

EMPIRICAL CHARACTERISTIC FUNCTION-BASED TESTS FOR MULTIVARIATE STABLE DISTRIBUTIONS

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Abstract

We consider goodness-of-fit testing for multivariate stable distributions. The proposed test statistics exploit a characterizing property of the characteristic function of these distributions and are consistent under some conditions. The asymptotic distribution is derived under the null hypothesis as well as under local alternatives. Conditions for an asymptotic null distribution free of parameters and for affine invariance are provided. Computational issues are discussed in detail and simulations show that with proper choice of the user parameters involved, the new tests lead to powerful omnibus procedures for the problem at hand.

Keywords: Characteristic function; Characterization; Goodness-of-fit; Multivariate stable distribution.

1 Introduction

Let X be a p -variate ($p \geq 1$) random vector and $\varphi(t)$ denote its characteristic function (CF). It is well known (see Sato, 1999, eqn. 13.1) that X follows a multivariate stable distribution if for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^p$ such that

$$(\varphi(t))^a = \varphi(bt)e^{it'c} \quad t \in \mathbb{R}^p. \quad (1.1)$$

The law of X is called strictly stable if (1.1) holds with $c = 0$. In this paper relation (1.1) will be exploited to construct goodness-of-fit tests for multivariate stable distributions with special attention devoted to tests for symmetric stable distributions, and to Cauchy and normal distributions in particular.

Previous related work closely connected to the approach followed here is that of Csörgő (1989), Henze & Zirkler (1990), Henze & Wagner (1997), Epps (1999), Gürtler & Henze (2000), Matsui & Takemura (2005, 2008), Pudielko (2005), Arcones (2007), and Jiménez-Gamero et al. (2009), where empirical and parametric CFs are compared by means of some

distance function. Here however we use the characterization (1.1) in the construction of the test statistics. This approach has a few advantages: first, there is no need to specify a parametric form of the CF in the testing, which often results in computational simplicity. Also, depending on the case at hand, one can avoid estimating a location parameter which implies that in the context of composite goodness-of-fit testing we only need to estimate the covariance matrix (or scatter matrix) from the data. Moreover it will be seen that appropriate choices of a and b in (1.1) allow us to obtain extremely powerful tests for a large range of alternatives.

The structure of the paper is as follows. In Section 2 we will recall some basic features of the CF of multivariate stable distributions and discuss the characterization in (1.1). In Section 3 the test statistics are introduced, the case of testing for the normal and Cauchy distribution are emphasized, and the asymptotic properties of the proposed test statistic are derived. Section 4 proposes an affine-invariant version of the new test statistic, presents computational strategies, and analyses the effect of user-specified parameters. In Section 5 we present simulations results on the power of the tests. Some conclusions and discussion is contained in Section 6. Several technical arguments are collected in an Appendix.

2 Multivariate stable distributions

First we introduce some basic notation and abbreviations. We will indicate the norm of a vector $x = (x_1, \dots, x_p)'$ as $|x| = \left(\sum_{j=1}^p x_j^2\right)^{1/2}$, and the inner product of two vectors x, y by $x'y$. The identity matrix of dimension $p \times p$ will be denoted by I_p . We shall write $o_p(1)$ for an asymptotically negligible quantity. The notation $\mathcal{N}(\delta, \Sigma)$ will be used for a normal distribution with mean δ and covariance matrix equal to Σ . Finally ‘independent and identically distributed’ will be abbreviated to i.i.d.. Further, more specialized, notation will be introduced in Section 3.

The following discussion, based on previous results in the literature, will show that formula (1.1) actually characterizes multivariate stable distributions.

Note first of all that φ is an infinitely divisible CF: in fact, for any $n > 0$, $a/n > 0$, $(\varphi(t))^{a/n}$ is a CF; it follows then that $(\varphi(t))^a = [(\varphi(t))^{a/n}]^n$, i.e. that φ is an infinitely divisible CF. Next, from (1.1) there exists $a_1, a_2 > 0$ with $a_1 + a_2 = a$, $b_1, b_2 > 0$, and $c_1, c_2 \in \mathbb{R}^p$ such that $(\varphi(t))^a = (\varphi(t))^{a_1}(\varphi(t))^{a_2} = \varphi(tb_1)e^{it'c_1}\varphi(tb_2)e^{it'c_2}$, from which we have

$$\varphi(tb_1)e^{it'c_1}\varphi(tb_2)e^{it'c_2} = \varphi(bt)e^{it'c} \quad (2.1)$$

for some $b > 0$ and $c \in \mathbb{R}^p$. If F denotes the distribution function of a p -variate random vector X and $*$ the convolution operator, (2.1) can be seen to be equivalent to

$$F\left(\frac{X - c'_1}{b'_1}\right) * F\left(\frac{X - c'_2}{b'_2}\right) = F\left(\frac{X - c'}{b'}\right) \quad (2.2)$$

with $b_1 = 1/b'_1$ and $c_1 = -c'_1/b'_1$ and similarly for the other terms. It has been originally shown by Lévy (1937) and Feldheim (1937) that the class of distributions satisfying (2.2) have a multivariate log CF

$$\log \varphi(t) = iP_1(t) - \frac{1}{2}P_2(t) + \int \left(e^{it'w} - 1 - \frac{it'w}{1 + w'w} \right) \frac{dr}{r^{\alpha+1}} d\mu(w) \quad (2.3)$$

where P_1 and P_2 are homogeneous polynomials of degree one and two, $w \in \mathbb{R}^p$, $r = |w|$, $0 < \alpha \leq 2$ and μ a finite measure defined on and integrable over the surface defined by the unit sphere. Integration is taken over the p -sphere. Press (1972) further developed equation (2.3) to obtain the more familiar form

$$\log \varphi(t) = i\delta't - \frac{1}{2} \sum_{j=1}^m (t'\Sigma_j t)^{\alpha/2} [1 + i\beta(t)] \quad (2.4)$$

with

$$\beta(t, \alpha) = \begin{cases} -\tan\left(\frac{\pi\alpha}{2}\right) \frac{\sum_{j=1}^m (t'\Sigma_j t)^{\alpha/2} \frac{w'_j t}{|w'_j t|}}{\sum_{j=1}^m (t'\Sigma_j t)^{\alpha/2}} & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} \frac{\sum_{j=1}^m (t'\Sigma_j t)^{1/2} \frac{w'_j t}{|w'_j t|} \log |w'_j t|}{\sum_{j=1}^m (t'\Sigma_j t)^{1/2}} & \text{if } \alpha = 1, \end{cases}$$

where Σ_j is a positive definite matrix of rank r_j , $1 \leq r_j \leq p$, $j = 1, 2, \dots, m$ and no two Σ_j 's are proportional; for further details and derivation of the formula see Press (1972).

By imposing a symmetry condition around a p -vector δ , i.e. requiring that $\varphi(t)$ satisfies $e^{-i\delta't}\varphi(t) = e^{i\delta't}\varphi(-t)$ for all $t \in \mathbb{R}^p$, implies that $\beta = 0$, identically in t ; thus a multivariate stable distribution symmetric around δ has $\log \varphi(t) = i\delta't - \frac{1}{2} \sum_{j=1}^m (t'\Sigma_j t)^{\alpha/2}$, which may also be parametrized in a slightly different way as

$$\varphi(t) = e^{i\delta't - (t'\Sigma t)^{\alpha/2}}, \quad t \in \mathbb{R}^p, \quad (2.5)$$

where Σ is assumed to be positive definite; see also Samorodnitsky & Taqqu (1994, §5.2).

If $\alpha = 2$, then we have from (2.5) the CF of a multivariate normal distribution with mean vector δ and covariance matrix 2Σ . The case of the multivariate Cauchy distribution symmetric around δ is given by (2.5) when $\alpha = 1$. In this last case as well as for all $\alpha \in (0, 2)$, the matrix Σ will be understood as a general *scatter* matrix.

From the results above it follows that definition (1.1) is characteristic of general multivariate stable distributions. In particular, for the normal and Cauchy cases which we will consider in more detail we state the following propositions which are easily verified.

Proposition 2.1. *Formula (1.1) holds for $b = \sqrt{a}$ and $c = \delta(a - \sqrt{a})$ if and only if X follows a multivariate normal distribution with mean $\delta \in \mathbb{R}^p$.*

Proposition 2.2. *Formula (1.1) holds for $b = a$ and $c = 0$ if and only if X follows a multivariate Cauchy distribution with location parameter $\delta \in \mathbb{R}^p$.*

Note that the multivariate Cauchy distribution is strictly stable for each $\delta \in \mathbb{R}^p$, while the multivariate normal distribution is strictly stable only if $\delta = 0$.

It is straightforward to see from (2.5) that Propositions 2.1 and 2.2 can be generalized to the family of symmetric stable distributions with arbitrary index $\alpha \in (0, 2]$. Specifically we have the following:

Proposition 2.3. *Formula (1.1) holds for $b = a^{1/\alpha}$ and $c = \delta(a - a^{1/\alpha})$ if and only if X follows a multivariate stable distribution with index $\alpha \in (0, 2]$ which is symmetric around $\delta \in \mathbb{R}^p$.*

Prop. 2.3 implies that the random variable X follows a symmetric stable distribution of index α if the equation

$$(\varphi(t))^a - \varphi(bt)e^{it^c} = 0, \quad (2.6)$$

holds for each $a > 0$, identically in $t \in \mathbb{R}^p$, with $b = a^{1/\alpha}$ and $c = \delta(a - a^{1/\alpha})$.

3 Goodness-of-fit tests

Let $X_j := (X_{j1}, \dots, X_{jp})'$, $j = 1, \dots, n$, denote independent copies of X , and denote by $\mathbb{S}_\alpha(\delta, \Sigma)$ the distribution with CF given by (2.5). Suppose that on the basis of X_j , $j = 1, \dots, n$, we wish to test the null hypothesis

$$\mathcal{H}_0 : \text{The law of } X \text{ is } \mathbb{S}_\alpha(\delta, \Sigma) \text{ for fixed } \alpha \in (0, 2], \text{ and for some } (\delta, \Sigma) \in \mathbb{R}^p \times \mathcal{M}^p,$$

where \mathcal{M}^p denotes the set of all symmetric positive definite matrices of dimension $p \times p$.

