

# Boundary Liouville Field Theory: Boundary Three Point Function

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## Abstract

Liouville field theory is considered on domains with conformally invariant boundary conditions. We present an explicit expression for the three point function of boundary fields in terms of the fusion coefficients which determine the monodromy properties of the conformal blocks.

## 1 Introduction

Liouville theory seems to be a universal building block that appears in various contexts such as noncritical string theory, two-dimensional gravity or D-brane physics. It is also closely related to the  $SL(2)$  or  $SL(2)/U(1)$  WZNW models which are interesting as solvable models for string theory on noncompact curved backgrounds. From a more general point of view one may regard Liouville theory as a prototype for an interacting conformal field with noncompact target space. It should therefore serve as a natural starting point for the development of techniques for the exact solution of such conformal field theories.

In the case of Liouville theory with periodic boundary conditions we now have a relatively complete characterization [14]: Knowledge of the spectrum of the theory and three point functions of primary fields allow one to consistently reconstruct arbitrary expectation values of local fields on the sphere or cylinder.

Our understanding is less satisfactory in the case of Liouville theory on two-dimensional domains with boundary such as the infinite strip, the upper half plane or the disk: One would again expect the theory to be fully characterized in terms of a finite set of structure functions together with the knowledge of the spectrum of the theory on the strip. Consistency of the reconstruction of the

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theory from these fundamental data requires them to satisfy consistency conditions very similar to those formulated by Cardy and Lewellen [3] in the case of rational conformal field theories. A part of these data has been determined and some of the basic consistency conditions have been verified [9] [15] [10]. What is missing are the determination of the three point function of boundary operators and the verification that these data satisfy the full set of conditions ensuring consistency of the reconstruction of the theory. The aim of the present paper is to propose an explicit expression for the three point function of boundary operators as the solution to one of the most important consistency conditions expressing the associativity of the product of boundary operators.

The structure of this paper is as follows: The following section gathers those results on Liouville theory that we will use in the present paper. The third section then contains our proposal for the three point function of boundary operators. It is based on the observation [13] that an ansatz for that three point function in terms of the fusion coefficients naturally leads to a solution of the consistency condition that expresses the associativity of the product of boundary operators. It remains to fix the remaining freedom by imposing certain normalization conditions.

Some concluding remarks are made in section 4, and the appendix contains some technical points used in the main text.

## 2 Requisites

### 2.1 Liouville theory on the sphere

Let us begin by recalling some results on Liouville theory that will be relevant for the subsequent discussion, see [14] for more details and references:

LFT on the sphere is semiclassically defined by the following action

$$\mathcal{A}_L = \int \left( \frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right) d^2x, \quad (1)$$

with the following boundary condition on the Liouville field  $\phi$

$$\phi(z, \bar{z}) = -Q \log(z\bar{z}) + O(1) \quad \text{at } |z| \rightarrow \infty. \quad (2)$$

The parameter  $b$  is related to Planck's constant  $\hbar$  via  $b^2 = \hbar$ , the scale parameter  $\mu$  is often called the cosmological constant, and  $Q$  is the background charge

$$Q = b + 1/b.$$

It was first proposed in [4] that Liouville theory can be quantized as a conformal field theory with a space of states that decomposes as follows into irreducible unitary highest weight representations  $\mathcal{V}_\alpha$  of the Virasoro algebra:

$$\mathcal{H} = \int_{\mathbb{S}} d\alpha \mathcal{V}_\alpha \otimes \mathcal{V}_\alpha, \quad \mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+. \quad (3)$$

The highest weight  $\Delta_\alpha$  of the representation  $\mathcal{V}_\alpha$  was parametrized as  $\Delta_\alpha = \alpha(Q - \alpha)$ . The action of the Virasoro algebra on  $\mathcal{H}$  is generated by the modes of the energy momentum tensor:

$$\begin{aligned} T(z) &= -(\partial\phi)^2 + Q\partial^2\phi, \\ \bar{T}(\bar{z}) &= -(\bar{\partial}\phi)^2 + Q\bar{\partial}^2\phi. \end{aligned}$$

The central charge of the Virasoro algebra is then given in terms of  $b$  via

$$c_L = 1 + 6Q^2.$$

The local observables can be generated from the fields  $V_\alpha(z, \bar{z})$  which semiclassically ( $b \rightarrow 0$ ) correspond to exponential functions  $e^{2\alpha\phi(z, \bar{z})}$  of the Liouville field. The fields  $V_\alpha(z, \bar{z})$  transform as primary fields under conformal transformations with conformal weight  $\Delta_\alpha$ . Thanks to conformal symmetry, the fields  $V_\alpha(z, \bar{z})$  are fully characterized by the three point functions

$$C(\alpha_3, \alpha_2, \alpha_1) = \lim_{z_3 \rightarrow \infty} |z_3|^{4\Delta_{\alpha_3}} \langle 0 | V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_2}(1, 1) V_{\alpha_1}(0, 0) | 0 \rangle.$$

An explicit formula for the three point function was proposed in [5, 16]<sup>3</sup>

$$C(\alpha_3, \alpha_2, \alpha_1) = \left[ \pi \mu \gamma(b^2) b^{2-2b^2} \right]^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}} \frac{\Upsilon_0 \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_1 + \alpha_3 - \alpha_2) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1)}, \quad (4)$$

where  $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ ,  $\Upsilon_0 = \text{res}_{x=0} \frac{d\Upsilon_b(x)}{dx}$ .

These pieces of information indeed amount to a full characterization of Liouville theory on the sphere or cylinder: Multipoint correlation functions can be factorized into three point functions by summing over intermediate states. Let us consider as prototypical example the four point function  $\langle 0 | \prod_{i=1}^4 V_{\alpha_i}(z_i, \bar{z}_i) | 0 \rangle$ . Such four point functions may be represented by summing over intermediate states from the spectrum (3) iff the variables  $\alpha_4, \dots, \alpha_1$  are restricted to the range<sup>4</sup>

$$\begin{aligned} 2|\text{Re}(\alpha_1 + \alpha_2 - Q)| < Q, & 2|\text{Re}(\alpha_1 - \alpha_2)| < Q, \\ 2|\text{Re}(\alpha_3 + \alpha_4 - Q)| < Q, & 2|\text{Re}(\alpha_3 - \alpha_4)| < Q. \end{aligned} \quad (5)$$

Inserting a complete set of intermediate states between  $\langle 0 | V_{\alpha_4} V_{\alpha_3}$  and  $V_{\alpha_2} V_{\alpha_1} | 0 \rangle$  would lead to an expression of the following form:

$$\begin{aligned} \langle 0 | V_{\alpha_4}(z_4, \bar{z}_4) V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_1}(z_1, \bar{z}_1) | 0 \rangle = \\ = \int_0^\infty dP C(\alpha_4, \alpha_3, Q/2 - iP) C(Q/2 + iP, \alpha_2, \alpha_1) |\mathcal{F}^s(\Delta_{\alpha_i}, \Delta, z_i)|^2 \end{aligned} \quad (6)$$

$\mathcal{F}^s(\Delta_{\alpha_i}, \Delta, z_i)$  is the s-channel conformal block which is completely determined by the conformal symmetry (although no closed formula is known for it in general).

$$\begin{aligned} \mathcal{F}^s(\Delta_{\alpha_i}, \Delta, z_i) = (z_4 - z_2)^{-2\Delta_2} (z_4 - z_1)^{\Delta_2 + \Delta_3 - \Delta_1 - \Delta_4} (z_4 - z_3)^{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4} \\ \times (z_3 - z_1)^{\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3} \mathcal{F} \left( \begin{array}{cc} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{array} \middle| P \middle| \eta \right) \end{aligned}$$

where  $\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_4)(z_1 - z_3)}$  and  $\Delta_{\alpha_i} = \alpha(Q - \alpha)$ ,  $\Delta = \frac{Q^2}{4} + P^2$ . Locality of the fields  $V_\alpha$  or associativity of the operator product expansion would lead to an alternative representation for

<sup>3</sup>see Appendix for some definitions and properties of the special functions used in this article

<sup>4</sup>It turns out [14] that the four-point function defined in the range (5) permits a meromorphic continuation to generic values of  $\alpha_4, \dots, \alpha_1$ .

