On Discerning Controllers for Switching Linear Systems

Marco Baglietto, Giorgio Battistelli and Pietro Tesi

Abstract—This paper presents recent developments in the study of non-autonomous switching linear systems. For such systems, we address the issue of how to systematically design linear controllers allowing the active process mode to be observable from closed-loop data. A parametrization of the set of such discerning controllers is given and compatibility with stabilization objectives investigated. It is finally shown how a given family of discerning controllers can be implemented as a single hybrid controller which preserves the discerning capability of the original controllers.

I. INTRODUCTION

Recent years have seen intensive research into dynamic systems that are characterized by the interaction of both continuous and discrete dynamics, commonly referred to as hybrid systems [1]. Switching linear systems represent a special class of hybrid systems, namely those systems composed of several linear subsystems (modes) and a switching signal that specifies the active subsystem at each instant of time. In many contexts, when the switching signal is not available for measurements, it may be of interest to estimate the current active subsystem using the available input/output data. A natural question therefore arises about the possibility of achieving exact mode identification. Such a problem is usually referred in the literature as the mode observability problem [2]-[6].

One of the main motivations for studying mode observability is control reconfiguration. In fact, many physical systems can be represented by switching between locally valid configurations, where such different configurations can be introduced to model multiple operating conditions or possible sensor, actuator, and plant faults (see [1],[7]-[9]). To cope with these situations, a viable solution is to resort to switching control by: (i) designing a family of controllers so that, in each configuration, the process performs satisfactorily when controlled by at least one of the candidate controllers; (ii) and then monitoring in real time the process input/output data so as to determine the controller to be applied at each instant of time. It is important to point out that, in most of the control architectures of the type described above, controller selection is based on the idea of certainty equivalence, which amounts to applying at each instant of time the controller designed for the model that best fits the available data [8], [10]. Mode-observability has therefore clear implications on certainty-equivalence-based control, since it addresses the question of whether or not a certain process mode can be identified from the measured data. In particular, it was recently shown in [11], [12] that, by resorting to mode-observability considerations, it is possible to construct suitable control reconfiguration mechanisms capable of ensuring input-to-state stability under slow-on-the-average process mode switching.

In the literature, mode-observability has been widely studied in the autonomous system case and the reader is referred to [13] for an overview of the subject. On the other hand, the study of mode-observability for non-autonomous systems still poses many challenges. In this context, most of the existing studies [4], [6], [14], [15] have been devoted to establish connections between input selection and mode observability, in close analogy with the problem of input selection for parameter identification [16], by introducing the notions of controlled-discernibility and discerning controls. While such results provide a useful theoretical background, their practical applicability is still limited by the fact that they deal with the special case where mode observability is the unique control objective. In practice, however, a control law not only has to satisfy other control objectives besides mode observability (most notably stabilization) but also it may be required to fulfill additional constraints (for example only controllers with specific structures like PID might be of interest).

The issue of mode-observability for feedback control architectures has been considered in [11] for the case where the control action is generated by linear time-invariant controllers. In particular, [11] shows that, under mild conditions, the set of controllers ensuring mode observability (i.e., the set of discerning controllers) is generic. Building on such a result, the present paper aims at addressing in a systematic way the issue of mode-observability for feedback control architectures in the case of SISO plants. More specifically, the main contributions are as follows: (i) a parametrization of the set of discerning controllers is given together with an explicit geometric characterization; (ii) constraints in the controller structure are taken into account and connections with stabilization objectives investigated; (iii) it is shown how a given family of discerning controllers can be implemented as a single hybrid controller which preserves the discerning capability of the original controllers.

The remainder of the paper is organized as follows. In Section II, the problem set up is described and the notion of discerning controllers introduced. In Section III, conditions for the existence of discerning controllers are given and a
parametrization of the set of discerning controllers is derived. Section IV deals with the problem of realizing a hybrid controller with guaranteed mode observability properties. Finally, Section V concludes the paper.

