structure. In Section 4, we construct an observer based on the backstepping method, and we provide an explicit expression for linear reaction-diffusion equations in Section 5. Our observer is numerically demonstrated to confirm the effectiveness of the proposed method in Section 6. Finally, we summarize this paper in Section 7.

Notation. I denotes the interval (0, 1) ⊂ R and I is the closure of I in R, that is, I = [0, 1]. The Sobolev space on an interval I with exponent 2 is denoted by H^k(I). For a function f on I, we denote its the L^2 norm and the H^k norm by ∥f∥ and ∥f∥_{H^k}, respectively. We use (·)' to denote the derivative of a function of a single spatial variable.

2. PROBLEM SETTING

We consider the distributed parameter system represented by the following 1-dimensional parabolic partial differential equation defined on I × (0, ∞):

\[
\frac{∂q}{∂t}(x, t) = \frac{∂^2 q}{∂x^2}(x, t) + λ(x)q(x, t),
\]

where q(x, t) ∈ H^2(I) is the state. It is assumed that a > 0, λ ∈ C^1(\overline{I}), and the boundary conditions are given by

\[
\frac{∂q}{∂x}(0, t) + aq(0, t) = 0, \quad q(1, t) = u(t),
\]

where α ∈ R. Note that general parabolic PDEs which contain the first spatial derivative of the state can be transformed into (1) with a slight modification as mentioned in [Smyshlyaev and Krstic, 2004]. It should be emphasized that the output is a weighted spatial average of q given by

\[
y(t) = \int_0^1 h(ξ)q(ξ, t)dξ,
\]

where h is a positive-valued function on \overline{I}, which is referred to as the weighting function. The class of h will be determined later.

Parabolic PDEs are an important class in engineering, since they are mathematical models of various physical phenomena such as heat conduction and other diffusive processes [Bird et al., 1960]. A linearized model of chemical reactors are also contained in this class [Delattre et al., 2004], and hence the system (1)–(3) is worth investigating. The purpose of this paper is to construct a state observer for the system (1)–(3) based on the backstepping method.

If the output y(t) is the boundary value of the state, the error system associated with a standard Luenberger-type observer of (1)–(2) has a triangular structure. This structure enables us to apply the backstepping method. Unfortunately, if the output has a form represented by (3), the associated error system does not have the triangular structure. Indeed, a Luenberger-type observer for the system (1)–(2) with the output (3) is given by

\[
\frac{∂\tilde{q}}{∂t}(x, t) = a\frac{∂^2 \tilde{q}}{∂x^2}(x, t) + λ(x)\tilde{q}(x, t) - l(x)\left(y(t) - \int_0^1 h(ξ)\tilde{q}(ξ, t)dξ\right),
\]

where l(x) is the observer gain. Subtracting the above equation from (1) leads to the following error equation:

\[
\frac{∂q}{∂t}(x, t) = a\frac{∂^2 q}{∂x^2}(x, t) + λ(x)q(x, t) + l(x)\left(y(t) - \int_0^1 h(ξ)\tilde{q}(ξ, t)dξ\right),
\]

where \tilde{q} := q - \hat{q} is the estimation error. It is easily observed that the third term in the right-hand side of the error equation depends on the value of \hat{q} at almost all points of I. This means that the error system fails to possess the triangular structure due to feedback of the output error. The lack of the triangular structure makes it impossible for us to obtain an exponentially stabilizing observer gain l(x) based on backstepping. This is the most crucial problem to be solved for our purpose.

3. KEY IDEA

As seen in the preceding section, backstepping does not provide a stabilizing observer gain directly. The present section explains our strategy to overcome this problem.

3.1 Integral transformation

Let us define a state transformation q → φ by

\[
φ(x, t) = (Tq(ξ, t))(x) := \int_0^x h(ξ)q(ξ, t)dξ.
\]

Then, the output (3) can be expressed in terms of φ as

\[
y(t) = \int_0^1 h(ξ)q(ξ, t)dξ = φ(1, t).
\]

This means that the output is the boundary value of the new state. Hence, we can design an observer for φ based on backstepping if the transformed system is described by a parabolic PDE. The following theorem guarantees that the transformed system is still parabolic under an assumption about the weighting function h.