$\langle 0 | \prod_{i=1}^4 V_{\alpha_i}(z_i, \bar{z}_i) | 0 \rangle$  as sum over  $t$ -channel conformal blocks  $\mathcal{F}^t$ :

$$\begin{aligned} \langle 0 | V_{\alpha_4}(z_4, \bar{z}_4) V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_1}(z_1, \bar{z}_1) | 0 \rangle &= \\ &= \int_0^\infty dP C(\alpha_4, Q/2 - iP, \alpha_1) C(Q/2 + iP, \alpha_3, \alpha_2) |\mathcal{F}^t(\Delta_{\alpha_i}, \Delta, z_i)|^2 \end{aligned} \quad (7)$$

For the equivalence of the two representations (6) and (7) it is crucial that there exist [14] invertible fusion transformations between s- and t-channel conformal blocks, defining the fusion coefficients:

$$\mathcal{F}^s(\Delta_{\alpha_i}, \Delta_{\alpha_{21}}, z_i) = \int_{\mathbb{S}} d\alpha_{32} F_{\alpha_{21}\alpha_{32}} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} \mathcal{F}^t(\Delta_{\alpha_i}, \Delta_{\alpha_{32}}, z_i). \quad (8)$$

In [11], an explicit formula for the fusion coefficients was proposed in terms of the Racah-Wigner coefficients for an appropriate continuous series of representations of the quantum group  $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$  with deformation parameter  $q = e^{i\pi b^2}$ . This formula was subsequently [14] confirmed by direct calculation. The resulting expression for the fusion coefficients is the following:

$$\begin{aligned} F_{\sigma_2\beta_3} \begin{bmatrix} \beta_2 & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} &= \\ &= \frac{\Gamma_b(2Q - \beta_1 - \beta_2 - \beta_3) \Gamma_b(\beta_2 + \beta_3 - \beta_1) \Gamma_b(Q + \beta_2 - \beta_1 - \beta_3) \Gamma_b(Q + \beta_3 - \beta_2 - \beta_1)}{\Gamma_b(2Q - \sigma_1 - \beta_1 - \sigma_2) \Gamma_b(\sigma_1 + \sigma_2 - \beta_1) \Gamma_b(Q - \beta_1 - \sigma_2 + \sigma_1) \Gamma_b(Q - \beta_1 - \sigma_1 + \sigma_2)} \\ &\times \frac{\Gamma_b(Q - \beta_3 - \sigma_1 + \sigma_3) \Gamma_b(\beta_3 + \sigma_1 + \sigma_3 - Q) \Gamma_b(\sigma_1 + \sigma_3 - \beta_3) \Gamma_b(\sigma_3 + \beta_3 - \sigma_1)}{\Gamma_b(Q - \beta_2 - \sigma_2 + \sigma_3) \Gamma_b(\beta_2 + \sigma_2 + \sigma_3 - Q) \Gamma_b(\sigma_2 + \sigma_3 - \beta_2) \Gamma_b(\sigma_3 + \beta_2 - \sigma_2)} \\ &\times \frac{\Gamma_b(2Q - 2\sigma_2) \Gamma_b(2\sigma_2)}{\Gamma_b(Q - 2\beta_3) \Gamma_b(2\beta_3 - Q)} \frac{1}{i} \int_{-i\infty}^{i\infty} ds \frac{S_b(U_1 + s) S_b(U_2 + s) S_b(U_3 + s) S_b(U_4 + s)}{S_b(V_1 + s) S_b(V_2 + s) S_b(V_3 + s) S_b(Q + s)}, \end{aligned} \quad (9)$$

where:

$$\begin{aligned} U_1 &= \sigma_2 + \sigma_1 - \beta_1, & V_1 &= Q + \sigma_2 - \beta_3 - \beta_1 + \sigma_3, \\ U_2 &= Q + \sigma_2 - \beta_1 - \sigma_1, & V_2 &= \sigma_2 + \beta_3 + \sigma_3 - \beta_1, \\ U_3 &= \sigma_2 + \beta_2 + \sigma_3 - Q, & V_3 &= 2\sigma_2, \\ U_4 &= \sigma_2 - \beta_2 + \sigma_3. \end{aligned}$$

An important identity satisfied by the fusion coefficients is the so-called pentagon equation, which follows from a similar identity satisfied by the Racah-Wigner coefficients mentioned previously [12].

$$\begin{aligned} \int_{\mathbb{S}} d\delta_1 F_{\beta_1\delta_1} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \beta_2 & \alpha_1 \end{bmatrix} F_{\beta_2\gamma_2} \begin{bmatrix} \alpha_4 & \delta_1 \\ \alpha_5 & \alpha_1 \end{bmatrix} F_{\delta_1\gamma_1} \begin{bmatrix} \alpha_4 & \alpha_3 \\ \gamma_2 & \alpha_2 \end{bmatrix} \\ = F_{\beta_2\gamma_1} \begin{bmatrix} \alpha_4 & \alpha_3 \\ \alpha_5 & \beta_1 \end{bmatrix} F_{\beta_1\gamma_2} \begin{bmatrix} \gamma_1 & \alpha_2 \\ \alpha_5 & \alpha_1 \end{bmatrix}. \end{aligned}$$

## 2.2 Liouville theory on domains with boundary

One is also interested in understanding Liouville theory on a simply connected domain  $\Gamma$  with a nontrivial boundary  $\partial\Gamma$ . For definiteness, we will only consider the conformally equivalent cases where  $\Gamma$  is either the unit disk, the upper half plane or the infinite strip.

Semiclassically, one may define the theory by means of the action

$$A_{\text{bound}} = \int_{\Gamma} \left( \frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right) d^2x + \int_{\partial\Gamma} \left( \frac{Qk}{2\pi} + \mu_B e^{b\phi} \right) dx, \quad (10)$$

where  $k$  is the curvature of the boundary  $\partial\Gamma$  and  $\mu_B$  is the so-called boundary cosmological constant. For the description of exact results in the quantum theory it was found to be useful [9] to parametrize  $\mu_B$  by means of a variable  $\sigma$  that is related to  $\mu_B$  via

$$\cos\left(2\pi b\left(\sigma - \frac{Q}{2}\right)\right) = \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)}. \quad (11)$$

Requiring  $\mu_B$  to be real one finds the two following regimes for the parameter  $\sigma$ :

1. if  $\frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)} > 1$ , then  $\sigma$  is of the form  $\sigma = Q/2 + iP$
2. if  $\frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)} < 1$ , then  $\sigma$  is real.

Anticipating that all relevant objects will be found to possess meromorphic continuations w.r.t. the boundary parameters  $\sigma$ , we shall discuss only the first regime explicitly in the following.