II. PROBLEM FORMULATION

Consider the feedback configuration shown in Figure 1. The process to be controlled is modeled by a switching linear system

\[
\begin{aligned}
\dot{x} &= Ax + bu \\
y &= cx
\end{aligned}
\]  

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R} \) is the input, \( y \in \mathbb{R} \) is the output and \( \rho : \mathbb{R}_{\geq 0} \mapsto \{1, 2, \ldots, N\} \) is the switching signal, i.e. the signal (right continuous) which identifies the index of the active system at each instant of time, assumed to be unknown. \( A_i, b_i, \) and \( c_i, i \in \{1, 2, \ldots, N\} \), are constant matrices of appropriate dimensions. In the sequel, we shall denote by \( P_i \) the linear time-invariant (LTI) system with state-space representation \( \{A_i, b_i, c_i\} \).

The control action is realized by means of a finite family \( \mathscr{C} = \{C_j, j = 1, 2, \ldots, M\} \) of LTI controllers, where each controller generates a candidate feedback signal \( u_j \). The control signal applied to the process at each instant of time is then \( u = u_\sigma \), where \( \sigma : \mathbb{R}_{\geq 0} \mapsto \{1, 2, \ldots, M\} \) is the controller switching signal. Hereafter, we let

\[
\begin{aligned}
\dot{q}_j &= F_j q_j + g_j y \\
u_j &= h_j q_j + k_j y
\end{aligned}
\]

be a state-space representation of \( C_j \) where \( q_j \in \mathbb{R}^{n_{q_j}} \) is the state; \( F_j, g_j, h_j, \) and \( k_j \) are constant matrices of appropriate dimensions.

The problem of interest is to design \( \mathscr{C} \) so as to satisfy the following control objectives:

- **O1. stabilization**, i.e., for each \( i \in \{1, 2, \ldots, N\} \), there must exist at least one candidate controller \( C_j \) able to internally stabilize \( P_i \);
- **O2. global discernability**.

Objective **O1** is the minimum requirement in order for \( \mathscr{C} \) to contain at least one stabilizing controller for any possible process mode. In this respect, stabilizability of \( (A_i, b_i) \) and detectability of \( (c_i, A_i) \), \( i \in \{1, 2, \ldots, N\} \), are trivial necessary and sufficient conditions for the solvability of **O1** and need not be discussed further.

Objective **O2** is described hereafter. In this respect, some preliminary definitions are needed. Define

\[
\Phi_{i/j} = \begin{bmatrix} A_i + b_i k_j c_i & b_i h_j \\ g_j c_i & F_j \end{bmatrix}, \quad \Delta_{i/j} = \begin{bmatrix} k_i c_i & h_j \\ c_i & 0 \end{bmatrix}
\]

and let \( z_{i/j}(\cdot, t_0, x_{i,0}, q_{j,0}) = \Delta_{i/j} e^{\Phi_{i/j}(t-t_0)} \psi(x_{i,0}, q_{j,0}) \). As can be recognized, the latter is the value at time \( t \) of the input/output data of \( P_i \) when fed-back by \( C_j \), the state of \( P_i \) and \( C_j \) at time \( t_0 \) is \( x_{i,0} \) and \( q_{j,0} \), respectively. With these definitions in place, the following notion of discernibility can be introduced.

**Definition 1**: Consider a process as in (1). A feedback controller \( C_j \) as in (2) is said to be discerning if, for all pairs \((i, \ell)\) of different process modes, the following property holds:

\[
z_{i/j}(\cdot, t_0, x_{i,0}, q_{j,0}) - z_{\ell/j}(\cdot, t_0, x_{\ell,0}, q_{j,0}) \neq 0 \tag{3}
\]

a.e. on \( I \) for all \( I := [t_0, t_0+T) \) with \( T > 0 \) and all nonzero quadruples of vectors \((x_{i,0}, q_{j,0}, q_{j,0}', x_{\ell,0}')\), where by “a.e.” we mean “almost everywhere”, i.e. everywhere except on a set with zero Lebesgue measure. A family \( \mathscr{C} \) of feedback controllers as in (2) is said to be **globally discerning** if every element of \( \mathscr{C} \) is discerning.