**Theorem 1.** Let a > 0 and let λ : \overline{I} → R be a function in C^1(\overline{I}). Assume that h : \overline{I} → R is a solution of the following ODE with an initial condition:

\[
ah'(x) + λ(x)h(x) = γh(x), \quad h'(0) + a h(0) = 0,
\]

where γ ∈ R is chosen so that h(x) > 0 for all x ∈ \overline{I}. Then, the transformation T defined by (4) is an invertible linear operator in L^2(I) which converts the system (1)–(2) into the system described by the following PDE:

\[
\frac{∂φ}{∂t}(x, t) = a\frac{∂^2 φ}{∂x^2}(x, t) - 2a\frac{h'(x)}{h(x)} φ(x, t) + γφ(x, t)
\]

\[
φ(0, 0) = (Tq_0)(x) = \int_0^x h(ξ)q_0(ξ)dξ
\]

with the boundary conditions

\[
φ(0, t) = 0, \quad \frac{∂φ}{∂x}(1, t) = h(1)u(t).
\]

**Remark 2.** We restrict the class of the weighting function h to positive-valued solutions of (5) and thus we can not handle arbitrary observation of the form (3). This is a weakness of the present approach. However, (5) contains the design parameter γ. Furthermore, we can specify the value of h at a point, since (5) contains only one condition.

**Remark 3.** If γ is an eigenvalue of the system operator, a solution of (5) is a corresponding eigenfunction. This causes a lack of the observability. However, this is not of our interest since eigenfunctions have zeros in \overline{I}.
3.2 Proof of Theorem 1

We need two technical lemmas to prove Theorem 1. Their proofs are omitted due to space limitation. The first lemma is related with the invertibility of $T$.

Lemma 4. Consider the linear operator $T$ defined in (4). Let $V$ be defined by $V = \{ f \in H^1(I) \mid f(0) = 0 \}$. If $h \in L_\infty(I)$, then the range of $T$ is included in $V$, that is, $\mathcal{R}(T) \subset V$. Furthermore, if $1/h$ is also in $L_\infty(I)$, $T$ is a bijection from $L^2(I)$ to $V$ and its inverse is given by

$$T^{-1}g = \frac{1}{h} \frac{\partial \phi}{\partial x}$$

with the domain $\mathcal{D}(T^{-1}) = V \subset L^2(I)$.

In Theorem 1, the weighting function is characterized as a positive-valued solution of an ODE with a design parameter $\gamma$. It is not obvious that there exists $\gamma \in \mathbb{R}$ such that a solution is a positive-valued function. We can prove the existence of such $\gamma$ by the aid of Sturm-Liouville theory [Coddington and Levinson, 1955, Zettl, 2005].

Lemma 5. Let $\lambda : \mathcal{T} \rightarrow \mathbb{R}$ be a function in $C^1(\mathcal{T})$, $a$ be a positive constant, and $\alpha, \gamma$ be real constants. Assume that $\mu_0 \in \mathbb{R}$ is the maximum eigenvalue of the boundary value problem expressed as

$$af''(x) + \lambda(x)f(x) = \mu f(x), \quad x \in I,$$

$$f(0) + \alpha f(0) = 0, \quad f(1) = 0.$$

Then, for any $h_0 > 0$, a solution of (5) satisfying $h(0) = h_0$ is a positive-valued function on $\mathcal{T}$ if and only if $1 > \gamma > \mu_0$.

This lemma guarantees that there exists $\gamma \in \mathbb{R}$ such that a solution of (5) is a positive-valued function, since the first eigenvalue of the Strum-Liouville operator is bounded.

We are now ready to prove Theorem 1.