As already mentioned, the characterizations in Section 2 may be used for constructing goodness-of-fit tests by using the empirical CF of X defined as

$$\varphi_n(t) = \frac{1}{n} \sum_{j=1}^n e^{it'X_j}. \quad (3.1)$$

Specifically we suggest to replace in the left-hand side of (2.6), $\varphi(\cdot)$ by $\varphi_n(\cdot)$, b by $a^{1/\alpha}$ and c by $\widehat{\delta}_n(a - a^{1/\alpha})$ where $\widehat{\delta}_n$ denotes a consistent estimator of the location parameter δ . However since $\mathbb{S}_\alpha(\delta, \Sigma)$ is invariant under affine transformations of the type $X \mapsto AX + d$, with $d \in \mathbb{R}^p$ and A a non-singular $p \times p$ matrix, it is natural to use in the test statistic the empirical CF

$$\phi_n(t) = \frac{1}{n} \sum_{j=1}^n e^{it'\widehat{Y}_j}, \quad (3.2)$$

corresponding to the standardized data $\widehat{Y}_j = \widehat{\Sigma}_n^{-1/2}(X_j - \widehat{\delta}_n)$, where $\widehat{\delta}_n$ and $\widehat{\Sigma}_n$ denote consistent estimators of the corresponding parameters. In this way we reduce the test to the standard case of testing $Y \in \mathbb{S}_\alpha(0, I_p)$, with $Y = \Sigma^{-1/2}(X - \delta)$, i.e. of testing for symmetric stability with location zero and scatter matrix equal to the identity matrix I_p .

To introduce our test statistic let $\vartheta = (\vartheta_1, \vartheta_2)$, where the vector $\vartheta_1 = (a, \alpha) \in (0, \infty) \times (0, 2]$ is known and the parameter $\vartheta_2 = (\delta, \Sigma) \in \mathbb{R}^p \times \mathcal{M}^p$ is assumed to be unknown. In view of (2.6) we suggest to reject the null hypothesis \mathcal{H}_0 for large values of the test statistic

$$\Delta_{n,w}(\vartheta_1) = n \int_{\mathbb{R}^p} |D_n(\vartheta_1; t)|^2 w(t) dt, \quad (3.3)$$

where

$$D_n(\vartheta_1; t) = (\phi_n(t))^a - \phi_n(a^{1/\alpha}t), \quad (3.4)$$

and $w(\cdot)$ denotes a non-negative weight function.

The test for multivariate normality corresponds to $\alpha = 2$ in (3.3) in which case one typically uses in the place of $\widehat{\delta}_n$ (resp. $2\widehat{\Sigma}_n$) the sample mean (resp. the sample covariance matrix) as estimator of δ (resp. 2Σ).

In turn, for $\alpha = 1$ in (3.3) a test for the multivariate Cauchy null hypothesis results. Note that in this case the characterization in Prop. 2.2 does not involve the location parameter δ . However, this fact alone does not immediately imply that we do not need to use a location standardization by $\widehat{\delta}_n$, in the same manner as the non-occurrence of the matrix Σ in Prop. 2.3 does not imply that we do not need to standardize the data by using $\widehat{\Sigma}_n$. In fact, generally, if we do not standardize, the asymptotic null distribution of the test statistic $\Delta_{n,w}(\vartheta_1)$ will depend on the true values of δ and Σ . The case of the Cauchy distribution however is peculiar with respect to location. To see this notice that for $\vartheta_1 = (a, 1)$ we have

$$D_n(\vartheta_1; t) = (\phi_n(t))^a - \phi_n(at) = e^{-it'\widehat{\Sigma}_n^{-1/2}\widehat{\delta}_na} \left[(\varphi_n(\widehat{\Sigma}_n^{-1/2}t))^a - \varphi_n(\widehat{\Sigma}_n^{-1/2}ta) \right] \quad (3.5)$$

by simple algebra. This last equation implies that the quantity $|D_n(\vartheta_1; t)|^2$ employed in the test statistic in (3.3) is location invariant, which in turn means that the value of $\Delta_{n,w}(\vartheta_1)$ does not depend on the value of δ . For this reason we will simply use $\widehat{Y}_j = \widehat{\Sigma}_n^{-1/2}X_j$ as standardized data in the case of testing for the Cauchy distribution. Moreover the maximum likelihood (ML) estimator will be used as estimator of Σ . Note at this point that ML estimation is a standard tool, certainly for normal data, but also in the context of the multivariate Cauchy and stable distributions; see Auderset et al. (2005) and Nolan (2013) for discussion and further references. The use of ML estimators has also been suggested by Matsui & Takemura (2005, 2008) who show good performance of these estimators in comparison to other methods.

3.1 Behavior of the test statistic under the null hypothesis

In this subsection, we study the behavior of $\Delta_{n,w}(\vartheta_1)$ under the null hypothesis. Here and all along the paper, we consider estimators $\widehat{\vartheta}_{2,n} := (\widehat{\delta}_n, \widehat{\Sigma}_n)$ converging in probability to the true, but unknown parameter $\vartheta_2 = (\delta, \Sigma)$. In addition we assume that $\widehat{\delta}_n$ admits the

Bahadur representation

$$\sqrt{n}(\widehat{\delta}_n - \delta) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \Pi(\vartheta_2; X_j) + r_n, \quad (3.6)$$

where r_n is a p -dimensional random vector which tends in probability to 0, and $\Pi(\vartheta_2; \cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a linear function such that for all $\epsilon > 0$,

$$\int_{\mathbb{R}^p} \Pi(\vartheta_2; x) dF(x) = 0, \quad \int_{\mathbb{R}^p} |\Pi(\vartheta_2; x)|^{2+\epsilon} dF(x) < \infty. \quad (3.7)$$

Furthermore we consider positive continuous weight functions $w(\cdot)$ satisfying,

$$w(t) \geq 0, \quad t \in \mathbb{R}^p, \quad 0 < \int_{\mathbb{R}^p} w(t) dt, \quad \int_{\mathbb{R}^p} |t|^2 w(t) dt < \infty, \quad (3.8)$$

(with the first relation holding except possibly in a set of measure zero), and

$$\int_{\mathbb{R}^p} \zeta(t'x) w(x) dx = 0, \quad t \in \mathbb{R}^p, \quad (3.9)$$

for any odd real-valued function ζ ($\zeta(x) = -\zeta(-x)$, $x \in \mathbb{R}$).

Remark 3.1. *Weight functions satisfying (3.9) yield test statistics whose limit distributions are more tractable. Some examples can be found among symmetric functions around 0.*

Remark 3.2. *From (3.6), by classical arguments, $\sqrt{n}(\widehat{\delta}_n - \delta)$ converges in distribution to a zero-mean Gaussian random vector with covariance matrix*

$$\Omega(\vartheta_2) = \int_{\mathbb{R}^p} \Pi(\vartheta_2; x) \Pi'(\vartheta_2; x) dF(x).$$

Theorem 3.3. *Assume that (3.6)-(3.9) hold. Then, under \mathcal{H}_0 , for all $\vartheta \in (0, \infty) \times (0, 2] \times \mathbb{R}^p \times \mathcal{M}^p$,*

$$\Delta_{n,w}(\vartheta_1) = |\Sigma^{1/2}| \int_{\mathbb{R}^p} S_n^2(\vartheta; t) w(t' \Sigma^{1/2}) dt + o_P(1), \quad (3.10)$$

where for all $t \in \mathbb{R}^p$,

$$\begin{aligned} S_n(\vartheta; t) = & \frac{1}{\sqrt{n}} \sum_{j=1}^n \left\{ a \widetilde{\varphi}^{a-1}(t) [\cos(t' X_j^*) + \sin(t' X_j^*) - \widetilde{\varphi}(t)] \right. \\ & \left. - [\cos(a^{1/\alpha} t' X_j^*) + \sin(a^{1/\alpha} t' X_j^*) - \widetilde{\varphi}(a^{1/\alpha} t)] - t' \Pi(\vartheta_2; X_j^* + \delta) \Psi(\vartheta; t) \right\}, \end{aligned}$$

with $\widetilde{\varphi}$ standing for the CF of $X_j^* = X_j - \delta$, $j \geq 1$, and $\Psi(\vartheta; t) = a \widetilde{\varphi}^a(t) - a^{1/\alpha} \widetilde{\varphi}(a^{1/\alpha} t)$, $t \in \mathbb{R}^p$.

Proof: See Section 7.

Example. Assume X is a p -dimensional Gaussian random vector with mean δ and covariance matrix 2Σ . Then $\hat{\delta}_n$ can be taken to be the sample mean and can be written in the form (3.6) with $\Pi(\vartheta; x) = x - \delta$, $x \in \mathbb{R}^p$, and $r_n = 0 \in \mathbb{R}^p$. It is also easy to see that $\Omega(\vartheta_2) = 2\Sigma$ and that, since $\alpha = 2$ in this case, one has

$$\Psi(\vartheta; t) = (a - a^{1/2})e^{at'\Sigma t}, \quad t \in \mathbb{R}^p.$$

The study of the limit distribution of $\Delta_{w,n}(\vartheta_1)$ will be carried out in conjunction with that of the process $S_n(\vartheta; \cdot)$. This process can be considered as a random element in a Fréchet space $C(\mathbb{R}^p \rightarrow \mathbb{R})$ of real-valued continuous function defined on \mathbb{R}^p endowed with the metric

$$\rho(u, v) = \sum_{j \geq 1} 2^{-j} \frac{\rho_j(u, v)}{1 + \rho_j(u, v)},$$

where for all $j \geq 1$, $\rho_j(u, v) = \max_{|t| \leq j} |u(t) - v(t)|$. In our first main result we provide the limit distribution of $S_n(\vartheta; \cdot)$ under \mathcal{H}_0 , while in the second result we give the limit distribution corresponding to $\Delta_{n,w}(\vartheta_1)$.