We can interpret Definition 1 as follows. Let \( C_j \) be switched-on in feedback with the process over an interval \( I = [t_0, t] \) with \( t > t_0 \), i.e. \( u(\tau) = u_j(\tau) \) for all \( \tau \in I \). Suppose now that over the same time interval the process mode is constant, i.e. \( \rho(\tau) = i \) for all \( \tau \in I \), for some \( i \in \{1, 2, \ldots, N\} \). Then

\[
z(\tau) = z_{i/j}(\tau, t_0, x_{i,0}, q_{j,0}) \quad \text{for all } \tau \in I \tag{4}
\]

where \( x_{i,0} = x(t_0) \) and \( q_{j,0} = q_{j}(t_0) \). The reason for requiring \( C_j \) to be discerning is therefore intuitively clear: under discernibility, when the switching signal \( \rho \) is constant, there is only one process mode consistent with the observed data, i.e. there is only one index \( i \) satisfying (4) while any other index \( \ell \) cannot be consistent with the observed data in accordance with (3). Discernibility has therefore clear implications on certainty-equivalence-based control, which amounts to applying at each instant of time the controller designed for the model that best fits the available data \([8, 10]\). In fact, it opens up the possibility to identify (exactly) the active process mode by suitably processing \( z \).

**Remark 1**: In the literature on switching systems, the problem of identifying the active process mode from data is usually referred to as the “mode-observability” problem \([2, 6]\). Definition 1 follows the same lines, by extending the notion of mode-observability to the presence of the
feedback interconnection. The term “discernibility”, as originally introduced in [6], is usually adopted so as to indicate that the mode-observability problem is addressed in a non-autonomous case where the interest is to characterize controls allowing process mode identification.

III. MAIN RESULTS

Assuming that \( O_1 \) is solvable, the interest is to find conditions under which \( O_1 \) and \( O_2 \) are jointly solvable, in particular, which conditions the process and the controllers must obey for \( \mathcal{G} \) to satisfy the desired set of specifications.

The notation for this section is in the main standard. We let \( \mathbb{R}[s] \) denote the ring of polynomials with reals coefficients. Given \( a(s), b(s) \in \mathbb{R}[s] \) we denote by \( \gcd\{a(s), b(s)\} \) the greatest common divisor of \( a(s) \) and \( b(s) \). To avoid heavy notation, given a signal \( u \) and a polynomial \( a(s) \), by the notation \( a(s)u \) we mean the action of the differential operator polynomial \( a(s)|_{s=d/dt} \) on \( u \) [17].

A. Preliminary considerations

For the sake of analysis, it is convenient to recast discernibility in terms of observability of a suitable interconnected system. More specifically, the following result descend directly from Definition 1.

**Proposition 1:** A controller \( C_j \) is discerning if and only if, for all pairs \( (i, \ell) \) of different process modes, the interconnected system of Figure 2 is observable from the output

\[
z_{i/j} - z_{\ell/j} = \begin{bmatrix} u_{i/j} - u_{\ell/j} \\ y_{i/j} - y_{\ell/j} \end{bmatrix}.
\]

In view of Proposition 1, it is immediate to see that the following are necessary conditions for \( \mathcal{G} \) to be globally discerning.

A1. The pair \((A_i, c_i)\) is observable, \( i \in \{1, 2, \ldots, N\} \).

D1. The pair \((F_j, h_j)\) is observable, \( j \in \{1, 2, \ldots, M\} \).