Proof. From Lemma 5, we can always obtain a positive-valued solution of (5) for sufficiently large $\gamma$. Since $h \in C^2(\mathcal{T})$ and $h(x) > 0$ for all $x \in \mathcal{T}$, $h$ and $1/h$ belong to $L_\infty(I)$. Then, the invertibility follows from Lemma 4.

Next, we show that the boundary value of $\phi$ satisfies the boundary condition (7). Since $h$ is a solution of (5), $h \in L_\infty(I)$. Furthermore, the positivity of $h$ implies that $1/h \in L^2(I)$. Hence, Lemma 4 guarantees that $\phi(., t) \in V \cap H^2(I)$ and that $\partial \phi/\partial x(x, t) = h(x)q(x, t)$.

By substituting $x = 0$ and $x = 1$ into $\phi$ and $\partial \phi/\partial x$, respectively, we have (7).

Finally, we show that $\phi$ satisfies (6). The first temporal derivative of $\phi$ is given by

$$\frac{\partial \phi}{\partial t}(x, t) = \int_0^x h(\xi) \left( a \frac{\partial^2 q}{\partial x^2}(\xi, t) + \lambda(\xi)q(\xi, t) \right) d\xi$$

$$= ah(x)\frac{\partial q}{\partial x}(x, t) - ah'(x)q(x, t)$$

$$+ \int_0^x \left( ah''(\xi) + \lambda(\xi)h(\xi) \right) q(\xi, t) d\xi.$$

We used integration by parts twice. The third term in the right-hand side can be rewritten as

$$\int_0^x \left( ah''(\xi) + \lambda(\xi)h(\xi) \right) q(\xi, t) d\xi = \gamma \int_0^x h(\xi)q(\xi, t) d\xi = \gamma \phi(x, t).$$

Lemma 4 leads to

$$q(x, t) = \frac{1}{h(x)} \frac{\partial \phi}{\partial x}(x, t),$$

and the second derivative of $\phi$ is given by

$$\frac{\partial^2 \phi}{\partial x^2} = h'(x)q(x, t) + h(x)\frac{\partial q}{\partial x}(x, t).$$

Substituting these formulas into the expression of the first temporal derivative of $\phi$ yields (6), which implies that the theorem holds.

The right boundary condition of $\phi$ originates from that of $q$. On the other hand, the left condition is due to a property of the transformation. The left boundary condition of $q$ is included in equation (6) and we can reconstruct it by evaluating $\partial \phi/\partial t$ at $x = 0$.

4. BACKSTEPPING OBSERVER: GENERAL CASE

In the previous section, we have showed that an invertible integral transformation converts the original system (1)–(3) into another parabolic system with boundary observation. We construct an observer for the original system based on the backstepping observer for the converted system.

4.1 Preliminary

The equation of the converted system (6) contains the first spatial derivative term. We first simplify the system so that the first spatial derivative does not appear in the state equation. Let $\varphi$ be a new state variable defined by

$$\varphi(x, t) = \phi(x, t) \exp \left( \int_0^x \frac{-2a h'\xi(\xi)}{h(\xi)} d\xi \right)$$

$$= \phi(x, t) \exp \left( \ln \frac{h(0)}{h(x)} \right) = \frac{h(0)}{h(x)} \phi(x, t).$$

This is a well-known transformation to eliminate the first spatial derivative term. In our setting, it becomes a simple form because the coefficient of the first spatial derivative of the state is a special form. It is not so hard to show that $\varphi$ satisfies the following parabolic PDE:

$$\frac{\partial \varphi}{\partial t}(x, t) = a \frac{\partial^2 \varphi}{\partial x^2}(x, t) + \lambda_\gamma(x)\varphi(x, t)$$

$$\varphi(x, 0) = \phi(x, 0),$$

where $\lambda_\gamma$ is a function in $C^1(\mathcal{T})$ defined by

$$\lambda_\gamma(x) := 2\gamma - \lambda(x) - 2a \left( \frac{h'(x)}{h(x)} \right)^2.$$