Theorem 3.4. *Under \mathcal{H}_0 , $\{S_n(\vartheta; \cdot) : n \geq 1\}$ converges weakly in $C(\mathbb{R}^p \rightarrow \mathbb{R})$ to a zero-mean Gaussian process $S(\vartheta; \cdot)$ with covariance kernel*

$$\begin{aligned} \Gamma(\vartheta; s, t) = & a^2 [\tilde{\varphi}(t)\tilde{\varphi}(s)]^{a-1} [\tilde{\varphi}(t-s) - \tilde{\varphi}(t)\tilde{\varphi}(s)] - a\tilde{\varphi}^{a-1}(t) [\tilde{\varphi}(t-a^{1/\alpha}s) - \tilde{\varphi}(t)\tilde{\varphi}(a^{1/\alpha}s)] \\ & - a\tilde{\varphi}^{a-1}(s) [\tilde{\varphi}(a^{1/\alpha}t-s) - \tilde{\varphi}(s)\tilde{\varphi}(a^{1/\alpha}t)] + \tilde{\varphi}[a^{1/\alpha}(t-s)] - \tilde{\varphi}(a^{1/\alpha}t)\tilde{\varphi}(a^{1/\alpha}s) \\ & - a\tilde{\varphi}^{a-1}(s) \int_{\mathbb{R}^p} [\cos(s'x) + \sin(s'x) - \tilde{\varphi}(s)] t' \Pi(\vartheta_2; x + \delta) \Psi(\vartheta; t) d\tilde{F}(x) \\ & - a\tilde{\varphi}^{a-1}(t) \int_{\mathbb{R}^p} [\cos(t'x) + \sin(t'x) - \tilde{\varphi}(t)] s' \Pi(\vartheta_2; x + \delta) \Psi(\vartheta; s) d\tilde{F}(x) \\ & - \int_{\mathbb{R}^p} [\cos(a^{1/\alpha}s'x) + \sin(a^{1/\alpha}s'x) - \tilde{\varphi}(a^{1/\alpha}s)] t' \Pi(\vartheta_2; x + \delta) \Psi(\vartheta; t) d\tilde{F}(x) \\ & - \int_{\mathbb{R}^p} [\cos(a^{1/\alpha}t'x) + \sin(a^{1/\alpha}t'x) - \tilde{\varphi}(a^{1/\alpha}t)] s' \Pi(\vartheta_2; x + \delta) \Psi(\vartheta; s) d\tilde{F}(x) \\ & + \Psi(\vartheta; s)\Psi(\vartheta; t)s'\Omega(\vartheta_2)t, \quad s, t \in \mathbb{R}^p, \end{aligned} \tag{3.11}$$

where \tilde{F} and $\tilde{\varphi}$ denote respectively, the cumulative distribution function and the CF of X_j^* , $j \geq 1$, $\Pi(\vartheta; \cdot)$ is given in (3.6) and $\Psi(\vartheta; t)$ is defined in Theorem 3.3.

Proof: See Section 7.

Corollary 3.5. *Assume that the conditions of Theorem 3.3 hold. Then $\Delta_{n,w}(\vartheta_1)$ converges in distribution to*

$$\Delta_w(\vartheta_1) = |\Sigma^{1/2}| \int_{\mathbb{R}^p} S^2(\vartheta; t) w(t' \Sigma^{1/2}) dt, \quad (3.12)$$

where $S(\vartheta; \cdot)$ is the Gaussian process defined in Theorem 3.4.

Proof: The proof can be established in the same lines as the proof of (2.17) of Henze & Wagner (1997). ■

Our tests will be shown to be affine invariant under certain conditions (see Section 4). Then the asymptotic null distribution is free of the parameters δ and Σ . With slight abuse of terminology we shall call such tests distribution-free. It is true that a test statistic may be distribution-free even without affine invariance provided that it is based on the standardized data $\widehat{Y}_j = \widehat{\Sigma}_n^{-1/2}(X_j - \widehat{\delta}_n)$; see for instance Quiroz & Dudley (1991). In what follows we explore this possibility of a non-invariant but distribution-free test statistic.

Corollary 3.6. *Assume that the conditions of Theorem 3.3 hold. Then, for $\alpha = 1$ and $\alpha = 2$, the random variable $\Delta_w(\vartheta_1)$ defined by (3.12) is distribution-free.*

Proof: By the change of variable $t = s' \Sigma^{-1/2}$, one has :

$$\Delta_w(\vartheta_1) = \int_{\mathbb{R}^p} S^2(\vartheta; \Sigma^{-1/2} s) w(s) ds.$$

For the Cauchy case which corresponds to $\alpha = 1$, it can be checked easily that $\Psi(\vartheta; t) = 0$, $t \in \mathbb{R}^p$. Then, the covariance kernel of the zero-mean Gaussian process $S(\vartheta; \Sigma^{-1/2} t)$ is given by

$$\begin{aligned} & a^2 \left[\widetilde{\varphi}(t) \widetilde{\varphi}(s) \right]^{a-1} \left[\widetilde{\varphi}(t-s) - \widetilde{\varphi}(t) \widetilde{\varphi}(s) \right] - a \widetilde{\varphi}^{a-1}(t) \left[\widetilde{\varphi}(t-as) - \widetilde{\varphi}(t) \widetilde{\varphi}(as) \right] \\ & - a \widetilde{\varphi}^{a-1}(s) \left[\widetilde{\varphi}(at-s) - \widetilde{\varphi}(s) \widetilde{\varphi}(at) \right] + \widetilde{\varphi}[a(t-s)] - \widetilde{\varphi}(at) \widetilde{\varphi}(as) \quad s, t \in \mathbb{R}^p, \end{aligned}$$

where $\widetilde{\varphi}$ stands for the CF of the $Y_j := \Sigma^{-1/2}(X_j - \delta)$, $j \geq 1$. Note that this case does not require any Bahadur representation for $\widehat{\delta}_n$.

The case $\alpha = 2$ corresponds to the Gaussian example mentioned earlier. From this, $\Psi(\vartheta; t) = (a - a^{1/2}) e^{at' \Sigma t}$, $t \in \mathbb{R}^p$ and $\Pi(\vartheta; x) = x - \delta$, $x \in \mathbb{R}^p$ and $\Omega(\vartheta_2) = 2\Sigma$. Then, the covariance

kernel of $S(\vartheta; \Sigma^{-1/2}t)$ is given by

$$\begin{aligned}
& a^2 \left[\tilde{\varphi}(t)\tilde{\varphi}(s) \right]^{a-1} \left[\tilde{\varphi}(t-s) - \tilde{\varphi}(t)\tilde{\varphi}(s) \right] - a\tilde{\varphi}^{a-1}(t) \left[\tilde{\varphi}(t-a^{1/2}s) - \tilde{\varphi}(t)\tilde{\varphi}(a^{1/2}s) \right] \\
& - a\tilde{\varphi}^{a-1}(s) \left[\tilde{\varphi}(a^{1/2}t-s) - \tilde{\varphi}(s)\tilde{\varphi}(a^{1/2}t) \right] + \tilde{\varphi}[a^{1/2}(t-s)] - \tilde{\varphi}(a^{1/2}t)\tilde{\varphi}(a^{1/2}s) \\
& - a(a-a^{1/2})\tilde{\varphi}^{a-1}(s) \int_{\mathbb{R}^p} \left[\cos(s'x) + \sin(s'x) - \tilde{\varphi}(s) \right] t'xe^{a|t|^2} d\tilde{F}(x) \\
& - a(a-a^{1/2})\tilde{\varphi}^{a-1}(t) \int_{\mathbb{R}^p} \left[\cos(t'x) + \sin(t'x) - \tilde{\varphi}(\vartheta; t) \right] s'xe^{a|s|^2} d\tilde{F}(x) \\
& - (a-a^{1/2}) \int_{\mathbb{R}^p} \left[\cos(a^{1/2}s'x) + \sin(a^{1/2}s'x) - \tilde{\varphi}(a^{1/2}s) \right] t'xe^{a|t|^2} d\tilde{F}(x) \\
& - (a-a^{1/2}) \int_{\mathbb{R}^p} \left[\cos(a^{1/2}t'x) + \sin(a^{1/2}t'x) - \tilde{\varphi}(a^{1/2}t) \right] s'xe^{a|s|^2} d\tilde{F}(x) \\
& + (a-a^{1/2})^2 e^{a(|t|^2+|s|^2)} s't, \quad s, t \in \mathbb{R}^p,
\end{aligned}$$

where $\tilde{\varphi}$ and \tilde{F} are respectively the CF and the cumulative distribution function of Y_j , $j \geq 1$.

Since in both cases, the zero-mean process $S(\vartheta; \Sigma^{-1/2}t)$ has a covariance kernel free of the unknown δ and Σ , it follows that $\Delta(\vartheta_1)$ is distribution-free. \blacksquare

Now, assume that $|\Sigma^{1/2}|w(t'\Sigma^{1/2})$ is the density function of some positive measure η_Σ with support \mathbb{R}^p . Denote by $\mathcal{L}_2 = L_2(\eta_\Sigma)$ the collection of functions g defined on \mathbb{R}^p such that $\int_{\mathbb{R}^p} g^2(t)d\eta_\Sigma(t) < \infty$. For $g_1, g_2, g \in \mathcal{L}_2$, $\langle g_1, g_2 \rangle = \int_{\mathbb{R}^p} g_1(t)g_2(t)d\eta_\Sigma(t)$ and $\|g\|_{\mathcal{L}_2} = \langle g, g \rangle^{1/2}$ respectively stand for the usual inner product and norm on \mathcal{L}_2 .

From our assumptions, the function $\Gamma(\vartheta; s, t)$ defined by (3.11) satisfies $\int_{\mathbb{R}^p} \Gamma(\vartheta; t, t)d\eta_\Sigma(t) < \infty$. Thus, it is a positive semidefinite kernel. Consequently, the integral operator ∇_Γ defined on \mathcal{L}_2 by

$$\nabla_\Gamma g(s) = \int_{\mathbb{R}^p} \Gamma(\vartheta; s, t)g(t)d\eta_\Sigma(t) \quad (3.13)$$

admits eigenvalues $\lambda_1, \lambda_2, \dots$ sorted so that $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, and eigenfunctions f_1, f_2, \dots which form an orthonormal basis for \mathcal{L}_2 .