Notice that throughout the paper we use the letter “A” to refer to assumptions that have to be satisfied by the process and the letter “D” to refer to design conditions that must be satisfied by the controller. Supposing that both conditions A1 and D1 hold, we can equivalently represent each plant and controller mode by resorting to the following input/output descriptions

\[
U_i(s)y = Q_i(s)u, \quad i \in \{1, 2, \ldots, N\}, \quad (5)
\]

\[
R_j(s)u = S_j(s)y, \quad j \in \{1, 2, \ldots, M\}, \quad (6)
\]  

where \( A_i \) and \( F_j \) are such that \( \det(sI - A_i) = U_i(s) \) and \( \det(sI - F_j) = R_j(s) \), respectively.

**Remark 2:** Here and in the sequel, it is therefore understood that possible cancellations in the polynomials \( U_i(s) \) and \( Q_i(s) \) and/or in the polynomials \( R_j(s) \) and \( S_j(s) \) do only refer to the uncontrollable dynamics of \( \mathcal{P}_i \) and \( \mathcal{C}_j \), respectively.

We are now in position to derive necessary and sufficient conditions for discernibility.

**Theorem 1:** Let A1 holds. A controller \( C_j \) of the form (6) is discerning if and only if, for all pairs \((i, \ell)\) of different process modes, the following conditions hold:

A2. \( U_i(s)Q_\ell(s) \neq U_\ell(s)Q_i(s) \).

A3. \( \gcd\{U_i(s), U_\ell(s), Q_i(s), Q_\ell(s)\} = 1 \).

D2. \( \text{rank} \begin{bmatrix} -Q_i(s) & U_i(s) \\ -Q_\ell(s) & U_\ell(s) \\ -R_j(s) & S_j(s) \end{bmatrix} = 2 \) for any \( s \in \mathbb{C} \).

We now make some observations concerning Theorem 1. Assumption A2 amounts to saying that, for all pairs \((i, \ell)\) of different process modes, the (non-reduced) transfer functions of \( \mathcal{P}_i \) and \( \mathcal{P}_\ell \) are different. Assumption A3 requires that, for all pairs \((i, \ell)\) of different process modes, \( \mathcal{P}_i \) and \( \mathcal{P}_\ell \) do not have common uncontrollable dynamics. Consistent with the observations made in Remark 2, A3 does not rule out the possibility that for some \( i \in \{1, 2, \ldots, N\} \) the polynomials \( U_i(s) \) and \( Q_i(s) \) have common factors. As for D2, this condition seemingly depends also on the open-loop process features. Nonetheless, we will see in the next section that under A1-A3 a controller satisfying D2 always exists and, given an adequately chosen controller structure, a controller is discerning for almost all values of its parameters.
B. Admissible controller parametrizations

Consider a parametric family of controllers of the form (6) with

\[ R_j(s) = R_j(s, \rho_j) = \sum_{m=1}^{M_j} \rho_{jm} R_{jm}(s) + R_{j0}(s) \]  
(7a)

\[ S_j(s) = S_j(s, \sigma_j) = \sum_{n=1}^{N_j} \sigma_{jn} S_{jn}(s) + S_{j0}(s) \]  
(7b)

where \( \rho_j = \text{col}(\rho_{j1}, \ldots, \rho_{jM_j}) \) and \( \sigma_j = \text{col}(\sigma_{j1}, \ldots, \sigma_{jN_j}) \) are free design parameters, whereas \( R_{j0}(s) \), \( R_{jm}(s) \), \( S_{j0}(s) \), \( S_{jn}(s) \) are fixed polynomials with \( R_{j0}(s) \) monic, \( \text{deg} R_{jm}(s) < \text{deg} R_{j0}(s) \) for \( m = 1, \ldots, M_j \), and \( \text{deg} S_{jn}(s) \leq \text{deg} R_{j0}(s) \) for \( n = 0, \ldots, N_j \).