The boundary conditions of $\varphi$ are given by

$$\varphi(0, t) = 0, \quad \frac{\partial \varphi}{\partial x}(1, t) + \frac{h(1)}{h(0)} \varphi(1, t) = h(0)u(t)$$

and the output is expressed as

$$y(t) = \int_0^1 h(\xi)q(\xi, t) d\xi = \phi(1, t) = \frac{h(1)}{h(0)} \varphi(1, t).$$

Instead of the absence of the first spatial derivative in the PDE, the boundary condition at $x = 1$ becomes a little bit complicated. However, since the term which contains $\varphi(1, t)$ can be expressed with the output $y(t)$, it does not matter for observer construction.
4.2 Observer synthesis

We here construct an observer for (8)–(10). A Luenberger-type observer is given by
\[
\frac{\partial \hat{\phi}}{\partial t}(x,t) = a \frac{\partial^2 \hat{\phi}}{\partial x^2}(x,t) + \lambda_\gamma(x)\hat{\phi}(x,t) - l(x) \left( \frac{h(1)}{h(0)} \gamma(t) - \phi(1,t) \right),
\]
\[\hat{\phi}(0,t) = 0,
\]
\[\frac{\partial \hat{\phi}}{\partial x}(1,t) + \beta_\gamma(y) = u(t) - l_b \left( \frac{h(1)}{h(0)} \gamma(t) - \phi(1,t) \right),
\]
where \(\beta_\gamma := h(0)h'(1)/h(1)^2\). Subtracting the above observer dynamics from (8)–(9) yields the following error dynamics:
\[\frac{\partial \tilde{\phi}}{\partial t}(x,t) = a \frac{\partial^2 \tilde{\phi}}{\partial x^2}(x,t) + \lambda_\gamma(x)\tilde{\phi}(x,t) + l(x)\tilde{\phi}(1,t)
\]
\[\tilde{\phi}(0,t) = 0,
\]
\[\frac{\partial \tilde{\phi}}{\partial x}(1,t) = l_b \tilde{\phi}(1,t).
\]
The third term in the right-hand side depends on the value of \(\tilde{\phi}\) only at \(x = 1\). This means that the error equation has the upper triangular structure. Hence, we can find a stabilizing observer gain \(l(x)\) and \(l_b\) based on backstepping.

Here, we follow the result proposed by Smyshlyaev and Krstic [2005]. First, we find a transformation of the form
\[\tilde{\phi}(x,t) = \hat{\phi}(x,t) + \int_x^1 p(x,y)\psi(y,t)dy,
\]
which converts the error system into the following exponentially stable system:
\[\frac{\partial \psi}{\partial t}(x,t) = a \frac{\partial^2 \psi}{\partial x^2}(x,t) + c \psi(x,t),
\]
\[\psi(0,t) = 0,
\]
\[\frac{\partial \psi}{\partial x}(1,t) = 0,
\]
where \(c > 0\) is a design parameter which determines convergent rate. This is achieved if
\[l(x) = a \frac{\partial p}{\partial y}(x,1), \quad l_b = p(1,1),
\]
where \(p\) is a solution of the following PDE:
\[a \frac{\partial^2 p}{\partial y^2}(x,y) = \frac{\partial^2 p}{\partial x^2}(x,y) + \lambda_\gamma(x) + \frac{c}{a} p(x,y),
\]
\[p(0,y) = 0, \quad p(x,0) = -\frac{1}{2a} \int_0^x (\lambda_\gamma(\xi) + c) d\xi.
\]
The existence, the uniqueness, and the smoothness of a solution of the above PDE are proved in [Liu, 2003, Smyshlyaev and Krstic, 2004]. A map from \(\tilde{\phi}\) to \(\hat{\phi}\) is a bounded operator on \(L^2(I)\) and its inverse is also bounded due to the boundedness of the integral kernel. Hence, the exponential stability of \(\psi\)-system implies that of \(\hat{\phi}\)-system. Furthermore, this method guarantees the exponential stability of the error dynamics in \(H^1(I)\) under assumption about the initial data.