Corollary 3.7. *Under the conditions of Theorem 3.3, $\Delta_{n,w}(\vartheta_1)$ has asymptotically the same distribution as $\sum_{j \geq 1} \lambda_j \chi_j^2$, where χ_j^2 , $j \geq 1$, are i.i.d. random variables following a chi-squared distribution with one degree of freedom.*

Proof: From our assumptions, the Gaussian process $S(\vartheta; \cdot)$ defined in Theorem 3.4 is a random element of \mathcal{L}_2 . It has the following Karhunen-Loève representation

$$S(\vartheta; t) = \sum_{j=1}^{\infty} N_j f_j(t), \quad t \in \mathbb{R}^p,$$

where for all $j \geq 1$, $N_j = \langle S(\vartheta; \cdot), f_j \rangle$ are independent zero-mean Gaussian random variables with variances λ_j . It follows from this that $\|S(\vartheta; \cdot)\|_{\mathcal{L}_2}^2 = \sum_{j=1}^{\infty} N_j^2$. Recall that $E(N_j^2) = \lambda_j \geq 0$, $j \geq 1$. For nil λ_j 's, the corresponding N_j 's are nil in probability. For positive λ_j 's, one can observe that $Z_j = N_j/\sqrt{\lambda_j}$, $j \geq 1$, are iid standard Gaussian random variables. Thus,

$$\Delta_w(\vartheta_1) = \int_{\mathbb{R}^p} S^2(\vartheta; t) d\eta_{\Sigma}(t) = \|S(\vartheta; \cdot)\|_{\mathcal{L}_2}^2 = \sum_{j=1}^{\infty} \lambda_j Z_j^2.$$

The result then follows from Corollary 3.5. ■

In practice the distribution of $\sum_{j=1}^{\infty} \lambda_j \chi_j^2$ is approximated by that of $\sum_{j=1}^J \lambda_j \chi_j^2$, for an arbitrary large J . However, since the λ_j 's are unknown they have to be estimated. In the present setting, one can estimate them by considering the eigenvalues of the operator $\nabla_{\widehat{\Gamma}_n}$, where $\widehat{\Gamma}_n(s, t) = \widehat{\Gamma}((\vartheta_1, \widehat{\vartheta}_{2,n}); s, t)$ is any consistent estimator of $\Gamma(\vartheta; s, t)$. More explicitly, one may estimate the λ_j 's by the $\widehat{\lambda}_j$'s from the Fredholm integral equations

$$\nabla_{\widehat{\Gamma}_n} \widehat{f}_j = \widehat{\lambda}_j \widehat{f}_j, \quad j \geq 1.$$

A natural estimator of $\Gamma(\vartheta; s, t)$ can be obtained by taking the empirical counterpart in the expression given in (3.11), in which $\vartheta = (\vartheta_1, \vartheta_2)$ is replaced by $(\vartheta_1, \widehat{\vartheta}_{2,n})$. The computation of the cumulative distribution function of $\sum_{j=1}^J \widehat{\lambda}_j \chi_j^2$ may be carried out by using the formulas in Matsui & Takemura (2008) p. 556, or the results in Subsection 3.3 of Deheuvels & Martynov (1996). From this, an approximation of the critical value of the test can then be obtained.

3.2 Behavior of the test statistic under local alternatives

In this subsection, we study the behavior of $\Delta_{n,w}(\vartheta_1)$ under the sequence of alternatives \mathcal{H}_1^n that the law of X has density $f(1 + n^{-1/2}h)$ where f is the density function of $\mathbb{S}_{\alpha}(\delta, \Sigma)$ and h is a function such that $\int f(x)h(x)dx = 0$.

We first state a contiguity result useful for the study of the power of tests under local alternatives. For more on the theory of contiguity, the interested reader can refer to Le Cam (1986).

Proposition 3.8. *Assume that $\sigma^2 = \int h^2(x)f(x)dx < \infty$. Then the hypotheses \mathcal{H}_0 and \mathcal{H}_1^n are contiguous.*

Proof: As in the proof of Theorem 3.1 of Henze & Wagner (1997), the log-likelihood ratio of \mathcal{H}_1^n against \mathcal{H}_0 , $\Lambda_n(X_1, \dots, X_n)$, can be written under \mathcal{H}_0 as

$$\Lambda_n(X_1, \dots, X_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n h(X_j) - \frac{1}{2n} \sum_{j=1}^n h^2(X_j) + o_P(1). \quad (3.14)$$

Then under \mathcal{H}_0 , by the law of large numbers, the second term in the right-hand side of (3.14) converges to $-\sigma^2/2$ and by the central limit theorem, the first term converges in distribution to a zero-mean Gaussian random variable with variance σ^2 . The contiguity of \mathcal{H}_0 and \mathcal{H}_1^n then follows from Proposition 7 of Le Cam (1986). ■

Theorem 3.9. *Assume that the assumptions of Theorem 3.3 and Proposition 3.8 hold. Then, under \mathcal{H}_1^n , $S_n(\vartheta; \cdot)$ converges weakly in $C(\mathbb{R}^p \rightarrow \mathbb{R})$ to a Gaussian process $\tilde{S}(\vartheta; \cdot)$ with mean function*

$$c(\vartheta; t) = \int_{\mathbb{R}^p} \left\{ a\tilde{\varphi}^{a-1}(t) [\cos(t'x) + \sin(t'x)] - [\cos(a^{1/\alpha}t'x) + \sin(a^{1/\alpha}t'x)] - t'\Pi(\vartheta_2; x + \delta)\Psi(\vartheta; t) \right\} h(x + \delta)f(x + \delta)dx, \quad t \in \mathbb{R}^p, \quad (3.15)$$

and covariance kernel $\Gamma(\vartheta; s, t)$, $s, t \in \mathbb{R}^p$ defined in Theorem 3.4.

Proof: In Section 7, where the details are postponed, one studies the finite-dimensional distributions of $S_n(\vartheta; \cdot)$ and its tightness under \mathcal{H}_1^n . ■

Corollary 3.10. *Under the conditions of Theorem 3.9, under \mathcal{H}_1^n , $\Delta_{n,w}(\vartheta_1)$ converges in distribution to*

$$\tilde{\Delta}_w(\vartheta_1) = |\Sigma^{1/2}| \int_{\mathbb{R}^p} \tilde{S}^2(\vartheta; t)w(t'\Sigma^{1/2})dt,$$

where $\tilde{S}(\vartheta; \cdot)$ is the Gaussian process defined in Theorem 3.9.

Proof: By contiguity, equation (3.10), which holds under \mathcal{H}_0 , also holds under \mathcal{H}_1^n . Then, using Theorem 3.9 and the reasoning of the proof of Theorem 3.2 of Henze & Wagner (1997) one can establish this result. ■

Corollary 3.11. *Under the conditions of Corollary 3.10, and under \mathcal{H}_1^n , $\Delta_{n,w}(\vartheta_1)$ has asymptotically the same distribution as $\sum_{j \geq 1} \lambda_j \chi_j^2(\xi_j)$, where $\chi_j^2(\xi_j)$, $j \geq 1$, are independent random variables following non-central chi-squared distributions with one degree of freedom and non-centrality parameter ξ_j^2 . For the non-centrality parameter we have $\xi_j = \lambda_j^{-1} \langle c(\vartheta; \cdot), f_j(\cdot) \rangle$ where the λ_j 's and $f_j(\cdot)$'s stand for the eigenvalues and eigenfunctions of the operator ∇_Γ defined in (3.13).*

Proof: It is easy to see that from Corollary 3.10, the processes $\tilde{S}(\vartheta; \cdot)$ and $c(\vartheta; \cdot) + S(\vartheta; \cdot)$ have the same distribution. Recall $\tilde{S}(\vartheta; \cdot)$ is the Gaussian process defined in Theorem 3.9 and $c(\vartheta; \cdot)$ is given by (3.15). As in the proof of Theorem 3.9, the Karhunen-Loève decomposition of $\tilde{S}(\vartheta; \cdot)$ is given by:

$$\tilde{S}(\vartheta; t) = \sum_{j \geq 1} \tilde{N}_j f_j(t), \quad t \in \mathbb{R}^p,$$

where for all $j \geq 1$, $\tilde{N}_j = N_j + \xi_j$, with the N_j 's defined in the proof of Corollary (3.7). It is clear that for all $j \geq 1$, $\tilde{N}_j \sim \mathcal{N}(\xi_j, \lambda_j)$, and $\tilde{\Delta}_w(\vartheta_1) = \|\tilde{S}(\vartheta; \cdot)\|_{\mathcal{L}_2}^2 = \sum_{j \geq 1} \lambda_j \chi_j^2(\xi_j)$. The result is then obtained by the application of Corollary 3.10.

Remark 3.12. *The distribution of $\sum_{j \geq 1} \lambda_j \chi_j^2(\xi_j)$ can be approximated by using the same tools as for $\sum_{j \geq 1} \lambda_j \chi_j^2$ (see the end of the previous subsection). This can allow for the approximation of the local power of the test, which depends upon the weight function w . An optimal choice of this function can be the one that maximizes the local power. However, the way w is linked to the λ_j 's is not explicit. Therefore, maximizing the local power with respect to w may not be an easy task.*

3.3 Consistency under fixed alternatives

We now consider the consistency of the test which rejects the null hypothesis \mathcal{H}_0 for large values of $\Delta_{n,w}(\vartheta_1)$. To this end assume that the weight function satisfies (3.8)

Proposition 3.13. *Let $X \in \mathbb{R}^p$ denote an arbitrary random variable and assume that*

$$(\widehat{\delta}_n, \widehat{\Sigma}_n) \rightarrow (\widetilde{\delta}, \widetilde{\Sigma}) \in \mathbb{R}^p \times \mathcal{M}^p, \quad (3.16)$$

in probability. Then under a fixed alternative \mathcal{H}_1 ,

$$\frac{\Delta_{n,w}(\vartheta_1)}{n} \rightarrow \left| \widetilde{\Sigma}^{1/2} \right| \int_{\mathbb{R}^p} |D(\vartheta_1; \tau)|^2 w(\widetilde{\Sigma}^{1/2} \tau) d\tau, \quad (3.17)$$

in probability, where for all $t \in \mathbb{R}^p$, $D(\vartheta_1; t) = (\widetilde{\varphi}(t))^a - \widetilde{\varphi}(a^{1/\alpha} t)$, with $\widetilde{\varphi}(\cdot)$ denoting the CF of $X - \widetilde{\delta}$.