The above-defined parametrization can capture different class of controllers possibly defined with respect to specific performance goals. E.g., given a process mode \( i \), it may be desirable to cancel an open-loop zero of \( P_i \) located at \( s_0 \in \mathbb{C} \) so as to diminish overshoot in the closed-loop response. This is achieved with controller \( C_i \) by letting \( R_{jm} = s_0^{m_j} - s_0 \), \( m = 0, \ldots, M_j = m_j - 1 \), where \( m_j \) is a given controller order. The case where no specific controller structure is a priori imposed is obtained by letting

\[ R_{jm} = s_0^{m_j} - s_0, \quad m = 0, \ldots, M_j = m_j \]
\[ S_{j0} = 0, \quad S_{jn} = s_0^{m_j} - n, \quad n = 1, \ldots, N_j = m_j + 1 \]  
(8)

In connection with D2, a sensible preliminary question is whether the controller parametrization is allowed to be arbitrary. The answer to that question is in the negative, except in some cases. To see this, let us define the following sets

\[ \Lambda_{i,\ell} \triangleq \left\{ s \in \mathbb{C} : \text{rank} \begin{bmatrix} -Q_i(s) & U_i(s) \\ -Q_\ell(s) & U_\ell(s) \end{bmatrix} = 1 \right\} \]  
(9)

\[ \mathcal{V}_{i,\ell} \triangleq \{ a(s) \in \mathbb{R}[s] : a(s) = 0 \text{ for some } s \in \Lambda_{i,\ell} \} \]  
(10)

with \( i, \ell \in \{1, 2, \ldots, N\} \).

In words, \( \mathcal{V}_{i,\ell} \) is the set of polynomials with at least one zero located within \( \Lambda_{i,\ell} \). Notice that under A1 the set \( \Lambda_{i,\ell} \) is finite for all pairs of different process modes. It is then clear that D2 places constraints on the admissible controller parametrizations. In this respect, the set of admissible controller parametrizations is captured in the next lemma.

**Lemma 1**: Let assumptions A1-A3 hold. Consider a controller \( C_j \) of the form (6) with \( R_{jm}(s) = 0, \ldots, M_j \) and \( S_{jn}(s) = 0, \ldots, N_j \). Then condition D2 can be attained only if

1. \( \gcd \{ Q_i(s), Q_\ell(s), R_{jm}(s) \} \notin \mathcal{V}_{i,\ell} \) \hspace{1cm} (11a)
2. \( \gcd \{ U_i(s), U_\ell(s), S_{jn}(s) \} \notin \mathcal{V}_{i,\ell} \) \hspace{1cm} (11b)
3. \( \gcd \{ R_{jm}(s), S_{jn}(s) \} \notin \mathcal{V}_{i,\ell} \) \hspace{1cm} (11c)

Each condition in (11) admits a simple interpretation: conditions (11a) and (11b) amount to requiring that, at least for some choice of the parameters, the controller does not cancel any zero or, respectively, pole common to \( P_i \) and \( P_\ell \); condition (11c) simply means that pole-zero cancellations in the controller transfer function should not belong to \( \Lambda_{i,\ell} \).

Notice that such conditions are automatically satisfied when \( \Lambda_{i,\ell} = \emptyset \) and when no specific controller structure is a priori imposed, as in (8).

C. Existence and genericity of discerning controllers

Having determined the admissible controller parametrizations, one has to prove that under (11) there exist at least one vector \( \text{col}(\rho_j, \sigma_j) \) for which \( C_j \) is discerning. The result which follows actually proves something more, namely that under (11) condition D2 holds for “almost all” choices of the vector \( \text{col}(\rho_j, \sigma_j) \). To this end, it is convenient to introduce the following notion.

**Definition 2**: A subset \( \Theta \) of a topological space is said to be generic when the following two conditions hold: for any \( \theta \in \Theta \) there exists a neighborhood of \( \theta \) contained in \( \Theta \); for any \( \theta \notin \Theta \) every neighborhood of \( \theta \) contains an element of \( \Theta \). Here, by “neighborhood of \( \theta \)” we mean an open set \( \mathcal{V} \) containing \( \theta \).