We have constructed an observer which estimates \(\phi\). We next estimate the original state \(\tilde{\phi}\) by using the estimate \(\hat{\phi}\) of \(\phi\). Let \(\tilde{\phi}\) be an estimate of \(\tilde{\phi}\) defined by
\[\tilde{\phi}(x,t) = \frac{1}{h(0)} \left( T^{-1}(h(\phi(\cdot,t))) \left( x \right) \right) = \frac{h'(x)}{h(0)h(x)} \phi(x,t) + \frac{1}{h(0)} \frac{\partial \tilde{\phi}(x,t)}{\partial x}.
\]
Then, the estimation error \(\tilde{\phi} := \phi - \tilde{\phi}\) satisfies
\[\tilde{\phi}(x,t) = \frac{h'(x)}{h(0)h(x)} \tilde{\phi}(x,t) + \frac{1}{h(0)} \frac{\partial \tilde{\phi}(x,t)}{\partial x}.
\]

4.3 Convergence property

The next theorem, which is the main result of this paper, shows that the estimation error \(\tilde{\phi} := \phi - \tilde{\phi}\) converges to 0 in \(L^2(I)\).

**Theorem 6.** Consider the error system (14)–(15). Let \(\lambda_\gamma\) be a function in \(C^1(\overline{T})\) and let \(h\) be a positive-valued function in \(C^1(\overline{T})\). If the observer gains are given by (16)–(17) for some \(c > 0\) in \(\mathbb{R}\), then, for any initial data \(\phi_0 \in H^2(I)\), there exists a unique \(\phi(x,0)\) satisfying \(\phi(\cdot,0) = \phi_0\) and \(\tilde{\phi}(\cdot,0)\) defined in (19) satisfies
\[\|\tilde{\phi}(\cdot,t)\| \leq C e^{-(c + \pi^2/4)} \|\phi_0\|_{H^1}.
\]
for some positive constant \(C\).

**Proof.** It is proved in [Smyshlyaev and Krstic, 2010] that, under the assumption in the theorem, there exist a positive constant \(C_0 \in \mathbb{R}\) such that
\[\|\tilde{\phi}(\cdot,t)\|_{H^1(I)} \leq C_0 e^{-(c + \pi^2/4)} \|\phi_0\|_{H^1}.
\]
Hence, it suffices to show that the \(L^2\) norm of \(\tilde{\phi}(\cdot,t)\) is bounded above by the \(H^1\) norm of \(\phi(\cdot,t)\). Recall that \(\|f + g\|^2 \leq 2(\|f\|^2 + \|g\|^2)\) for any \(f, g \in L^2(I)\). Hence, we get
\[\|\tilde{\phi}(\cdot,t)\|^2 \leq \frac{2}{h(0)^2} \left( \left\| \frac{h'}{h} \tilde{\phi}(\cdot,t) \right\|^2 + \left\| \frac{\partial \tilde{\phi}}{\partial x}(\cdot,t) \right\|^2 \right)
\]
\[\leq C_2 \left( \|\tilde{\phi}(\cdot,t)\|^2 + \left\| \frac{\partial \tilde{\phi}}{\partial x}(\cdot,t) \right\|^2 \right)
\]
\[= C_2 \|\tilde{\phi}(\cdot,t)\|_{H^1},
\]
where \(C_2\) is defined by
\[C_2 := \frac{2}{h(0)^2} \max \left\{ 1, \max_{x \in I} \left( \frac{h'(x)}{h(x)} \right)^2 \right\}.
\]
This completes the proof. \(\square\)

The uniqueness of \(\tilde{\phi}\) follows from that of \(\tilde{\phi}\). Hence, if the original PDE (1)–(2) has a unique solution \(\phi\) and the weighting function \(h\) is a positive-valued solution of (5), then the observer represented by (11)–(13) and (18) gives a unique estimate of \(\phi\).