Proof: We can write from (3.3)

$$\frac{\Delta_{n,w}(\vartheta_1)}{n} = \int_{\mathbb{R}^p} |D_n(\vartheta_1; t)|^2 w(t) dt, \quad (3.18)$$

where $D_n(\cdot, \cdot)$ is defined by (3.4). Since $|\phi_n(t)| \leq 1$, we have $|\phi_n(t)^a| \leq 1$; then it follows that,

$$|D_n(\vartheta_1; t)|^2 \leq 4, \quad \forall a > 0. \quad (3.19)$$

Also recall $\phi_n(\cdot)$ defined by (3.2), and notice that $\phi_n(t) = e^{-i\widehat{\delta}_n' \tau_n} \varphi_n(\tau_n)$, with $\tau_n = \widehat{\Sigma}_n^{-1/2} t$, and $\varphi_n(\cdot)$ defined in (3.1). Then under assumption (3.16) and due to the uniform convergence of the empirical CF (see Ushakov, 1999, Theorem 3.2.1) we have

$$\phi_n(t) \rightarrow e^{-i\widetilde{\delta}' \tau} \varphi(\tau) = \widetilde{\varphi}(\tau), \quad (3.20)$$

and

$$\phi_n(a^{1/\alpha} t) \rightarrow e^{-ia^{1/\alpha} \widetilde{\delta}' \tau} \varphi(a^{1/\alpha} \tau) = \widetilde{\varphi}(a^{1/\alpha} \tau), \quad (3.21)$$

almost surely. Equations (3.20) and (3.21) imply that $|D_n(\vartheta_1; t)|^2 \rightarrow |D(\vartheta_1; \tau)|^2$, which together with (3.19), and Lebesgue's theorem of dominated convergence yield the result in (3.17), and the proof is complete ■

Due to (3.8) and for fixed a , the right-hand side of (3.17) is positive unless $D(\vartheta_1; t) = 0$, identically in t . This however implies that (2.6) holds with $b = a^{1/\alpha}$ and $c = \delta(a - a^{1/\alpha})$, which according to Proposition 2.3 is true if and only if X follows a multivariate symmetric stable distribution. Consequently the test that rejects the null hypothesis for large values of $\Delta_{n,w}(\vartheta_1)$ is consistent against each fixed alternative distribution.

4 Computation and affine invariance

In this section we discuss and compare some strategies of computation of the test statistics. First of all we will present closed computational formulas in the case that the parameter a figuring in (2.6) is equal to an integer ≥ 2 ; this choice, besides being convenient from the computational point of view, will not essentially change the power properties of the procedures as it will be clear from the further analysis and simulations.

We will develop a general formulation which, by proper choice of parameters, covers all tests discussed here. To this end, let \sum_{j_1, \dots, j_a} denote the multiple sum $\sum_{j_1=1}^n \cdots \sum_{j_a=1}^n$ and notice that the quantity defined by (3.4) may be written as

$$D_n(\vartheta_1; t) = \frac{1}{n^a} \sum_{j_1, \dots, j_a} e^{it'(\widehat{Y}_{j_1} + \cdots + \widehat{Y}_{j_a})} - \frac{1}{n} \sum_{j_1} e^{ia^{1/\alpha} t' \widehat{Y}_{j_1}}. \quad (4.1)$$

Following some further algebra, we get

$$\begin{aligned} |D_n(\vartheta_1; t)|^2 &= \frac{1}{n^{2a}} \sum_{j_1, \dots, j_{2a}} \cos\left(t'(\widehat{Y}_{j_1} + \cdots + \widehat{Y}_{j_a} - \widehat{Y}_{j_{a+1}} - \cdots - \widehat{Y}_{j_{2a}})\right) \\ &\quad + \frac{1}{n^2} \sum_{j_1, j_2} \cos\left(a^{1/\alpha} t'(\widehat{Y}_{j_1} - \widehat{Y}_{j_2})\right) - \frac{2}{n^{a+1}} \sum_{j_1, \dots, j_{a+1}} \cos\left(t'(\widehat{Y}_{j_1} + \cdots + \widehat{Y}_{j_a} - a^{1/\alpha} \widehat{Y}_{j_{a+1}})\right). \end{aligned} \quad (4.2)$$

Then if we employ (4.2) in the definition of the test statistic in (3.3) we readily obtain

$$\begin{aligned} \Delta_{n,w}(\vartheta_1) &= \frac{1}{n^{2a-1}} \sum_{j_1, \dots, j_{2a}} I_w\left(\widehat{Y}_{j_1} + \cdots + \widehat{Y}_{j_a} - \widehat{Y}_{j_{a+1}} - \cdots - \widehat{Y}_{j_{2a}}\right) \\ &\quad + \frac{1}{n} \sum_{j_1, j_2} I_w\left(a^{1/\alpha}(\widehat{Y}_{j_1} - \widehat{Y}_{j_2})\right) - \frac{2}{n^a} \sum_{j_1, \dots, j_{a+1}} I_w\left(\widehat{Y}_{j_1} + \cdots + \widehat{Y}_{j_a} - a^{1/\alpha} \widehat{Y}_{j_{a+1}}\right), \end{aligned} \quad (4.3)$$

where

$$I_w(x) = \int_{\mathbb{R}^p} \cos(t'x) w(t) dt.$$

Following Henze & Wagner (1997), the specific choice $w(t) = \exp[-\gamma|t|^2]$, $\gamma > 0$, will be given special emphasis since in this case we have $I_w(x) = \left(\frac{\pi}{\gamma}\right)^{p/2} e^{-\frac{|x|^2}{4\gamma}}$, which clearly

facilitates computations by allowing a closed formula for the test statistic in (4.3). As it will be seen this weight function is also suitable when the test statistic is computed directly via equation (3.3) by means of numerical integration. For the purposes of the following analysis we will write $\Delta_{n,\gamma}(a)$ for the test statistic figuring in (4.3) corresponding to $w(t) = \exp[-\gamma|t|^2]$.

Remark 4.1. Equation (4.3) with $\widehat{Y}_j = \widehat{\Sigma}_n^{-1/2} X_j$ provides a general computational formula for the test statistic corresponding to the null hypothesis \mathcal{H}_0 with $\alpha = 1$ (Cauchy null hypothesis) while the same formula with $\widehat{Y}_j = \widehat{\Sigma}_n^{-1/2}(X_j - \widehat{\delta}_n)$ corresponds to the test statistic for the null hypothesis \mathcal{H}_0 with $\alpha = 2$ (Gaussian null hypothesis).

Besides computational simplicity, the choice of the weight function $w(t) = \exp[-\gamma|t|^2]$ is further suggested by the important property of *affine invariance*. To see this write $\Delta_{n,\gamma}(a) := \Delta_{n,\gamma}(a; X_1, \dots, X_n)$ for the test statistic based on the observations X_1, \dots, X_n , and likewise for $\widehat{\delta}_n$ and $\widehat{\Sigma}_n$. Then we have the following:

Proposition 4.2. *If for each $d \in \mathbb{R}^p$ and each non-singular $p \times p$ matrix A , the estimators $(\widehat{\delta}_n, \widehat{\Sigma}_n)$ of (δ, Σ) are such that*

$$\widehat{\delta}_n(AX_1 + d, \dots, AX_n + d) = A\widehat{\delta}_n(X_1, \dots, X_n) + d,$$

and

$$\widehat{\Sigma}_n(AX_1 + d, \dots, AX_n + d) = A\widehat{\Sigma}_n(X_1, \dots, X_n)A',$$

then the test statistic in (4.3) with weight function $w(t) = \exp[-\gamma|t|^2]$, satisfies

$$\Delta_{n,\gamma}(a; AX_1 + d, \dots, AX_n + d) = \Delta_{n,\gamma}(a; X_1, \dots, X_n),$$

for each integer value of a .

Proof: From (4.3) it is easy to see by simple algebra that the test statistic $\Delta_{n,\gamma}(a)$ depends on the observations only via $D_{jk} = (X_j - \widehat{\delta}_n)' \widehat{\Sigma}_n^{-1} (X_k - \widehat{\delta}_n)$, where $\widehat{\delta}_n = \widehat{\delta}_n(X_1, \dots, X_n)$ and $\widehat{\Sigma}_n = \widehat{\Sigma}_n(X_1, \dots, X_n)$. Naturally if we have data $\widetilde{X}_j = AX_j + d$, $j = 1, \dots, n$, then the test statistic will depend on $\widetilde{D}_{jk} = (\widetilde{X}_j - \widetilde{\delta}_n)' \widetilde{\Sigma}_n^{-1} (\widetilde{X}_k - \widetilde{\delta}_n)$, where $\widetilde{\delta}_n = \widehat{\delta}_n(\widetilde{X}_1, \dots, \widetilde{X}_n)$ and $\widetilde{\Sigma}_n = \widehat{\Sigma}_n(\widetilde{X}_1, \dots, \widetilde{X}_n)$. The proof follows since clearly $\widetilde{D}_{jk} = D_{jk}$, under the standing assumptions.