**Lemma 2**: Let assumptions A1-A3 hold. Consider a controller \( C_j \) of the form (6) with \( R_{jm}(s) = 0, \ldots, M_j \) and \( S_{jn}(s) = 0, \ldots, N_j \), satisfying (11). Then the set \( \mathcal{V}_j \) of vectors \( \text{col}(\rho_j, \sigma_j) \in \mathbb{R}^{M_j+N_j} \) for which condition D2 holds is generic. In particular its complement \( \overline{\mathcal{V}}_j \) is at most a finite union of linear varieties in \( \mathbb{R}^{M_j+N_j} \).

As can be easily verified, the set \( \overline{\mathcal{V}}_j \) is given by

\[ \overline{\mathcal{V}}_j = \bigcup_{i, \ell \neq j \in \{1 \ldots, N\}} \overline{\mathcal{V}}_{i,\ell,j} \]

where

\[ \overline{\mathcal{V}}_{i,\ell,j} \triangleq \bigcup_{\lambda \in \Lambda_{i,\ell}} \left\{ \text{col}(\rho_j, \sigma_j) \in \mathbb{R}^{M_j+N_j} : \begin{aligned} U_i(\lambda) R_j(\lambda, \rho_j) - Q_i(\lambda) S_j(\lambda, \sigma_j) &= 0 \\ U_\ell(\lambda) R_j(\lambda, \rho_j) - Q_\ell(\lambda) S_j(\lambda, \sigma_j) &= 0 \end{aligned} \right\} \]

**Remark 3**: The form of \( \overline{\mathcal{V}}_j \) suggests a simple procedure for enforcing discernibility. Indeed, \( U_i(s) R_j(s) - Q_i(s) S_j(s) \)

Given two finite sets of polynomials \( \{ a_p(s) \}_{p=0}^{P} \) and \( \{ b_q(s) \}_{q=0}^{Q} \), we use the notations \( \gcd_{p,q} \{ a_p(s) \} \) and \( \gcd_{p,q} \{ b_q(s) \} \) to mean \( \gcd \{ a_0(s), \ldots, a_P(s) \} \) and \( \gcd \{ b_0(s), \ldots, b_Q(s) \} \), respectively.
is easily recognized to be the characteristic polynomial of the closed-loop resulting from the feedback interconnection of \( P_j \) and \( C_j \). Thus, a sufficient condition for enforcing discernibility simply consists in designing \( C_j \) so that, for all pairs of different process modes, the corresponding closed-loop characteristic polynomials are coprime. □

D. Fulfillment of the control objectives

By the above results, it turns out that whenever assumptions \( A1-A3 \) are satisfied, it is always possible to design \( \mathcal{C} \) achieving the desired set of specifications. To see this, let \( w = \text{col}(\rho_j, \sigma_j; j \in M) \) be the global parameter vector associated with \( \mathcal{C} \). Then, the following theorem can be stated which answers the question on the joint fulfillment of the control objectives \( O1 \) and \( O2 \).

**Theorem 2:** Consider a process as in (1) and let assumptions \( A1-A3 \) hold. Consider a controller \( C_j \) of the form (6) with \( R_{jm}(s), m = 0, \ldots, M_j, \) and \( S_{jm}(s), n = 0, \ldots, N_j, \) satisfying (11). Assume that \( \mathcal{C} \) is able to stabilize the plant, in the sense that there exists at least one value of \( w \) for which objective \( O1 \) is achieved. Then, the set of values of \( w \) for which objectives \( O1 \) and \( O2 \) are jointly achieved has non-empty interior, i.e., it contains a ball of positive radius. □