5. BACKSTELLING OBSERVER: LINEAR REACTION-DIFFUSION EQUATIONS

We must solve an ODE and a PDE to construct the observer derived in the previous section. If the coefficient \(\lambda_\gamma\) is a constant function and \(a = 0\), we can obtain closed-form solutions. Furthermore, the parametrized family of the resulting weighting functions contains a constant function, and thus we can construct an observer when the output is a pure spatial average of the state. Let \(\lambda_\gamma(x)\) be a function
taking $\lambda_0$ at all points of $I$. Then, (5) becomes a second order ODE with constant coefficients. We can easily solve the ODE, and the general solution of (5) with $\lambda(x) = \lambda_0$ is given by

$$h(x) = h_0 \left(\cosh(\omega_\gamma x) - \frac{\alpha}{\omega_\gamma} \sinh(\omega_\gamma x)\right),$$

where $h_0$ is an arbitrary constant and $\omega_\gamma$ is defined by $\omega_\gamma = \sqrt{(\gamma - \lambda_0)/a}$. Note that $\omega_\gamma$ may be a pure imaginary number and that $h$ is a real-valued function even in such a case. If $\alpha = 0$, $h$ becomes a simpler form. There still remains freedom of choice of an arbitrary constant $h_0$. Here, we determine $h_0$ so that $h$ satisfies $\int_{-1}^{1} h(\xi)d\xi = 1$. Then, the expression of $h$ becomes

$$h(x) = \frac{\omega_\gamma}{\sinh(\omega_\gamma)} \cosh(\omega_\gamma x).$$

We can observe that the weighting function is completely parametrized by $\omega_\gamma$. This weighting function is a positive-valued function if and only if $\omega_\gamma^2 > -\pi^2/4$ or equivalently $\gamma > -\pi^2a/4 + \lambda_0$ holds. The latter one coincides with the condition in Lemma 5. Figure 1 illustrates the weighting functions for some $\omega_\gamma$'s. For each $\omega_\gamma$, the weighting function is a monotone function. As $\omega_\gamma^2$ increases or equivalently $\gamma$ increases in comparison to $\lambda$, the weighting function takes a larger value around the right boundary $x = 1$. A notable case is when $\omega_\gamma = 0$, that is, $\gamma = \lambda_0$. The resulting weighting function is a constant function. Hence, as mentioned in earlier in this section, we can take the pure spatial average of the state as the output. This is a natural choice in practical point of view.

We can also obtain the closed-form solution of (17). In this case, $\lambda_\gamma$ becomes

$$\lambda_\gamma(x) = \lambda_0 + \frac{2a\omega_\gamma^2}{\cosh^2(\omega_\gamma x)}.$$  

Fortunately, the PDE (17) with $\lambda_\gamma$ of the above form has already been solved by Snyshlyeiev and Krstic [2004] as

$$p(x, y) = -\lambda \frac{I_1(\sqrt{\lambda(y^2 - x^2)})}{\sqrt{\lambda(y^2 - x^2)}} - \omega_\gamma \tanh(\omega_\gamma x) I_0(\sqrt{\lambda(y^2 - x^2)}),$$

where $\lambda = (\lambda_0 + c)/a$ and $I_k$ is the modified Bessel function of order $k$. Calculating (16) yields the following closed-form observer gains:

$$l(x) = -a\lambda x \frac{I_2(\sqrt{\lambda(1-x^2)})}{1-x^2} - \omega_\gamma \lambda \tanh(\omega_\gamma x) I_1(\sqrt{\lambda(1-x^2)}) \sqrt{\lambda(1-x^2)}$$

$$l_b = -\frac{\lambda}{2} - \omega_\gamma \lambda \tanh(\omega_\gamma x).$$

It is worth pointing out that the observer gains do not directly depend on $\gamma$. The parameter $\gamma$ affects $l(x)$ and $l_b$ only through $\omega_\gamma$. We plot $l(x)$ for several $\omega_\gamma$'s in Fig. 2, where the parameters $a$ and $\lambda$ are set to be $a = 1$ and $\lambda = 9$. The observer gain $l(x)$ has a property opposite to that of the weighting function. As $\omega_\gamma$ decreases, $l(x)$ takes a larger value around the right boundary. Furthermore, the value of $l(x)$ at $x = 1$ tends to $+\infty$ as $\omega_\gamma^2$ goes to $-\pi^2/4$. In contrast, decrease of $l(x)$ due to increase of $\omega_\gamma$ is mild. This is mathematically caused by the function $\tanh(\omega_\gamma x)$ contained in the second term in the closed-form expression of $l(x)$.