Remark 4.3. *An affine invariant test for multivariate normality has been developed by Henze & Wagner (1997), while in Henze (2002) one may find an excellent review of affine invariant tests for normality. Here affine invariance generalizes beyond the case $\alpha = 2$ for our tests. Moreover it is evident from the reasoning above that in the computation of $\Delta_{n,\gamma}(a)$, we do not need to compute the square root of $\widehat{\Sigma}_n$.*

Although quite elegant and simple, the computation of $\Delta_{n,\gamma}(a)$ by means of equation (4.3) requires n^{2a} operations; this becomes soon intractable and Monte Carlo numerical evaluation of $\Delta_{n,\gamma}(a)$ can provide precise estimates with lower computational time. Table 1 below provides computational results of the test statistic $\Delta_{n,\gamma}(a)$ for the bivariate Cauchy null hypothesis, for $(\gamma, a) = (1, 2)$, and sample size $n = 10, 30, 50$ and 100 , with three computational strategies: [1] exact evaluation by (4.3); [2] a simple Monte Carlo rule, where the integral figuring in the right-hand side of (3.3) is estimated by $(1/m) \sum_{j=1}^m |D_n(\vartheta_1, N_j)|^2$ (with $\vartheta_1 = (a, \alpha) = (2, 1)$) where $N_j, j = 1, \dots, m$, is an i.i.d. sample from a bivariate zero-mean normal distribution with covariance matrix equal to $(1/2\gamma)\mathbf{I}_2$. This procedure is quickly implementable in most softwares; [3] a Quasi Monte Carlo approach where the integral is again evaluated as in [2] but using instead a deterministic sequence. In this case we have implemented the *NIntegrate* with the option *Quasi Monte Carlo* function of Mathematica[®] 8 software. In strategy [2] we set $m = 50000$ while in case [3] we allowed Mathematica[®] 8 to use up to 10^6 points if required; exploiting symmetries in the CF, the region of integration has been set as $t = (t_1, t_2)'$ with $0 < t_1 < \infty$ and $-\infty < t_2 < \infty$.

| <i>Method</i> → | Exact | | MC | | QMC | |
|-----------------|---------|---------|--------|---------|-------|---------|
| | Time | Value | Time | Value | Time | Value |
| $n = 10$ | 0.109 | 2.8249 | 4.524 | 2.8314 | 0.811 | 2.8210 |
| $n = 30$ | 9.157 | 2.60791 | 9.719 | 2.63472 | 1.576 | 2.60353 |
| $n = 50$ | 62.167 | 2.79927 | 14.96 | 2.82919 | 2.496 | 2.805 |
| $n = 100$ | 928.346 | 2.07131 | 25.678 | 2.08494 | 3.572 | 2.0739 |

Table 1: Cauchy test: Time (in seconds) and value of $\Delta_{n,1}(2)$ for the test statistic by three strategies of computation: by formula (4.3) (Exact); Monte Carlo (MC) and Quasi Monte Carlo (QMC).

As we see, both Monte Carlo strategies produce accurate estimates. The timing required by the exact formula becomes soon quite large with the sample size n . On the other hand, the QMC method implemented with Mathematica[®] 8 is much faster with respect to the other two methods.

Now we pass to the discussion of the choice of the user-specified parameters γ and a of the test statistic figuring in eqn. (4.3), with $w(t) = e^{-\gamma|t|^2}$. Although we have mostly considered the case $a = 2$, generally speaking this value may not be an optimal choice for the test statistic $\Delta_{n,\gamma}(a)$. In what follows we investigate the behavior of the test statistic $\Delta_{n,\gamma}(a)$ as a function of a . To this end and in view of Prop. 3.13 we analyze the asymptotic behavior of the proposed test statistics based on the quantity

$$\Delta_\gamma(a) = \int_{\mathbb{R}^p} |(\varphi(t))^a - \varphi(a^{1/\alpha}t)|^2 e^{-\gamma|t|^2} dt, \quad (4.4)$$

i.e., we consider this measure of deviation for distributions in their standard form with $\delta = 0$ and $\Sigma \equiv \mathbf{I}_p$). For the univariate case $p = 1$, Figure 1 reports, as a function of a

and for different choices of γ , the values of $\Delta_\gamma(a)$ corresponding to the test statistic for the Cauchy null hypothesis, and for the case of Student- t , symmetric stable, normal, and logistic distribution, as alternatives.

As it can be noted, all cases are quite similar, (we noted the same behavior even with other parameter values), i.e., the limit of the test statistic, under each alternative and for $a > 1$, increases with a . Inspection of the graphs suggests that the power of the tests should generally increase quite sharply with values of $a > 2$, e.g. $a = 4$ or $a = 6$.

Also, a large value of γ reduces the size of the limiting test statistic and should result in lower power. Here the reason for this behavior is probably due to the excessive weight that large values of γ place on informative sections of (4.4). Note that a value of $\gamma = 0.025$ gives the highest values of the test statistics under the alternative. These graphs however do not consider variability of the test statistic and a larger value of γ should contribute to reduce excessive oscillations of the estimates $\phi_n(t)$ for large t .

In the case of the test for normality we might expect a similar behavior given the close analogy of the two test statistics. Some cases are reported in Figure 2. As we see low values of γ and high values of a obtain the largest values for $\Delta_\gamma(a)$. Note that in comparison with the Cauchy test, even larger a values should be used. Nevertheless the case $a = 6$ seems to yield high power values. As a final overall comment we would suggest that a value around $a = 6$ coupled with a small value of γ should yield a powerful test statistic for normality as well as for the Cauchy null hypothesis. This suggestion will be investigated by means of simulations.

5 Monte Carlo analysis

We perform here a simulation study in order to investigate the actual performance of the test statistics under various alternatives. Specifically, we analyze univariate and bivariate tests for the Cauchy null hypothesis, bivariate stability and bivariate normality. However, while for the Cauchy and normal cases we consider a general composite hypothesis, in the bivariate stable case the simple hypothesis where $\delta = 0$ and $\Sigma = I_2$, the 2×2 identity matrix, is considered. This last choice is dictated by the extremely large computing time required by the power simulations with an estimation step added in the stable case. The simulations were carried out with a number of *MC* Monte Carlo samples of size n , and correspond to a $q\%$ significance level. In each case we report the percentage of rejection of the null hypothesis \mathcal{H}_0 rounded to the nearest integer.

In the simulations the following alternatives are considered:

- 1) Student- t distributions with ν degrees of freedom, denoted with t_ν ;
- 2) symmetric α -stable distributions indicated with $\mathcal{S}_\alpha := \mathbb{S}_\alpha(0, I_p)$;

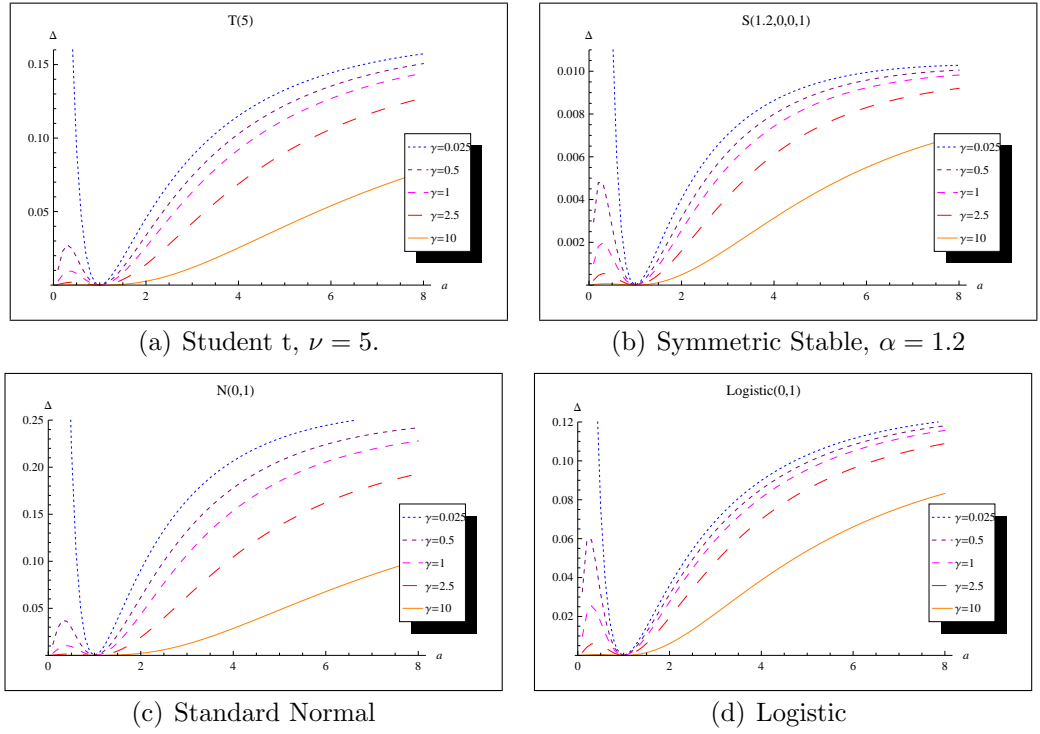


Figure 1: Value of $\Delta_\gamma(a)$ for the Cauchy null hypothesis under different alternatives .

- 3) the standard normal distribution;
- 4) the generalized Burr-Pareto logistic distribution with normal marginals, with parameters λ and μ , denoted by $\mathcal{BP}(\lambda, \mu)$ (Cook & Johnson (1986)). Note that the case $\lambda \rightarrow \infty$ and $\mu = 0$ corresponds to independent normals;
- 5) a bivariate normal mixture, denoted $\mathcal{NM}(\kappa, \delta, \rho_1, \rho_2)$ obtained by $\kappa \mathcal{N}_1(0, \rho_1) + (1 - \kappa) \mathcal{N}_2(\delta, \rho_2)$ and where $\mathcal{N}_i(\delta_i, \rho_i)$ indicates a bivariate normal distribution with (δ_i, δ_i) mean vector and covariance matrix with unit diagonal and ρ_i off diagonal, $i = 1, 2$.

In Table 2 we report rejection rates at a 10%-significance level corresponding to the test statistic for the univariate Cauchy null hypothesis.

Note that power increases quite sharply with a and with γ . However further simulation results not shown here confirm, as we expected from the discussion in the previous section, that a value of γ which is too large results in loss of power. The results are quite clear in the sense that for all the cases considered, a good suggestion seems to favor values around $a = 6$ and $\gamma = 2.5$.