We can summarize the main results of this section as follows:

i) If the controller parametrization is adequately chosen, a controller can be found which is also discerning. Indeed, in the unfavourable situation where the controller is not discerning for a certain pair of process modes, one can recover discernibility by small perturbations in the controller parameters. In particular, one can always discernability for all pairs of process modes since the number of process modes is finite by hypothesis;

ii) Since the controllers are designed independent of each other, the family \( \mathcal{C} \) inherits all the properties of the single controllers, and can thus be therefore built up to achieve the desired set of specifications;

iii) the unique feature of discernibility, sharply distinguishing it from other approaches to process mode-observability, is that it is independent of persistence of excitation conditions, it shows in particular that the question of process mode-observability can be successfully addressed at the control design stage, without the aid of probing signals.

IV. HYBRID REALIZATION

Suppose that a family \( \mathcal{C} \) of controllers is available which satisfies objectives \( O1 \) and \( O2 \). The control architecture of Figure 1 therefore ensures that at each instant of time it is possible to reconstruct the actve mode of the process from the measured data. Such a control architecture, however, is known to exhibit some deficiencies. Implementing each controller as a separate dynamical system raises the question of stability of the latent controllers, i.e. the controllers which are not in charge of the plant. It is then convenient to implement the multi-controller directly as a single controller with switched parameters, i.e.

\[
\begin{align*}
\dot{\theta} &= \tilde{\phi}_\sigma q + \phi_\sigma y \\
u &= \eta_\sigma q + \phi_\sigma y 
\end{align*}
\]

where \( \{\tilde{\phi}_\sigma, \phi_\sigma, \eta_\sigma, \phi_\sigma \} \) is parameter-dependent. Such a controller is of a hybrid type and relies on the idea of “state sharing”, as originally introduced in [8]. Hereafter, we show that a family of controllers satisfying objectives \( O1 \) and \( O2 \) can always be implemented in the form (12) while preserving the fulfillment of \( O1 \) and \( O2 \).

Let \( \mathcal{C} \) be a family of controllers satisfying \( O1 \) and \( O2 \), with each controller as in (6), i.e.

\[
R_j(s)u_j = S_j(s)y, \quad j \in \{1,2,\ldots,M\} \tag{13}
\]

Define \( n_q \triangleq \max_{j \in \{1,2,\ldots,M\}} \deg R_j(s) \). By the above arguments, we wish to implement this family of controllers as a hybrid system of the form (12) in such way that, for each \( j \in \{1,2,\ldots,M\} \), the realization \( \{\tilde{\phi}_j, \tilde{g}_j, \hat{h}_j, \tilde{t}_j \} \) yields a discerning controller with output \( u_j \) as in (13). The main complication arising in this case is that if \( \deg R_j(s) < n_q \) then \( \{\tilde{\phi}_j, \tilde{g}_j, \hat{h}_j, \tilde{t}_j \} \) will necessarily be non-minimal. Since by condition \( D1 \), \( \{\tilde{\phi}_j, \tilde{g}_j, \hat{h}_j, \tilde{t}_j \} \) must be observable, the idea is to search for observable uncontrollable realizations. One such realization can be obtained as follows. Let \( (k_j, \tilde{S}_j(s)), k_j \in \mathbb{R}, \tilde{S}_j(s) \in \mathbb{R}[s], \deg \tilde{S}_j(s) < \deg R_j(s), \) be the unique solution pair of

\[
\tilde{S}_j(s) = \tilde{S}_j(s) + k_j R_j(s) \tag{14}
\]