The Hamiltonian interpretation of the theory [15] is simplest in the case that  $\Gamma$  is the infinite strip. The associated Hilbert space  $\mathcal{H}_B$  was found in [15] to decompose as follows into irreducible representations of the Virasoro algebra:

$$\mathcal{H}^B = \int_{\mathbb{S}}^{\oplus} d\beta \mathcal{V}_\beta. \quad (12)$$

The highest weight states generating the subrepresentation  $\mathcal{V}_\beta$  in  $\mathcal{H}^B$  will be denoted  $|\beta; \sigma_2, \sigma_1\rangle$ , where  $\sigma_2$  ( $\sigma_1$ ) are the parameters of the boundary conditions associated to the left (right) boundaries of the strip. It was proposed in [9] [15] that the states  $|\beta; \sigma_2, \sigma_1\rangle$  satisfy a reflection relation of the form

$$|\beta; \sigma_2, \sigma_1\rangle = S(\beta; \sigma_2, \sigma_1) |Q - \beta; \sigma_2, \sigma_1\rangle. \quad (13)$$

which expresses the totally reflecting nature of the Liouville potential in (10). The following formula was given in [9] for the reflection coefficient  $S(\beta; \sigma_2, \sigma_1)$ :

$$S(\beta_3, \sigma_3, \sigma_1) = (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{2b}(Q-2\beta_3)} \times \frac{\Gamma_b(2\beta_3 - Q) S_b(\sigma_3 + \sigma_1 - \beta_3) S_b(2Q - \beta_3 - \sigma_1 - \sigma_3)}{\Gamma_b(Q - 2\beta_3) S_b(\beta_3 + \sigma_3 - \sigma_1) S_b(\beta_3 + \sigma_1 - \sigma_3)}. \quad (14)$$

In addition to the fields  $V_\alpha(z, \bar{z})$  localized in the interior of  $\Gamma$ , one may now also consider operators  $B_\beta^{\sigma_2\sigma_1}(x)$  that are localized at the boundary  $\partial\Gamma$ . The insertion point  $x$  may separate segments of the boundary with different boundary conditions  $\sigma_2$  and  $\sigma_1$ . The boundary fields  $B_\beta^{\sigma_2\sigma_1}(x)$  are required to be primary fields with conformal weight  $\Delta_\beta = \beta(Q - \beta)$ . They are therefore expected to create states  $|\beta; \sigma_2, \sigma_1\rangle$  and  $\langle\beta; \sigma_2, \sigma_1|$  via

$$\lim_{x \rightarrow 0} B_\beta^{\sigma_2\sigma_1}(x)|0\rangle = |\beta; \sigma_2, \sigma_1\rangle, \quad \lim_{x \rightarrow \infty} \langle 0|B_\beta^{\sigma_1\sigma_2}(x)|x|^{2\Delta_\beta} = \langle\beta; \sigma_2, \sigma_1|. \quad (15)$$

To fully characterize LFT on the upper half plane, one needs to determine some additional structure functions beside the bulk three point function  $C(\alpha_3, \alpha_2, \alpha_1)$ :

1. Bulk one point function [9]:

$$\langle 0|V_\alpha(z, \bar{z})|0\rangle = \frac{U(\alpha|\sigma)}{|z - \bar{z}|^{2\Delta_\alpha}}. \quad (16)$$

2. Boundary two point function [9]:

$$\langle 0|B_{\beta_1}^{\sigma_1\sigma_2}(x)B_{\beta_2}^{\sigma_2\sigma_1}(0)|0\rangle = \frac{\delta(\beta_2 + \beta_1 - Q) + S(\beta_1, \sigma_2, \sigma_1)\delta(\beta_2 - \beta_1)}{|x|^{2\Delta_{\beta_1}}}. \quad (17)$$

Let us remark that requiring the prefactor of the first delta-distribution on the right hand side of (17) to be unity partially fixes the normalization of boundary operators. The appearance of the second term in (17) is a consequence of the reflection property (13).

3. bulk-boundary two point function [10]:<sup>5</sup>

$$\langle 0|V_\alpha(z, \bar{z})B_\beta^\sigma(x)|0\rangle = \frac{R(\alpha, \beta|\sigma)}{|z - \bar{z}|^{2\Delta_\alpha - \Delta_\beta} |z - x|^{2\Delta_\beta}} \quad (18)$$

4. Boundary three point function:

$$\langle 0|B_{\beta_3}^{\sigma_1\sigma_3}(x_3)B_{\beta_2}^{\sigma_3\sigma_2}(x_2)B_{\beta_1}^{\sigma_2\sigma_1}(x_1)|0\rangle = \frac{C_{\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}}{|x_{21}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{32}|^{\Delta_2 + \Delta_3 - \Delta_1} |x_{31}|^{\Delta_3 + \Delta_1 - \Delta_2}}.$$

Taking advantage of the reflection property (13), we shall consider instead of  $C_{\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  the related quantity

$$C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} \equiv C_{Q-\beta_3, \beta_2, \beta_1}^{\sigma_3\sigma_2\sigma_1} \equiv S^{-1}(\beta_3; \sigma_1, \sigma_3)C_{\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}. \quad (19)$$

The present note will be devoted to the determination of this last structure function.

## 3 Boundary three point function

### 3.1 Associativity condition

The basic consistency condition that the three-point function of boundary operators has to satisfy expresses the associativity of the product of boundary fields. Let us consider the 4 point function of boundary operators. Inserting a complete set of intermediate states between the first two and the last two fields leads to an expansion into conformal blocks of the following form:<sup>6</sup>

$$\begin{aligned} & \left\langle B_{Q-\beta_4}^{\sigma_1\sigma_4}(x_4)B_{\beta_3}^{\sigma_4\sigma_3}(x_3)B_{\beta_2}^{\sigma_3\sigma_2}(x_2)B_{\beta_1}^{\sigma_2\sigma_1}(x_1) \right\rangle = \\ & = \int_{\mathbb{S}} d\beta_{21} C_{\beta_4|\beta_3\beta_{21}}^{\sigma_4\sigma_3\sigma_1} C_{\beta_{21}|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} \mathcal{F}^s(\Delta_{\beta_i}, \Delta_{\beta_{21}}, x_i). \end{aligned}$$

By using either cyclicity of correlation functions or associativity of the operator product expansion one would get a second expansion (t-channel):

$$\begin{aligned} & \left\langle B_{Q-\beta_4}^{\sigma_1\sigma_4}(x_4)B_{\beta_3}^{\sigma_4\sigma_3}(x_3)B_{\beta_2}^{\sigma_3\sigma_2}(x_2)B_{\beta_1}^{\sigma_2\sigma_1}(x_1) \right\rangle = \\ & = \int_{\mathbb{S}} d\beta_{32} C_{\beta_4|\beta_3\beta_{32}}^{\sigma_4\sigma_2\sigma_1} C_{\beta_{32}|\beta_3\beta_2}^{\sigma_4\sigma_3\sigma_2} \mathcal{F}^t(\Delta_{\beta_i}, \Delta_{\beta_{32}}, x_i). \end{aligned}$$

<sup>5</sup>the bulk one point function is a special case of the bulk-boundary coefficient with  $\beta = 0$ .

<sup>6</sup>As in the discussion of the four point function of bulk fields, we shall restrict ourselves to the case where  $Re(\beta_i)$ ,  $i = 1 \dots 4$  are close enough to  $Q/2$ . In this case,  $\beta_{21}$  is of the form  $Q/2 + iP$ . It turns out a posteriori that the general case can be treated by meromorphic continuation.

Using the fusion transformations (8), the equivalence of the factorisation in the two channels can be rewritten:

$$\int_{\mathbb{S}} d\beta_{21} C_{\beta_4|\beta_3,\beta_{21}}^{\sigma_4\sigma_3\sigma_1} C_{\beta_{21}|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} F_{\beta_{21}\beta_3\beta_2} \begin{bmatrix} \beta_3 & \beta_2 \\ \beta_4 & \beta_1 \end{bmatrix} = C_{\beta_4|\beta_3\beta_2,\beta_1}^{\sigma_4\sigma_2\sigma_1} C_{\beta_3\beta_2|\beta_3\beta_2}^{\sigma_4\sigma_3\sigma_2}. \quad (20)$$

By means of the pentagon equation (4) it easy to verify that the following ansatz

$$C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = \frac{g(\beta_3; \sigma_3, \sigma_1)}{g(\beta_2; \sigma_3, \sigma_2)g(\beta_1; \sigma_2, \sigma_1)} F_{\sigma_2\beta_3} \begin{bmatrix} \beta_2 & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} \quad (21)$$

yields a solution to (20), as was noticed in [13]. The functions  $g(\beta; \sigma_2, \sigma_1)$  appearing are unrestricted by (20). Additional information is needed to determine them.

