A note on the not 3-choosability of some families of planar graphs

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Abstract

A graph $G$ is $L$-list colorable if for a given list assignment $L = \{L(v) : v \in V\}$, there exists a proper coloring $c$ of $G$ such that $c(v) \in L(v)$ for all $v \in V$. If $G$ is $L$-list colorable for any list assignment with $|L(v)| \geq k$ for all $v \in V$, then $G$ is said $k$-choosable. In [M. Voigt, A not 3-choosable planar graph without 3-cycles, Discrete Math. 146 (1995) 325–328] and [M. Voigt, A non-3-choosable planar graph without cycles of length 4 and 5, 2003, Manuscript], Voigt gave a planar graph without 3-cycles and a planar graph without 4-cycles and 5-cycles which are not 3-choosable. In this note, we give smaller and easier graphs than those proposed by Voigt and suggest an extension of Erdős’ relaxation of Steinberg’s conjecture to 3-choosability.

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1. Introduction

Let $G$ be a graph. Let $V(G)$ be its set of vertices and $E(G)$ be its set of edges. A proper vertex coloring of $G$ is an assignment $c$ of integers (or labels) to the vertices of $G$ such that $c(u) \neq c(v)$ if the vertices $u$ and $v$ are adjacent in $G$. A graph $G$ is $L$-list colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring $c$ of the vertices such that $\forall v \in V(G), c(v) \in L(v)$. If $G$ is $L$-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is said $k$-choosable.

In [4], Grötzsch proved that every planar graph without 3-cycles is 3-colorable and in 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [5]). In [7] and [8], Voigt proved that we cannot extend this result and this conjecture to list coloring: She gave a planar graph without 3-cycles having 166 vertices and a planar graph without 4-cycles and 5-cycles having 344 vertices which are not 3-choosable.

In this note we present smaller and easier graphs than those proposed by Voigt: We give a planar graph without 3-cycles having 128 vertices and a planar graph without 4-cycles and 5-cycles having 209 vertices which are not 3-choosable. Moreover we pose the problem of the sufficient conditions of 3-choosability of planar graphs.

2. A not 3-choosable planar graph without 3-cycles

Let $G(u, v)$ be the graph depicted by Fig. 1. Given the color $a$ and $b$ of the vertices $u$ and $v$ ($a \neq b$) and the list assignment $L_{a, b}$ given by Fig. 2, we cannot choose
for each vertex a color from its list such that the coloring obtained is proper: For the vertices $x_1$ and $x_4$, we have only one choice; we must color these vertices with the color 1 (since these two vertices are adjacent to $u$ and $v$ colored by $a$ and $b$). Now for the vertices $x_2$ and $x_3$, there are two cases: Either we color $x_2$ with 2 and $x_3$ with 3 or we color $x_2$ with 3 and $x_3$ with 2. In the first case (resp. second case), observe that we must then color the 5-cycle $y_1 y_2 y_3 y_4 y_5 y_1$ (resp. $z_1 z_2 z_3 z_4 z_5 z_1$) with the two colors 4 and 5 which is impossible.

Now, we construct the graph $G^*$ as follows: For each pair $(a, b) \in \{6, 7, 8\} \times \{9, 10, 11\}$, let $G(u_i, v_j)$ ($1 \leq i \leq 9$) be a copy of $G(u, v)$ with the list assignment $L_{a,b}$. So, we have nine copies of $G(u, v)$: $G(u_1, v_1)$ with the list assignment $L_{6,9}$, $G(u_2, v_2)$ with the list assignment $L_{6,10}$, $G(u_3, v_3)$ with the list assignment $L_{6,11}$, and so on. We then identify all the vertices $u_i$ (resp. $v_j$), $1 \leq i \leq 9$ to a vertex $u^*$ (resp. $v^*$).

We assign the list $\{6, 7, 8\}$ to the vertex $u^*$ and the list $\{9, 10, 11\}$ to the vertex $v^*$. Now, for any coloring of the vertices $u^*$ and $v^*$ with the colors $c_1$ and $c_2$, there exists a copy $G(u_j, v_j)$ with the list assignment $L_{c_1, c_2}$ which we cannot color. The graph $G^*$ contains $14 \times 9 + 2 = 128$ vertices and does not contain any 3-cycles.

3. A not 3-choosable planar graph without 4- and 5-cycles

By the same way, let $H(u, v)$ be the graph depicted by Fig. 3 and $L_{a,b}$ its list assignment given by Fig. 4. Given the colors $a$ and $b$ of the vertices $u$ and $v$, we cannot proper color the vertices of $H(u, v)$ with a color from their list: Given the colors $a$ and $b$ of the vertices $u$ and $v$, the vertices $x_i$, $y_i$ for $i = 1, 2$ must be colored with 2, 3 and so $z_i$ with 1. Now there are two cases: Either we color $z$ with 2 and $t$ with 3 or we color $z$ with 3 and $t$ with 2. In the first case (resp. second case), it is easy to see that we cannot color the 3-cycle $u_1 u_2 u_3$ (resp. $v_1 v_2 v_3$): We have only the two colors 4 and 5.