Obviously, \( k_j = 0 \) whenever \( \deg \tilde{S}_j(s) < \deg R_j(s) \). Now define

\[
\{\tilde{\phi}_j, \tilde{g}_j, \hat{h}_j, \tilde{t}_j \} = \{\tilde{\phi}_0 + \alpha_j \hat{h}_0, \beta_j, \hat{h}_0, k_j \tag{15}
\]

where \( (\hat{h}_0, \tilde{\phi}_0) \) is a \( n_q \)-dimensional parameter-independent observable pair, whereas \( \alpha_j \) and \( \beta_j \) are free assignable parameters. We can assume without loss of generality that \( (\hat{h}_0, \tilde{\phi}_0) \) has the observer form

\[
\tilde{\phi}_0 = \begin{bmatrix}
-f_{n_q-1} & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
f_1 & 0 & 0 & \cdots & 1 \\
f_0 & 0 & 0 & \cdots & 0
\end{bmatrix} \tag{16a}
\]

\[
\hat{h}_0 = [1 \ 0 \ 0 \ \cdots \ 0] \tag{16b}
\]

Next pick a strictly Hurwitz monic polynomial \( \phi_j(s) \) with \( \deg \phi_j(s) = n_q - \deg R_j(s) \) and such that

\[
\phi_j(s) \notin \mathcal{V}_{i,\ell} \tag{17}
\]

for all pairs \((i, \ell)\) of different process modes. Then, the parameters \( \alpha_j \) and \( \beta_j \) are defined as the unique solutions of

\[
\omega(s) - \hat{h}_0 \phi_j(sI - \tilde{\phi}_0) \alpha_j = R_j(s) \phi_j(s) \tag{18}
\]

and

\[
\hat{h}_0 \phi_j(sI - \tilde{\phi}_0) \beta_j = \tilde{S}_j(s) \phi_j(s) \tag{19}
\]
where we have defined \(\omega(s) = \det(sI - F_0)\). Existence and uniqueness of \(\alpha_j\) and \(\beta_j\) follows simply by noting that \(h_0 \in \mathbb{C}\{s\}^{n_q} = [s^{n_q-1} \ldots s 1]\) and that \(\deg R_j(s) \phi_j(s) - \omega(s) < n_q\), \(\deg S_j(s) \phi_j(s) < n_q\).

On that basis, one can draw the following conclusions. For each \(j \in \{1, 2, \ldots, M\}\), the realization \(\{\tilde{F}_j, g_j, h_j, \tau_j\}\) It is simple to see that such realization yields a controller with output \(u_j\) satisfying

\[
R_j(s) \phi_j(s) u_j = S_j(s) \phi_j(s) y
\]

thus as in (13) after cancellation of the Hurwitz polynomial \(\phi_j(s)\). This implies that \(\{\tilde{F}_j, g_j, h_j, \tau_j\}\) still satisfies objective O1. In connection with O2, we must prove that each realization \(\{\tilde{F}_j, g_j, h_j, \tau_j\}\) still provides a discerning controller. To see this, observe first that condition D1 follows directly from observability of \(\{\tilde{F}_j, h_j\}\). Accordingly, by Theorem 1 it remains to prove that the matrix

\[
\Theta_{i,j}(s) = \begin{bmatrix}
Q_j(s) & U_j(s) \\
Q_j(s) & U_j(s)
\end{bmatrix}
\]

has rank 2 for all \(s \in \mathbb{C}\). By (17), \(\Theta_{i,j}(s)\) has rank 2 for all \(s_0\) such that \(\phi_j(s_0) = 0\). In addition, \(\Theta_{i,j}(s)\) has rank 2 when \(\phi_j(s) \neq 0\) because the original controllers are discerning by hypothesis, i.e. they satisfy condition D2. This implies that \(\{\tilde{F}_j, g_j, h_j, \tau_j\}\) still satisfies objective O2.

V. Conclusions

In this paper we addressed the issue of mode-observability for non-autonomous switching linear systems. To this end, we introduced a notion of discerning controllers, related to the condition that a linear time-invariant controller of a given structure must obey in order to solve the mode-observability problem given a family of feedback-distinguishable process modes. We addressed both existence and genericity problems and provided an explicit characterization of all discerning controllers. It was finally shown that a given family of discerning controllers can always be implemented as a single hybrid dynamical system which preserves the discerning capability of the original controllers.

References