### 3.2 Determination of the function $g$

The boundary three point function  $C_{\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  should be meromorphic w.r.t. the variables  $\beta_3, \beta_2, \beta_1$ . This far-reaching assumption can be motivated in various ways: One may e.g. use arguments like those reviewed in section 3 of [14] concerning the path integral for Liouville theory. These arguments exhibit the analytic properties of correlation functions as a reflection of the asymptotic behavior of the Liouville path integral measure in the region  $\phi \rightarrow -\infty$  where the interaction terms vanish.<sup>7</sup>

Such considerations lead in particular to the identification of the residues for the poles of  $C_{\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  with certain correlation functions in free field theory, which generalize the so-called screening-charge constructions of [6]. The resulting prescription for the calculation of these residues was formulated in [9]. Most relevant for our purposes will be the observation that  $C_{\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  has a pole with residue 1 if  $\beta_1 + \beta_2 + \beta_3 = Q$ : The relevant correlation functions in free field theory do not contain any screening charges.

On the other hand, it seems worth observing that the fusion coefficients themselves are meromorphic functions of all six variables they depend on, see [12, Lemma 21]. This means that the function  $C_{\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  that is given by the expression (21) will be meromorphic iff the function  $g(\beta; \sigma_2, \sigma_1)$  is meromorphic w.r.t.  $\beta$ .

In the following, we shall consider the special boundary field  $B_{-b}^{\sigma\sigma}(x)$ , which corresponds to a degenerate representation of the Virasoro algebra. As pointed out in [9], it is in general not a trivial issue to decide when a boundary field that corresponds to a degenerate representation will satisfy the corresponding differential equations expressing null vector decoupling. Here, however, one may observe that one may create the boundary field  $B_{-b}^{\sigma\sigma}(x)$  by sending the bulk field  $V_{-b/2}$  to the boundary: It follows from the fact that  $V_{-b/2}$  satisfies a second order differential equation that the asymptotic behavior when  $V_{-b/2}$  approaches the boundary is described by a boundary field  $B_{-b}^{\sigma\sigma}(x)$  that satisfies a third order differential equation. This last fact also implies that the operator product expansion of  $B_{-b}^{\sigma\sigma}(x)$  with a generic boundary operator can only contain three types of contributions:

$$B_{\beta}^{\sigma_2\sigma_1}(x') B_{-b}^{\sigma_1\sigma_1}(x) = \sum_{s=-1}^1 c_s(\beta; \sigma_2; \sigma_1) |x' - x|^{\Delta_{\beta-sb} - \Delta_{\beta} - \Delta_{-b}} B_{\beta-sb}^{\sigma_2\sigma_1}(x) + \quad (22)$$

+ (descendants).

<sup>7</sup>An equivalent discussion can be carried out in the framework of canonical quantization by considering the asymptotic behavior of the Hamiltonian and its generalized eigenfunctions, see [14, Section 11] for such a discussion in the case of Liouville theory without boundary, and [15] for the basics of the corresponding treatment in the case of boundary conditions like (10).

One may then consider the vacuum expectation values of the product of operators that is obtained by multiplying (22) with the boundary fields  $B_{Q-\beta+sb}^{\sigma_1\sigma_3}$ ,  $s \in \{-, 0, +\}$ . Taking into account (17), one is led to identify the structure functions  $c_s(\beta; \sigma_2; \sigma_1)$  ( $s = +, 0, -$ ) with residues of the general three point function. As mentioned previously, the relevant residues can be represented as correlation functions in free field theory. The structure function  $c_+$  is nothing but a special case of the above-mentioned residue at  $\beta_1 + \beta_2 + \beta_3 = Q$ , which is 1.

This should be compared to what would follow from our ansatz (21). Let us note that it follows from appendix B.3 that the fusion coefficients indeed have a pole in the presently considered case. The corresponding residue is most easily calculated by means of the recursion relations that follow from (4), see appendix B.3 for some details. We find

$$F_{\sigma_1, \beta_2 - b} \begin{bmatrix} \beta_2 & -b \\ \sigma_3 & \sigma_1 \end{bmatrix} = \frac{\Gamma(1+b^2)}{\Gamma(1+2b^2)} \frac{\Gamma(2b\sigma_1)\Gamma(2b(Q-\sigma_1))}{\Gamma(b(Q-\beta_2+\sigma_3-\sigma_1))\Gamma(b(Q-\beta_2+\sigma_1-\sigma_3))} \times \quad (23)$$

$$\times \frac{\Gamma(b(Q-2\beta_2))\Gamma(b(Q-2\beta_2+b))}{\Gamma(b(\sigma_3+\sigma_1-\beta_2))\Gamma(b(2Q-\beta_2-\sigma_3-\sigma_1))}.$$

Our ansatz (21) together with  $c_+ \equiv 1$  therefore implies the following first order difference equation for  $g$ :

$$1 = \frac{g(\beta_2 - b; \sigma_3, \sigma_1)}{g(\beta_2; \sigma_3, \sigma_1)g(-b; \sigma_1, \sigma_1)} F_{\sigma_1, \beta_2 - b} \begin{bmatrix} \beta_2 & -b \\ \sigma_3 & \sigma_1 \end{bmatrix}. \quad (24)$$

This functional equation is solved by the following expression:

$$g(\beta; \sigma_3, \sigma_1) = \frac{f(\sigma_3, \sigma_1)^{b^{-1}\beta/2} \Gamma_b(Q) \Gamma_b(Q-2\beta) \Gamma_b(2\sigma_1) \Gamma_b(2Q-2\sigma_3)}{\Gamma_b(2Q-\beta-\sigma_1-\sigma_3) \Gamma_b(\sigma_1+\sigma_3-\beta) \Gamma_b(Q-\beta+\sigma_1-\sigma_3) \Gamma_b(Q-\beta+\sigma_3-\sigma_1)}, \quad (25)$$

where  $f(\sigma_3, \sigma_1)$  is an arbitrary function. Let us furthermore note that one may derive a second finite difference equation that is related to (24) by substituting  $b \rightarrow b^{-1}$  if one considers  $B_{-b}^{\sigma_1\sigma_1}$  instead of  $B_{-b}^{\sigma_1\sigma_1}$ . Taken together, these two functional equations allow one to conclude that our solution (25) is unique at least for irrational values of  $b$ .

It is finally useful to note that we now have two possible ways to calculate the structure function  $c_-(\beta; \sigma_2, \sigma_1)$ : On the one hand one may use our ansatz (21), using (25) and the following residue of the fusion coefficients:

$$F_{\sigma_1, \beta_2 + b} \begin{bmatrix} \beta_2 & -b \\ \sigma_3 & \sigma_1 \end{bmatrix} = \frac{\Gamma(1+b^2)}{\Gamma(1+2b^2)} \frac{\Gamma(2b\sigma_1)\Gamma(2b(Q-\sigma_1))\Gamma(2b\beta_2-2bQ)\Gamma(2b\beta_2-1)}{\Gamma(b(\beta_2+\sigma_3-\sigma_1))\Gamma(b(\beta_2+\sigma_1-\sigma_3))\Gamma(b(\sigma_3+\sigma_1+\beta_2-Q))\Gamma(b(\beta_2-\sigma_3-\sigma_1+Q))}$$

On the other hand,  $c_-(\beta; \sigma_2, \sigma_1)$  is one of the cases where a representation in terms of free field correlation functions is available [9]:

$$c_-(\beta; \sigma_2, \sigma_1) = -\frac{4\mu}{\pi} \frac{\Gamma(1+b^2)}{\Gamma(-b^2)} \times \Gamma(b(2\beta_2-Q))\Gamma(2b\beta_2-1)\Gamma(1-2b\beta_2)\Gamma(1-b(2\beta_2+b)) \quad (26)$$

$$\times \sin \pi b(Q + \beta_2 - \sigma_3 - \sigma_1) \sin \pi b(\beta_2 + \sigma_3 + \sigma_1 - Q)$$

$$\times \sin \pi b(\beta_2 + \sigma_3 - \sigma_1) \sin \pi b(\beta_2 + \sigma_1 - \sigma_3).$$