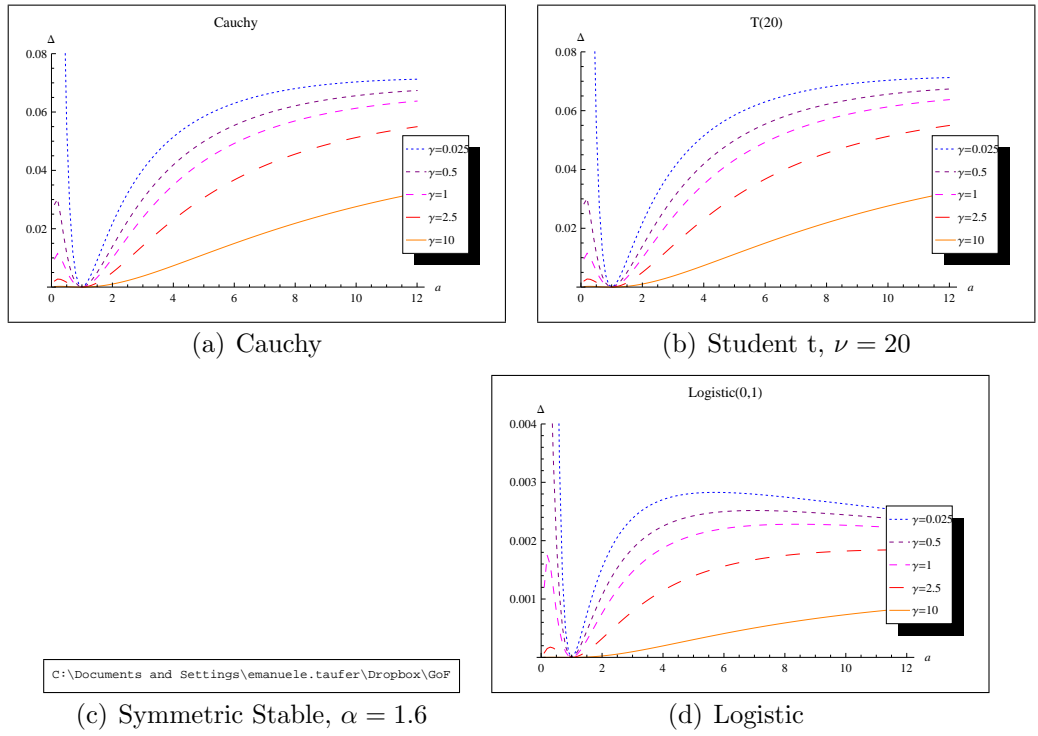


Figure 2: Value of $\Delta_\gamma(a)$ for the normal null hypothesis under different alternatives.

| $\gamma \rightarrow$ | 0.025 | | | 0.1 | | | 0.5 | | | 1 | | | 2.5 | | |
|----------------------|-------|----|----|-----|----|----|-----|----|----|----|----|-----|-----|----|-----|
| $a \rightarrow$ | 2 | 4 | 6 | 2 | 4 | 6 | 2 | 4 | 6 | 2 | 4 | 6 | 2 | 4 | 6 |
| t_1 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 9 | 10 |
| t_2 | 13 | 24 | 33 | 15 | 31 | 41 | 16 | 36 | 47 | 16 | 41 | 52 | 16 | 43 | 54 |
| t_3 | 16 | 40 | 56 | 20 | 51 | 67 | 27 | 61 | 74 | 31 | 66 | 77 | 32 | 70 | 81 |
| t_5 | 21 | 58 | 77 | 28 | 71 | 86 | 41 | 82 | 91 | 48 | 87 | 94 | 54 | 89 | 96 |
| t_{10} | 26 | 72 | 88 | 36 | 83 | 94 | 55 | 92 | 98 | 66 | 95 | 98 | 73 | 97 | 99 |
| $\mathcal{S}_{0.5}$ | 49 | 72 | 75 | 62 | 82 | 85 | 74 | 84 | 89 | 78 | 84 | 90 | 83 | 87 | 89 |
| $\mathcal{S}_{0.8}$ | 14 | 15 | 13 | 15 | 17 | 17 | 19 | 16 | 19 | 21 | 16 | 21 | 28 | 22 | 21 |
| $\mathcal{S}_{1.2}$ | 11 | 16 | 18 | 12 | 17 | 21 | 12 | 16 | 23 | 11 | 16 | 25 | 9 | 20 | 25 |
| $\mathcal{S}_{1.5}$ | 17 | 37 | 48 | 20 | 46 | 58 | 26 | 50 | 64 | 29 | 51 | 66 | 28 | 60 | 68 |
| $\mathcal{S}_{1.7}$ | 22 | 57 | 70 | 29 | 67 | 81 | 43 | 75 | 86 | 49 | 76 | 88 | 53 | 85 | 89 |
| $\mathcal{N}(0,1)$ | 33 | 83 | 94 | 47 | 92 | 98 | 72 | 96 | 99 | 81 | 97 | 100 | 89 | 99 | 100 |

Table 2: Percentage of rejection of the test for the univariate Cauchy null hypothesis; Sample size $n = 50$, $a = 2, 4, 6$, $\gamma = 0.025, 0.1, 0.5, 1, 2.5$; Significance level $q = 10\%$, $MC = 5000$ Monte Carlo trials.

| $n \rightarrow$ | 20 | | | | | | 50 | | | | | |
|---------------------|-----|----|----|----|-----|----|-----|-----|-----|-----|-----|-----|
| | 0.5 | | 1 | | 2.5 | | 0.5 | | 1 | | 2.5 | |
| $a \rightarrow$ | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 |
| t_1 | 10 | 10 | 9 | 9 | 11 | 10 | 9 | 8 | 9 | 9 | 9 | 9 |
| t_2 | 28 | 36 | 31 | 39 | 38 | 47 | 52 | 63 | 61 | 70 | 66 | 76 |
| t_4 | 44 | 58 | 50 | 62 | 58 | 68 | 80 | 90 | 87 | 94 | 92 | 96 |
| t_5 | 62 | 76 | 68 | 81 | 77 | 86 | 96 | 100 | 99 | 100 | 99 | 100 |
| t_{10} | 73 | 87 | 81 | 90 | 88 | 94 | 99 | 100 | 100 | 100 | 100 | 100 |
| $\mathcal{S}_{0.5}$ | 37 | 21 | 38 | 18 | 44 | 16 | 94 | 91 | 96 | 96 | 97 | 97 |
| $\mathcal{S}_{0.8}$ | 8 | 4 | 7 | 3 | 8 | 3 | 18 | 13 | 21 | 16 | 23 | 18 |
| $\mathcal{S}_{1.2}$ | 19 | 18 | 20 | 20 | 24 | 25 | 28 | 32 | 33 | 39 | 34 | 41 |
| $\mathcal{S}_{1.5}$ | 42 | 46 | 47 | 50 | 54 | 58 | 76 | 83 | 83 | 88 | 88 | 92 |
| $\mathcal{S}_{1.7}$ | 63 | 70 | 69 | 73 | 77 | 82 | 95 | 98 | 98 | 99 | 99 | 99 |

Table 3: Percentage of rejection of the test for the bivariate Cauchy null hypothesis; $n = 20, 50$, $a = 4, 6$, $\gamma = 0.5, 1, 2.5$; $q = 10\%$, $MC = 3000$.

The results of Table 2 are comparable with those appearing in Table 7 of Gurtler & Henze (2000) and Table 5 and 6 of Matsui & Takemura (2005) which consider CF-based tests for the univariate Cauchy distribution as well as other classical tests. We see that the choice of $a = 6$ and $\gamma = 2.5$ always yields greater power than the CF-based and other test statistics considered in those papers, and often by a wide margin. Moreover, the power of the test based on $\Delta_{n,2.5}(6)$ is also comparable with the power of the UMP invariant test against normality discussed in Gurtler & Henze (2000).

The results in Table 3 correspond to the test for the bivariate Cauchy distribution for which we are not aware of previous simulations reported in the literature. Here for stable alternatives there seems to be no uniform best choice of a and γ . Note however that for moderate sample size ($n = 50$), again, $a = 6$ and $\gamma = 2.5$ always yields highest power, or nearly so.

In Table 4 a test for the bivariate $\mathcal{S}_{1.5}$ is considered for which analogous considerations as those of Table 3 apply. Furthermore we might expect that the power of the corresponding test with estimated parameters will yield better power as it is often the case when comparing the power of a goodness-of-fit test for the simple hypothesis against the power of the same test for the composite hypothesis.

Table 5 reports power values of the 5%-significance level test for bivariate normality against Student- t and mixtures of normal distributions. The results reported are similar to those of Table 2 in the sense that the combination $a = 6$ and $\gamma = 2.5$ appears to be the best choice as it renders the highest power values in nearly all cases. These results can be compared to those in Tables 6.1 and 6.2 of Henze & Zirkler (1990) and Table 4 of Szekely & Rizzo (2005). As we see, the power values of Table 5, for the case $a = 6$ and $\gamma = 2.5$, are generally similar or higher compared to the values reported in those papers.

| $n \rightarrow$ | 20 | | | | | | 50 | | | | | |
|---------------------|-----|----|----|----|-----|----|-----|----|----|----|-----|----|
| | 0.5 | | 1 | | 2.5 | | 0.5 | | 1 | | 2.5 | |
| | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 |
| t_1 | 31 | 26 | 41 | 31 | 53 | 38 | 63 | 59 | 75 | 70 | 86 | 81 |
| t_2 | 15 | 18 | 16 | 17 | 16 | 18 | 19 | 21 | 19 | 21 | 19 | 21 |
| t_3 | 12 | 17 | 11 | 17 | 9 | 14 | 14 | 19 | 11 | 16 | 10 | 15 |
| t_5 | 11 | 19 | 10 | 18 | 6 | 14 | 16 | 26 | 14 | 26 | 11 | 25 |
| t_{10} | 12 | 21 | 10 | 20 | 6 | 13 | 25 | 44 | 25 | 47 | 19 | 44 |
| $\mathcal{S}_{0.8}$ | 55 | 45 | 68 | 54 | 81 | 63 | 93 | 91 | 98 | 96 | 99 | 98 |
| $\mathcal{S}_{1.2}$ | 17 | 14 | 22 | 17 | 27 | 19 | 30 | 28 | 36 | 32 | 47 | 39 |
| $\mathcal{S}_{1.7}$ | 10 | 12 | 9 | 11 | 6 | 9 | 12 | 15 | 9 | 16 | 12 | 16 |
| $\mathcal{S}_{1.9}$ | 12 | 14 | 11 | 14 | 6 | 11 | 24 | 30 | 14 | 35 | 30 | 39 |

Table 4: Percentage of rejection of the test for bivariate symmetric stability with $\alpha = 1.5$; $n = 20, 50$, $a = 4, 6$, $\gamma = 0.5, 1, 2.5$; $q = 10\%$, $MC = 3000$.