We construct the graph $H^*$ as follows: For each pair $(a, b) \in \{6, 7, 8\} \times \{9, 10, 11\}$, let $H(u_i, v_j)$ ($1 \leq i \leq 9$) be a copy of $H(u, v)$ with the list assignment $L_{a,b}$. So, we have nine copies of $H(u, v)$: $H(u_1, v_1)$ with the list assignment $L_{6,9}$, $H(u_2, v_2)$ with the list assignment $L_{6,10}$, $H(u_3, v_3)$ with the list assignment $L_{6,11}$, and so on. We then identify all the vertices $u_i$ (resp. $v_j$), $1 \leq i \leq 9$ to a vertex $u^*$ (resp. $v^*$). We assign the list $\{6, 7, 8\}$ to the vertex $u^*$ and the list $\{9, 10, 11\}$ to the vertex $v^*$. Now, for any coloring of the vertices $u^*$
and \(v^*\) with the colors \(c_1\) and \(c_2\), there exists a copy \(H(uj, vj)\) with the list assignment \(L_{c_1, c_2}\) which we cannot color. The graph \(H^*\) contains \(23 \times 9 + 2 = 209\) vertices and does not contain any 4- and 5-cycles.

4. Concluding remarks

In [6], Thomassen proved that every planar graph with girth 5 is 3-choosable. He gives a partial answer of the following problem:

**Problem 1.** What are the sufficient conditions for a planar graph to be 3-choosable?

In 1990, Erdös suggested the following relaxation of Steinberg’s conjecture: Which is the smallest integer \(i\) such that every graph without \(j\)-cycles for \(4 \leq j \leq i\) is 3-colorable. The best known result is \(i = 7\) [2]. The following problem is an extension of the relaxation of Erdös.

**Problem 2.** Which is the smallest integer \(i\) such that every graph without \(j\)-cycles for \(4 \leq j \leq i\) is 3-choosable?

Since there exists not 3-choosable planar graphs without 4- and 5-cycles (see Section 3), then \(i \geq 6\). Moreover we know that:

**Observation 1.** Every planar graph without \(j\)-cycles, \(4 \leq j \leq 9\), is 3-choosable.

And so, \(i \leq 9\). This observation is based on the following structural theorem due to Borodin [1]:

**Theorem 1.** ([1]) Let \(G\) be a plane graph without two triangles sharing an edge. Then the following statements are valid in which all numerical parameters are best possible:

1. \(\delta(G) \leq 4\) (where \(\delta(G)\) is the minimum degree of \(G\));
2. if \(\delta(G) \geq 3\), then there are adjacent vertices \(x, y\) such that \(d(x) + d(y) \leq 9\);
3. if \(\delta(G) \geq 3\), then there is either an \(i\)-face where \(4 \leq i \leq 9\) or 10-face incident with ten vertices of degree 3 and adjacent to five triangles.

**Proof of Observation 1.** Let \(H\) be a counterexample with the minimum order and \(L\) a list assignment such that there does not exist a proper coloring \(c\) such that \(\forall v \in V(H), c(v) \in L(v)\). Clearly, \(\delta(H) \geq 3\) and by Theorem 1.3, \(H\) has a 10-face \(f\) with all incident vertices of degree 3. By minimality of \(H\), the graph \(H'\) obtained from \(H\) by removing all the vertices incident to \(f\) is 3-choosable and so there exists a proper coloring \(c\) of \(H'\) such that \(\forall v \in V(H'), c(v) \in L(v)\). Now we can extend \(c\) to the whole graph \(H\). For each vertex incident to \(f\), we have two available colors and since even cycles are 2-choosable, we can extend \(c\) to \(H\). \(\square\)

Recently in [9], Zhang and Wu proved that:

**Theorem 2.** ([9]) Every planar graph without cycles of length 4, 5, 6, or 9 is 3-choosable.

We conjecture:

**Conjecture 1.** Every planar graph without cycles of length 4, 5, 6, is 3-choosable.

Let \(n_1\) (resp. \(n_2\)) be the minimum number of vertices of a planar graph without 3-cycles (resp. without 4- and 5-cycles) which is not 3-choosable. By our examples, we have \(n_1 \leq 128\) and \(n_2 \leq 209\). However, the first result is not the best known: In [3], Glebov et al. give the smallest known not 3-choosable planar graph without 3-cycles; their example contains 97 vertices. And so, \(n_1 \leq 97\).

**Problem 3.** What are the exact values of \(n_1\) and \(n_2\)?

Fig. 4. The list assignment \(L_{a,b}\) of the graph \(H(u,v)\).
References