It is a nontrivial check of the consistency of our approach that the expressions which one obtains by following these two ways are indeed consistent and agree if the function  $f(\sigma_3, \sigma_1)$  is chosen according to

$$f(\sigma_3, \sigma_1) = \pi\mu\gamma(b^2)b^{2-2b^2}. \quad (27)$$

By collecting the pieces, one finally arrives at the following expression for the three point function of boundary operators:

$$\begin{aligned} C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} &= \\ &= \frac{\Gamma_b(2Q - \beta_1 - \beta_2 - \beta_3)\Gamma_b(\beta_2 + \beta_3 - \beta_1)\Gamma_b(Q + \beta_2 - \beta_1 - \beta_3)\Gamma_b(Q + \beta_3 - \beta_1 - \beta_2)}{\Gamma_b(2\beta_3 - Q)\Gamma_b(Q - 2\beta_2)\Gamma_b(Q - 2\beta_1)\Gamma_b(Q)} \\ &\times (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{2b}(\beta_3 - \beta_2 - \beta_1)} \frac{S_b(\beta_3 + \sigma_1 - \sigma_3)S_b(Q + \beta_3 - \sigma_3 - \sigma_1)}{S_b(\beta_2 + \sigma_2 - \sigma_3)S_b(Q + \beta_2 - \sigma_3 - \sigma_2)} \\ &\times \frac{1}{i} \int_{-i\infty}^{i\infty} ds \frac{S_b(U_1 + s)S_b(U_2 + s)S_b(U_3 + s)S_b(U_4 + s)}{S_b(V_1 + s)S_b(V_2 + s)S_b(V_3 + s)S_b(Q + s)}, \end{aligned} \quad (28)$$

where the coefficients  $U_i, V_i$  and  $i = 1, \dots, 4$  are defined as

$$\begin{aligned} U_1 &= \sigma_1 + \sigma_2 - \beta_1, & V_1 &= Q + \sigma_2 - \sigma_3 - \beta_1 + \beta_3, \\ U_2 &= Q - \sigma_1 + \sigma_2 - \beta_1, & V_2 &= 2Q + \sigma_2 - \sigma_3 - \beta_1 - \beta_3, \\ U_3 &= \beta_2 + \sigma_2 - \sigma_3, & V_3 &= 2\sigma_2, \\ U_4 &= Q - \beta_2 + \sigma_2 - \sigma_3. \end{aligned}$$

### 3.3 Further consistency checks

1. One recovers the expression for the boundary reflection amplitude (14) from the boundary three point function the same way the bulk reflection amplitude was recovered from the bulk three point function in [16]: Using the fact that the fusion matrix depends on conformal weights only, and is thus invariant when  $\beta_i \rightarrow Q - \beta_i$ , one finds:

$$C_{Q-\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = \frac{g(Q - \beta_3, \sigma_3, \sigma_1)}{g(\beta_3, \sigma_3, \sigma_1)} C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} \quad (29)$$

From the expression (25) for the function  $g$ , one indeed finds formula (14) for  $S(\beta; \sigma_2, \sigma_1)$ .

2. One may explicitly check that the two-point function (17) is recovered by taking e.g. the limit  $\beta_1 \rightarrow 0$  if the three-point function:

$$\lim_{\beta_1 \rightarrow 0} C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = \delta(\beta_3 - \beta_2) + S(\beta_2; \sigma_3, \sigma_1)\delta(\beta_3 + \beta_2 - Q).$$

This is an easy consequence of the identity (36) proven in Appendix B.1.

3. With the help of symmetry properties of the fusion coefficients (see Appendix B.2), it is possible to check that the boundary three point function is invariant w.r.t. cyclic permutations.

### 3.4 Uniqueness

We are finally going to sketch an argument in favor of the uniqueness of our expression for the boundary three point function: Let us consider the associativity condition in the case that  $\sigma_1 = \sigma_2$  and that the boundary field  $B_{\beta_1}^{\sigma_1\sigma_1}(x_1)$  is replaced by the degenerate field  $B_{-b}^{\sigma_1\sigma_1}(x_1)$ . Due to (22), one finds that the associativity condition (20) gets replaced by

$$\sum_{\substack{\beta_{21}=\beta_2-sb \\ s \in \{-,0,+\}}} c_s(\beta_2; \sigma_3, \sigma_1) F_{\beta_{21}\beta_{32}} \begin{bmatrix} \beta_3 & \beta_2 \\ \beta_4 & -b \end{bmatrix} C_{\beta_4|\beta_3\beta_{21}}^{\sigma_4\sigma_3\sigma_1} = c_s(\beta_{32}; \sigma_4, \sigma_1) C_{\beta_{32}|\beta_3\beta_2}^{\sigma_4\sigma_3\sigma_1}, \quad (30)$$

where  $\beta_{32}$  takes the values  $\beta_4 + sb$ ,  $s \in \{-,0,+\}$ . This can be read as a system of finite difference equations for the general boundary three point function  $C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$ . By specializing to the case that  $\beta_{32} = \beta_4$ , one finds in particular a linear relation between the  $C_{\beta_4|\beta_3\beta_2-sb}^{\sigma_4\sigma_3\sigma_1}$ ,  $s \in \{-,0,+\}$ . Replacing the degenerate field  $B_{-b}^{\sigma_1\sigma_1}(x_1)$  by  $B_{-b^{-1}}^{\sigma_1\sigma_1}(x_1)$  leads to a similar second order finite difference equation which is related to the first by replacing  $b \rightarrow b^{-1}$  in the coefficients, as well as replacing  $\beta_2 - sb$  by  $\beta_2 - sb^{-1}$ .

It can be shown (see appendix C) that such self-dual systems of finite difference equations can for irrational  $b$  only have at most two linearly independent solutions. The relevant linear combination of these two solutions can be fixed e.g. by imposing the correct behavior w.r.t. the reflection  $\beta_2 \rightarrow Q - \beta_2$  as given by (13). In this way one arrives at the conclusion that the finite difference equations following from the associativity condition together with the reflection property (13) suffice to uniquely determine the dependence of  $C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  w.r.t. the variable  $\beta_1$ .

But one may of course repeat that line of arguments by replacing any of the four operators in the four point function of boundary fields by the degenerate fields  $B_{-b}^{\sigma}(x)$  or  $B_{-b^{-1}}^{\sigma}(x)$ , which would lead to finite difference equations that constrain the dependence of  $C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  w.r.t.  $\beta_3$  and  $\beta_2$ . This leads to the conclusion that indeed the associativity condition in the presence of degenerate fields together with the reflection property (13) uniquely determine the dependence of  $C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  w.r.t. all three variables  $\beta_3, \beta_2, \beta_1$ .

The remaining freedom consists of multiplication with an arbitrary function of the boundary parameters  $\sigma_3, \sigma_2, \sigma_1$ . This freedom is eliminated by requiring that the residue of the pole of  $C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$  at  $\beta_1 + \beta_2 + \beta_3 = Q$  should indeed be unity, as discussed in subsection 3.2.

## 4 Concluding remarks

We now have determined the last of the structure functions that one needs to completely characterize Liouville theory on domains with boundary. What remains to be done is the verification that the expressions that have been put forward indeed satisfy the full set of Cardy-Lewellen type [3] consistency conditions. Although some particularly important conditions have been verified (an analog of the Cardy condition [15], as well as the associativity condition studied in the present paper), it remains in particular to verify the conditions that link the boundary three point function with the bulk-boundary two-point function proposed in [10].