Here, we provide a qualitative explanation. The left boundary value of $\partial q/\partial x$ is fixed to 0 because $a = 0$. For a physical quantity satisfying a diffusion-type equation, its first spatial derivative corresponds to the flux. Since the flux determines flow of a physical quantity, $q$ can not change drastically on a neighborhood around $x = 0$. On the other hand, there is no restriction on $\partial q/\partial x$ at $x = 1$. Furthermore, the coefficient $\lambda$ is flat. Consequently, the value of $q$ on a neighborhood around $x = 1$ is dominant for the system (6)–(7) with $a = 0$ and $\lambda(x) = \lambda_0$. If $\omega_\gamma^2$ approaches $-\pi^2/4$, then the output is less informative. Therefore, the observer gain drastically increases to stabilize the error dynamics.

6. NUMERICAL SIMULATION

We now examine our observer by numerical simulation. The physical parameters are set as follows: $a = 1$, $\lambda(x) \equiv \lambda_0 = 4$, and $\alpha = 0$. In this case, the maximum eigenvalue of the system (1)–(2) is positive. Hence, the system is unstable. We set the input to be $u(t) = 0.2\sin(10\pi t)$, and the initial condition is given by $q_0(x) = \cos(5\pi x^2/2)$. Numerical schemes used to discretize the system are the

![Fig. 1. Weighting functions for some $\omega_\gamma$'s.](image1)

![Fig. 2. Observer gain $l(x)$ for some $\omega_\gamma$'s with $\bar{\lambda} = 9$.](image2)
Fig. 3. Distribution of the state $q$.

Fig. 4. Distribution of the estimate $\hat{q}$ of the state $q$.

sixth-order tridiagonal compact finite difference scheme [Lele, 1992] in space and the fourth-order Runge-Kutta scheme in time. The design parameters are $\gamma = 6$ and $c = 5$. Since $\omega_1^2 = 2$ and $\lambda = 9$, we can find the corresponding weighting function and observer gain from Fig. 1 and Fig. 2, respectively. The initial condition for the observer state is $\varphi(x, 0) \equiv 0$. For the observer, we employ a low-order and a low-resolution numerical schemes. The second order central difference scheme and the explicit Euler scheme are used in space and in time, respectively.

Figures 3 and 4 illustrate the distribution of the state $q(x, t)$ and its estimate $\hat{q}(x, t)$ by the proposed observer. At first, those two distributions seem different because of the effect of the initial error. However, the distribution of estimate $\hat{q}$ approaches that of $q$ after a while.

Time history of the $L^2$ norm of the estimation error $\|\tilde{q}(\cdot, t)\|_2$ is plotted in Fig. 5. We can observe that the estimation error immediately converges to 0 in terms of the $L^2$ norm.

7. CONCLUSION

We have investigated an observer based on backstepping for a class of 1-dimensional parabolic distributed parameter systems whose output is weighted spatial average of the state. To apply backstepping to observer construction, we employed an integral transformation which converts the system into a system with boundary observation. An estimate of the original state is obtained through the inverse transformation. The weighting function of the output is characterized by a solution of an ODE. Although this is an obstacle in practical application, various weighting functions are obtained since the ODE contains a design parameter. Furthermore, a family of solutions contains a constant function when the system is described by a linear diffusion-reaction equation. This enable us to take the pure spatial average as the output. The procedure for arbitrary weighting functions is still open problem.

REFERENCES