A beautiful characterization of the structure constants of certain classes of rational conformal field theories with boundaries has been given in [7], see also [2] for closely related results. It can be read as the statement that upon choosing a suitable normalization of the three point conformal blocks or chiral vertex operators, it becomes possible to recover all of the structure constants from the defining data of an associated modular tensor category. Validity of the Cardy-Lewellen conditions is

automatic in this formalism. What is not directly furnished by that formalism, though, is the explicit characterization for the necessary normalization of the three point conformal blocks.

It would certainly be nice to have at hand a similarly powerful formalism for non-rational conformal field theories such as Liouville theory. This should allow one in particular to carry out the missing proof that the structure functions satisfy the full set of Cardy-Lewellen type [3] consistency conditions. We will therefore try to verify whether our expression for the boundary three point function can be written in a form that one would expect to find in an extension of the formalism of [7] to non-rational CFT.

This turns out to be the case: Let us write the three point function in terms of the b-Racah-Wigner coefficients:

$$C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = \frac{g(\beta_3, \sigma_3, \sigma_1)}{g(\beta_2, \sigma_3, \sigma_2)g(\beta_1, \sigma_2, \sigma_1)} \frac{N(\sigma_3, \beta_2, \sigma_2)N(\sigma_2, \beta_1, \sigma_1)}{N(\sigma_3, \beta_3, \sigma_1)N(\beta_3, \beta_2, \beta_1)} \left\{ \begin{array}{c|c} \sigma_1 & \beta_1 \\ \beta_2 & \sigma_3 \end{array} \middle| \begin{array}{c} \sigma_2 \\ \beta_3 \end{array} \right\}_b \quad (31)$$

where [11]

$$N(\beta_3, \beta_2, \beta_1) = \frac{\Gamma_b(2\beta_1)\Gamma_b(2\beta_2)\Gamma_b(2Q - 2\beta_3)}{\Gamma_b(2Q - \beta_1 - \beta_2 - \beta_3)\Gamma_b(Q - \beta_1 - \beta_2 + \beta_3)\Gamma_b(\beta_1 + \beta_3 - \beta_2)\Gamma_b(\beta_2 + \beta_3 - \beta_1)} \quad (32)$$

It is easy to see that this can be rewritten as

$$C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = \frac{1}{g(\beta_3; \beta_2, \beta_1)} \left\{ \begin{array}{c|c} \sigma_1 & \beta_1 \\ \beta_2 & \sigma_3 \end{array} \middle| \begin{array}{c} \sigma_2 \\ \beta_3 \end{array} \right\}'_b, \quad (33)$$

where the b-Racah-Wigner coefficients that appear on the right hand side have been modified w.r.t. to those considered in [12] according to

$$\left\{ \begin{array}{c|c} \sigma_1 & \beta_1 \\ \beta_2 & \sigma_3 \end{array} \middle| \begin{array}{c} \sigma_2 \\ \beta_3 \end{array} \right\}'_b \equiv \frac{S_b(\sigma_3 + \beta_3 - \sigma_1)S_b(\beta_3 + \beta_2 - \beta_1)}{S_b(\sigma_3 + \beta_2 - \sigma_2)S_b(\sigma_2 + \beta_1 - \sigma_1)} \left\{ \begin{array}{c|c} \sigma_1 & \beta_1 \\ \beta_2 & \sigma_3 \end{array} \middle| \begin{array}{c} \sigma_2 \\ \beta_3 \end{array} \right\}_b. \quad (34)$$

By using the the counterpart of (42) for the b- Racah-Wigner coefficients one may write (33) as

$$C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = \frac{1}{g(\beta_3; \beta_2, \beta_1)} \left\{ \begin{array}{c|c} \bar{\beta}_2 & \bar{\beta}_1 \\ \sigma_1 & \sigma_3 \end{array} \middle| \begin{array}{c} \bar{\beta}_3 \\ \sigma_2 \end{array} \right\}'_b. \quad (35)$$

We consider (35) as an encouraging hint that a verification of the Cardy-Lewellen conditions should be possible along similar lines as followed in [7] for the case of rational CFT.

## 5 Acknowledgments

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## A Special functions

### A.1 $\Gamma_b$ function

The Double Gamma function introduced by Barnes [1] is defined by:

$$\log \Gamma_2(s|\omega_1, \omega_2) = \left( \frac{\partial}{\partial t} \sum_{n_1, n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right)_{t=0}$$

Definition:  $\Gamma_b(x) \equiv \frac{\Gamma_2(x|b, b^{-1})}{\Gamma_2(Q/2|b, b^{-1})}$ .

Functional relations:

$$\Gamma_b(x+b) = \frac{\sqrt{2\pi} b^{bx - \frac{1}{2}}}{\Gamma(bx)} \Gamma_b(x)$$

$$\Gamma_b(x+1/b) = \frac{\sqrt{2\pi} b^{-\frac{x}{b} + \frac{1}{2}}}{\Gamma(x/b)} \Gamma_b(x)$$

$\Gamma_b$  is a meromorphic function of  $x$ , which poles are located at  $x = -nb - mb^{-1}$ ,  $n, m \in \mathbb{N}$   
Integral representation convergent for  $0 < \text{Re}x$

$$\log \Gamma_b(x) = \int_0^{\infty} \frac{dt}{t} \left[ \frac{e^{-xt} - e^{-Qt/2}}{(1 - e^{-bt})(1 - e^{-t/b})} - \frac{(Q/2 - x)^2}{2} e^{-t} - \frac{Q/2 - x}{t} \right]$$

### A.2 $S_b$ function

Definition:  $S_b(x) \equiv \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$

Functional relations:

$$S_b(x+b) = 2\sin(\pi bx) S_b(x)$$

$$S_b(x+1/b) = 2\sin(\pi x/b) S_b(x)$$

$S_b(x)$  is a meromorphic function of  $x$ , which poles are located at  $x = -nb - mb^{-1}$ ,  $n, m \in \mathbb{N}$ , and which zeros are located at  $x = Q + nb + mb^{-1}$ ,  $n, m \in \mathbb{N}$ .

Integral representation convergent in the strip  $0 < \text{Re}x < Q$

$$\log S_b(x) = \int_0^{\infty} \frac{dt}{t} \left[ \frac{\sinh(\frac{Q}{2} - x)t}{2\sinh(\frac{bt}{2})\sinh(\frac{t}{2b})} - \frac{(Q-2x)}{t} \right]$$

### A.3 $\Upsilon_b$ function

Definition:  $\Upsilon_b(x)^{-1} \equiv \Gamma_b(x)\Gamma_b(Q-x)$

Functional relations:

$$\Upsilon_b(x+b) = \frac{\Gamma(bx)}{\Gamma(1-bx)} b^{1-2bx} \Upsilon_b(x)$$

$$\Upsilon_b(x+1/b) = \frac{\Gamma(x/b)}{\Gamma(1-x/b)} b^{2x/b-1} \Upsilon_b(x)$$

$\Upsilon_b(x)$  is an entire function of  $x$  which zeros are located at  $x = -nb - mb^{-1}$  and  $x = Q + nb + mb^{-1}$ ,  $n, m \in \mathbb{N}$ .

Integral representation convergent in the strip  $0 < \text{Re}x < Q$

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2} - x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}} \right]$$

## B Useful properties of the fusion coefficients

### B.1 Some limiting cases of the fusion coefficients

In this appendix we shall consider two important limiting cases of the fusion coefficients. We are going to show:

i) If  $\alpha_3 = \frac{Q}{2} + iP_3$ ,  $\alpha_t = \frac{Q}{2} + iP_t$  then

$$\lim_{\alpha_2 \rightarrow 0} F_{\alpha_1 \alpha_t} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} = \delta(P_t - P_3). \quad (36)$$

ii) Introduce  $\tilde{C}(\alpha_3, \alpha_2, \alpha_1) \equiv (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{b}} (\sum_{i=1}^3 \alpha_i - Q) \Upsilon_0^{-1} C(\alpha_3, \alpha_2, \alpha_1)$ . Then:

$$\lim_{\alpha_s \rightarrow 0} F_{\alpha_s \alpha_t} \begin{bmatrix} \alpha_2 & \alpha_1 \\ \alpha_2 & \alpha_1 \end{bmatrix} = \frac{\Gamma_b(2Q)}{\Gamma_b(Q)} \frac{S_b(2\alpha_t)}{S_b(2\alpha_t - Q)} \tilde{C}(\alpha_3, \alpha_2, \alpha_1). \quad (37)$$

To prove i), we will study the distribution on  $\mathcal{S}'(\mathbb{R} \times \mathbb{R})$  defined as

$$\begin{aligned} I_{\sigma_3, \sigma_1}(p_3, p_2) &\equiv \\ &\equiv \lim_{\beta_1 \rightarrow 0} \frac{1}{i} \int_{-i\infty}^{i\infty} ds \frac{S_b(U_1 + s)S_b(U_2 + s)S_b(U_3 + s)S_b(U_4 + s)}{S_b(V_1 + s)S_b(V_2 + s)S_b(V_3 + s)S_b(Q + s)} \Bigg|_{\beta_j = \frac{Q}{2} + ip_j; j=2,3}^{\sigma_1 = \sigma_2} \end{aligned}$$

It should be remarked that in sending  $\beta_1 \rightarrow 0$  some of the poles at  $V_1 + s = Q + nb$  and  $V_2 + s = Q + nb$  will cross the imaginary axis so that one has to deform the contour accordingly. If one first considers  $I_{\sigma_3, \sigma_1}(p_3, p_1)$  for  $p_1 \neq p_3$  one finds by changing the integration variable to  $t = \sigma_1 - \sigma_3 + \beta_3 + s$  that the integral simplifies to a special value of the  $b$ -hypergeometric function:

$$\begin{aligned} &\int_{-i\infty}^{i\infty} dt \frac{S_b(\beta_2 - \beta_3 + t)S_b(Q - \beta_2 - \beta_3 + t)}{S_b(2Q - 2\beta_3 + t)S_b(Q + t)} = \\ &= \frac{S_b(\beta_2 - \beta_3)S_b(Q - \beta_2 - \beta_3)}{S_b(2Q - 2\beta_3)} F_b(\beta_2 - \beta_3, Q - \beta_2 - \beta_3; 2Q - 2\beta_3; 0). \end{aligned}$$

This particular value of the  $b$ -hypergeometric function vanishes, as follows from the identity [12]

$$F_b(\alpha, \beta; \gamma; \frac{1}{2}(\gamma - \beta - \alpha - Q)) = e^{-2\pi i \alpha \beta} \frac{G_b(\gamma)G_b(\gamma - \alpha - \beta)}{G_b(\gamma - \alpha)G_b(\gamma - \beta)}$$

and the fact that  $G_b(\gamma - \alpha - \beta)$  has a zero for  $\gamma - \alpha - \beta = Q$ . One has thereby found that  $I_{\sigma_3, \sigma_1}(p_3, p_2)$  has support only for  $\beta_3 = \beta_2$ . In order to analyze the singular behavior near  $\beta_3 = \beta_2$  it will be useful to split off the residue contributions of the first poles that have crossed the real axis:

$$I_{\sigma_3, \sigma_1}(p_3, p_2) = I'_{\sigma_3, \sigma_1}(p_3, p_2) - \lim_{\epsilon \rightarrow 0} \left( \frac{S_b(\beta_3 + \beta_2 - Q)S_b(\beta_3 - \beta_2 - \epsilon)}{S_b(2\beta_3)} + \frac{S_b(\beta_2 - \beta_3 - \epsilon)S_b(Q - \beta_2 - \beta_3)}{S_b(2Q - 2\beta_3)} \right),$$

where  $I'_{\sigma_3, \sigma_1}(p_3, p_2)$  is defined by a contour that passes to the right of the poles at  $V_1 + s = Q$  and  $V_2 + s = Q$ . One observes that  $I'_{\sigma_3, \sigma_1}(p_3, p_2)$  is nonsingular at  $\beta_3 = \beta_2$ , and that  $S_b(x) \sim \frac{1}{2\pi x}$  near  $x = 0$ . The singular behavior near  $\beta_3 = \beta_2$  is therefore given by

$$-\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \left( \frac{1}{i(p_3 - p_2) - \epsilon} + \frac{1}{i(p_2 - p_3) - \epsilon} \right) = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \frac{2\epsilon}{(p_3 - p_2)^2 + \epsilon^2} = \delta(p_3 - p_2)$$

We have therefore found that  $I_{\sigma_3, \sigma_2}(p_3, p_2) = |S_b(2\beta_3)|^{-2} \delta(p_3 - p_2)$ . Our claim i) is an easy consequence of this fact.

In order to verify ii), one should observe that the prefactor of the integral in the expression for the fusion coefficients vanishes. However, the contour of integration gets pinched between the poles from the factors  $S_b(s + \alpha_s)$  and  $S_b^{-1}(s + Q)$  of the integrand in taking the limit. To isolate the singular contribution of the integral, one may deform the contour  $i\mathbb{R}$  into a contour that goes around the pole at  $s = 0$  in the right half plane plus a small circle around  $s = 0$ . Due to the vanishing prefactor, only the residue contribution survives in the limit. The rest is straightforward.

## B.2 Symmetries of the fusion coefficients

The fusion coefficients satisfy two types of symmetry relations: First, one may permute pairs of the variables  $\alpha_1, \dots, \alpha_4$ :

$$F_{\alpha_s \alpha_t} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} = F_{\alpha_s \alpha_t} \begin{bmatrix} \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{bmatrix} = F_{\alpha_s \alpha_t} \begin{bmatrix} \alpha_1 & \alpha_4 \\ \alpha_2 & \alpha_3 \end{bmatrix}. \quad (38)$$

These identities follow from similar identities for the b-Racah Wigner symbols:

$$\left\{ \begin{array}{cc|c} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t \end{array} \right\}_b = \left\{ \begin{array}{cc|c} \alpha_3 & Q - \alpha_4 & Q - \alpha_s \\ \alpha_1 & Q - \alpha_2 & Q - \alpha_t \end{array} \right\}_b \\ \left\{ \begin{array}{cc|c} \alpha_2 & \alpha_1 & \alpha_s \\ Q - \alpha_4 & Q - \alpha_3 & Q - \alpha_t \end{array} \right\}_b, \quad (39)$$

which are easily derived from the definition of the b-Racah Wigner symbols given in [12] taking into account the following properties of the b-Clebsch-Gordan coefficients:

$$\left( \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_b \right)^* = e^{-\pi i \alpha_2^* (Q - \alpha_2^*)} \begin{bmatrix} Q - \alpha_1^* & \alpha_2^* & Q - \alpha_3^* \\ x_1 - i\frac{Q}{2} & x_2 & x_3 - i\frac{Q}{2} \end{bmatrix}_b \\ = e^{+\pi i (\alpha_3^* (Q - \alpha_3^*) - \alpha_2^* (Q - \alpha_2^*) - \alpha_1^* (Q - \alpha_1^*))} \begin{bmatrix} \alpha_3^* & \alpha_1^* & \alpha_2^* \\ x_3 & x_2 & x_1 \end{bmatrix}_b. \quad (40)$$

Second, there are identities that exchange the two ‘‘internal indices’’ with a pair of ‘‘external indices’’

$$\begin{aligned} F_{\alpha_t \alpha_s} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 \end{bmatrix} F_{0\alpha_1} \begin{bmatrix} \alpha_4 & \alpha_t \\ \alpha_4 & \alpha_t \end{bmatrix} &= F_{\alpha_3 \alpha_1} \begin{bmatrix} \alpha_s & \alpha_2 \\ \alpha_4 & \alpha_t \end{bmatrix} F_{0\alpha_s} \begin{bmatrix} \alpha_4 & \alpha_3 \\ \alpha_4 & \alpha_3 \end{bmatrix}, \\ F_{\alpha_s \alpha_t} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} F_{\alpha_1 0} \begin{bmatrix} \alpha_t & \alpha_t \\ \alpha_4 & \alpha_4 \end{bmatrix} &= F_{\alpha_1 \alpha_3} \begin{bmatrix} \alpha_t & \alpha_2 \\ \alpha_4 & \alpha_s \end{bmatrix} F_{\alpha_s 0} \begin{bmatrix} \alpha_3 & \alpha_3 \\ \alpha_4 & \alpha_4 \end{bmatrix}. \end{aligned} \quad (41)$$

The first of these identities is obtained from the pentagon (4) by setting  $\beta_1 = \alpha_3$  and taking the limit  $\beta_2 \rightarrow 0$  with the help of (36). The second can be obtained from the first by taking into account

$$\begin{aligned} C(\alpha_4, \alpha_3, \alpha_s) C(Q - \alpha_s, \alpha_2, \alpha_1) F_{\alpha_s \alpha_t} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} &= \\ &= C(\alpha_4, \alpha_t, \alpha_1) C(Q - \alpha_t, \alpha_3, \alpha_2) F_{\alpha_t \alpha_s} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 \end{bmatrix}. \end{aligned} \quad (42)$$

This identity in turn follows via standard Moore-Seiberg type arguments [8] from the fundamental identity that assures crossing symmetry [11] [12], together with  $(\alpha = \frac{Q}{2} + iP, \alpha' = \frac{Q}{2} + iP')$

$$\int_{\mathbb{S}} d\beta F_{\alpha\beta} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} F_{\beta\alpha'} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 \end{bmatrix} = \delta(P - P'). \quad (43)$$

### B.3 Some residues of the fusion coefficients

If one considers the special cases where one of  $\alpha_1, \dots, \alpha_4$ , say  $\alpha_i$  equals  $-\frac{n}{2}b - \frac{m}{2}b^{-1}$  and where a triple  $(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{\alpha_{21}})$ ,  $(\Delta_{\alpha_{21}}, \Delta_{\alpha_2}, \Delta_{\alpha_1})$  which contains  $\Delta_{\alpha_i}$  satisfies the fusion rules of [FF], one will find that the right hand side of the fusion relation (8) reduces to a finite sum of terms selected by the fusion rules of [FF]. The fusion coefficients that multiply the conformal blocks are residues of the general fusion coefficients (9), as can be seen by a generalization of our calculation leading to (36). In order to derive explicit expressions for these residues, it is useful to observe that the pentagon equation (4) leads to recursion relations that determines the above-mentioned residues in terms of the following special case:

$$F_{s,s'}(\beta|\sigma_1, \sigma_2) \equiv F_{\sigma_1 - \frac{sb}{2}, \beta - \frac{s'b}{2}} \begin{bmatrix} \beta & -\frac{b}{2} \\ \sigma_2 & \sigma_1 \end{bmatrix},$$

where  $s, s' = \pm$ . The explicit expressions for these coefficients are:

$$\begin{aligned} F_{++} &= \frac{\Gamma(b(2\sigma_1 - b))\Gamma(b(b - 2\beta) + 1)}{\Gamma(b(\sigma_1 - \beta - \sigma_2 + b/2) + 1)\Gamma(b(\sigma_1 - \beta + \sigma_2 - b/2))} \\ F_{+-} &= \frac{\Gamma(b(2\sigma_1 - b))\Gamma(b(2\beta - b) - 1)}{\Gamma(b(\sigma_1 + \beta + \sigma_2 - 3b/2) - 1)\Gamma(b(\sigma_1 + \beta - \sigma_2 - b/2))} \\ F_{-+} &= \frac{\Gamma(2 - b(2\sigma_1 - b))\Gamma(b(b - 2\beta) + 1)}{\Gamma(2 - b(\sigma_1 + \beta + \sigma_2 - 3b/2))\Gamma(1 - b(\sigma_1 + \beta - \sigma_2 - b/2))} \\ F_{--} &= \frac{\Gamma(2 - b(2\sigma_1 - b))\Gamma(b(2\beta - b) - 1)}{\Gamma(b(-\sigma_1 + \beta + \sigma_2 - b/2))\Gamma(b(-\sigma_1 + \beta - \sigma_2 + b/2) + 1)} \end{aligned}$$

In subsection 3.2 we need the following fusion coefficients:

$$F_{\sigma_1, \beta_2 \pm b} \begin{bmatrix} \beta_2 & -b \\ \sigma_3 & \sigma_1 \end{bmatrix} = F_{\sigma_1, \beta_2 \pm b} \begin{bmatrix} -b & \beta_2 \\ \sigma_1 & \sigma_3 \end{bmatrix}. \quad (44)$$

The pentagon identity (4) then yields the formula that was used to calculate the expressions used in subsection 3.2:

$$F_{\sigma_2, \beta + sb} \begin{bmatrix} \beta & -b \\ \sigma_2 & \sigma_1 \end{bmatrix} = \sum_{t=\pm} \frac{F_{t+}(-\frac{b}{2}|\beta, \beta + s)}{F_{++}(-\frac{b}{2}|\sigma_2, \sigma_2)} F_{-t}(\beta|\sigma_2 - \frac{b}{2}, \sigma_1) F_{+, s-t}(\beta - \frac{tb}{2}|\sigma_2, \sigma_1). \quad (45)$$

## C Uniqueness of solutions of finite difference equations of the second order

Let us indicate how one may obtain statements on the uniqueness of such equations: We will consider functions  $f(x)$  that are analytic in some domain  $D$  that includes  $i[0, 2b^{-1}]$  and satisfy

$$(A_b(x)T^{2b} + B_b(x)T^b + C_b(x))f(x) = 0, \quad (46)$$

where  $T^b$  is the operator defined by  $T^b f(x) = f(x + b)$ , as well as the difference equation obtained by replacing  $b \rightarrow b^{-1}$ . We would like to show that there exist at most two linearly independent solutions. Assume having three solutions  $f_1, f_2, g$  of which  $f_1$  and  $f_2$  are linearly independent. One may consider

$$\det \begin{vmatrix} g & f_1 & f_2 \\ T^b g & T^b f_1 & T^b f_2 \\ T^{2b} g & T^{2b} f_1 & T^{2b} f_2 \end{vmatrix}. \quad (47)$$

The determinant vanishes due to the fact that each row is a linear combination of the two other by means of the difference equation (46). But this implies that also the columns must be linearly dependent, in particular

$$g = c_1(x)f_1 + c_2(x)f_2, \quad (48)$$

with coefficients  $c_1, c_2$  that might a priori depend on  $x$ . These coefficients are found as

$$c_1 = \frac{\mathcal{W}(g, f_2)}{\mathcal{W}(f_1, f_2)}, \quad c_2 = \frac{\mathcal{W}(g, f_1)}{\mathcal{W}(f_2, f_1)}, \quad \mathcal{W}(f, g) \equiv fT^b g - gT^b f. \quad (49)$$

$\mathcal{W}(f, g)$  can be seen as a q-analogue of the Wronskian relevant for second order it differential equations. By a direct calculation using (46) one finds that

$$T^b \mathcal{W}(f, g) = \frac{C_b(x)}{A_b(x)} \mathcal{W}(f, g) \quad \text{and} \quad T^b c_i = c_i, \quad i = 1, 2. \quad (50)$$

In a similar way one obtains  $T^{1/b} c_i = c_i, i = 1, 2$ . It then follows for irrational  $b$  that the  $c_i, i = 1, 2$  must be constant: From  $T^b c_i = c_i$  one finds periodicity of  $d_i(x) \equiv c_i(-ix)$  in the interval  $[0, 2b]$ , so that  $c_i$  can be represented by a Fourier-series. One may then use the equation  $T^{1/b} c_i = c_i$  to show vanishing of all Fourier-coefficients but the one of the constant mode.

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