| $n \rightarrow$ | 20 | | | | | | 50 | | | | | |
|-------------------------------|-----|----|----|----|-----|----|-----|-----|-----|-----|-----|-----|
| | 0.5 | | 1 | | 2.5 | | 0.5 | | 1 | | 2.5 | |
| | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 |
| t_1 | 92 | 94 | 95 | 96 | 97 | 97 | 100 | 100 | 100 | 100 | 100 | 100 |
| t_2 | 50 | 54 | 60 | 61 | 72 | 70 | 90 | 88 | 94 | 94 | 98 | 97 |
| t_4 | 14 | 17 | 21 | 21 | 34 | 31 | 42 | 33 | 44 | 45 | 69 | 62 |
| t_5 | 10 | 12 | 15 | 16 | 25 | 22 | 30 | 20 | 29 | 30 | 54 | 45 |
| t_{10} | 6 | 6 | 8 | 7 | 11 | 10 | 13 | 7 | 10 | 9 | 20 | 16 |
| $\mathcal{NM}(.5, 2, 0, 0)$ | 9 | 8 | 11 | 10 | 9 | 10 | 20 | 18 | 26 | 25 | 25 | 25 |
| $\mathcal{NM}(.5, 3, 0, 0)$ | 34 | 31 | 37 | 35 | 28 | 34 | 86 | 83 | 90 | 90 | 85 | 88 |
| $\mathcal{NM}(.5, 4, 0, 0)$ | 71 | 66 | 75 | 74 | 54 | 65 | 100 | 100 | 100 | 100 | 100 | 100 |
| $\mathcal{NM}(.5, 0, .9, 0)$ | 9 | 9 | 11 | 11 | 16 | 15 | 20 | 22 | 28 | 28 | 37 | 36 |
| $\mathcal{NM}(.5, .5, .9, 0)$ | 11 | 11 | 12 | 12 | 17 | 16 | 20 | 23 | 32 | 30 | 40 | 39 |

Table 5: Percentage of rejection of the test for bivariate normality; $n = 20, 50$, $a = 4, 6$, $\gamma = 0.5, 1, 2.5$; $q = 5\%$, $MC = 3000$.

| λ | μ | γ | | | λ | μ | γ | | |
|-----------|-------|----------|-----|-----|-----------|-------|----------|-----|-----|
| | | 0.5 | 1.0 | 2.5 | | | 0.5 | 1.0 | 2.5 |
| 0.1 | -1.0 | 68 | 81 | 93 | 1 | -1.0 | 7 | 10 | 13 |
| | -0.5 | 69 | 83 | 92 | | -0.5 | 6 | 9 | 11 |
| | 0.0 | 73 | 84 | 93 | | 0.0 | 7 | 8 | 11 |
| | 0.5 | 74 | 85 | 94 | | 0.5 | 6 | 10 | 12 |
| | 1.0 | 81 | 85 | 94 | | 1.0 | 7 | 9 | 13 |
| 0.5 | -1.0 | 10 | 15 | 22 | 10 | -1.0 | 5 | 6 | 7 |
| | -0.5 | 12 | 16 | 24 | | -0.5 | 5 | 6 | 7 |
| | 0.0 | 12 | 17 | 26 | | 0.0 | 4 | 6 | 6 |
| | 0.5 | 13 | 19 | 29 | | 0.5 | 4 | 6 | 6 |
| | 1.0 | 14 | 20 | 31 | | 1.0 | 4 | 7 | 7 |

Table 6: Percentage of rejection of the test for bivariate normality against $\mathcal{BP}(\lambda, \mu)$ distributions; $n = 50$, $a = 6$, $\gamma = 0.5, 1, 2.5$; $q = 5\%$ $MC = 3000$.

Table 6 reports an excerpt, for $a = 6$, of the power values of the 5%–significance level test for bivariate normality against the generalized Burr-Pareto logistic distribution for some choices of the parameters λ and μ . Overall simulation results again indicate that the choice $a = 6$ and $\gamma = 2.5$ obtain the highest power. Table 6 can be compared with Table 4 in Doornik & Hansen (2008) where we see that the power values of Table 6, for $\gamma = 2.5$, are generally similar or higher compared to all the tests considered there, with the exception of the Mardia (1970) test which performs quite well for this distribution.

Given that the simulation results reported here indicate $a = 6$ and $\gamma = 2.5$ as a good choice, nearly uniformly, Table 7 reports the critical values for the corresponding test $\Delta_{n,2.5}(6)$. The results indicate rapid convergence of the quantiles to their asymptotic values. The last row of the table indicates the quantiles approximated by a lognormal distribution obtained by equating expectation and second moment. As the results show a very good agreement for either tests, we would suggest using the log normal–derived quantiles for an approximate test. This would considerably simplify the application of the test in practice.

6 Conclusions

We have presented a class of weighted L2–type statistics with which we are able to address the problem of testing the composite goodness–of–fit for the family of multivariate symmetric stable distributions, for the first time in the literature. The test statistics are based solely on the empirical CF and are affine invariant under proper choice of the weight function and of the estimators of the unknown distributional parameters. The main theoretical properties of the test statistics are studied in detail. Among others it was shown that at least for testing the Cauchy and the normal null hypothesis these statistics are free of parameters even

| <hr/> <hr/> | | | | | <hr/> <hr/> | | | | |
|-------------|---------|------|---------|------|-------------|---------|-------|---------|-------|
| n | $p = 1$ | | $p = 2$ | | n | $p = 1$ | | $p = 2$ | |
| | 10% | 5% | 10% | 5% | | 10% | 5% | 10% | 5% |
| 10 | .275 | .330 | .513 | .574 | 10 | 2.291 | 2.615 | 2.225 | 2.548 |
| 30 | .266 | .342 | .505 | .579 | 30 | 2.557 | 3.004 | 2.105 | 2.354 |
| 50 | .264 | .334 | .484 | .555 | 50 | 2.594 | 2.996 | 2.061 | 2.273 |
| 100 | .256 | .324 | .493 | .569 | 100 | 2.593 | 3.015 | 2.043 | 2.245 |
| LNA | .261 | .358 | .498 | .577 | LNA | 2.577 | 2.993 | 2.036 | 2.190 |

Table 7: Critical values for the test for the normal null hypothesis (left table) and for the Cauchy null hypothesis (right table); $a = 6$, $\gamma = 2.5$, $MC = 5000$.

without affine invariance. Also the computational analysis carried out in conjunction with the Monte Carlo results reported narrows down the choice of good user-specified parameter values required in order to achieve a test procedure with high power. There are several directions for future research: One is to extend the test statistics to non-symmetric stable distributions with unspecified tail and asymmetry index. At the same time it would be interesting to consider the same problem not with simple i.i.d. data but with structured data possibly involving dependence as in the case of the stable GARCH model of Bonato (2012).

7 Appendix

In the proofs of our results, we repeatedly make use of the inequality

$$(v_1 + v_2 + \dots + v_m)^\ell \leq m^{\ell-1}(v_1^\ell + v_2^\ell + \dots + v_m^\ell), \quad (7.1)$$

where the v_i 's are non-negative numbers.

The proof of Theorem 3.3 rests on three preliminary lemmas that we state and establish in the sequel.

Denote by $\tilde{\varphi}_n$ the empirical characteristic functions of the X_j^* 's. For all $t \in \mathbb{R}^p$, define the stochastic process

$$\begin{aligned} R_n(\vartheta; t) &= a\tilde{\varphi}^{a-1}(t)\sqrt{n}[\tilde{\varphi}_n(t) - \tilde{\varphi}(t)] - \sqrt{n}[\tilde{\varphi}_n(a^{1/\alpha}t) - \tilde{\varphi}(a^{1/\alpha}t)] \\ &\quad - t' \sqrt{n}(\hat{\delta}_n - \delta) i\Psi(\vartheta; t), \end{aligned}$$

where i is the complex number satisfying $i^2 = -1$.

Lemma 7.1. *Let $\psi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a linear transformation. Assume that (3.8) holds. For all $n \geq 1$, define the random variables*

$$T_n^\psi = \int_{\mathbb{R}^p} \left| \psi(t)' \sqrt{n}(\hat{\delta}_n - \delta) \Psi(\vartheta; \psi(t)) \right|^2 w(t) dt$$

$$\begin{aligned}
U_n^\psi &= \int_{\mathbb{R}^p} |\sqrt{n} [\tilde{\varphi}_n(\psi(t)) - \tilde{\varphi}(\psi(t))]|^2 w(t) dt, & V_n^\psi &= (U_n^\psi)^{1/2} \\
W_n^\psi &= \int_{\mathbb{R}^p} \left| \psi(t)' \frac{1}{\sqrt{n}} \sum_{j=1}^n \Pi(\vartheta_2; X_j^* + \delta) \Psi(\vartheta; \psi(t)) \right|^2 dw(t) \\
Z_n^\psi &= \int_{\mathbb{R}^p} |R_n(\vartheta; \psi(t))|^2 w(t) dt.
\end{aligned}$$

Then the sequences $(T_n^\psi)_{n \geq 1}$, $(U_n^\psi)_{n \geq 1}$, $(V_n^\psi)_{n \geq 1}$, $(W_n^\psi)_{n \geq 1}$ and $(Z_n^\psi)_{n \geq 1}$ are tight.

Proof: Let $\psi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a linear transformation. Note that for all $t \in \mathbb{R}^p$, $|\psi(t)| \leq C|t|$ for some positive universal constant C . For the tightness of $(T_n^\psi)_{n \geq 1}$, write

$$T_n^\psi \leq C \left| \sqrt{n} (\hat{\delta}_n - \delta) \right|^2 \int_{\mathbb{R}^p} |t|^2 w(t) dt.$$

Then, by Remark 3.2 and the fact that the integral in the right-hand side of the last inequality is finite, $(T_n^\psi)_{n \geq 1}$ is tight. The tightness of $(W_n^\psi)_{n \geq 1}$ can be established using the same arguments.

We now turn to the tightness of $(U_n^\psi)_{n \geq 1}$. Denote by \bar{z} the conjugate of a complex number z . One can check that

$$\begin{aligned}
|\sqrt{n} [\tilde{\varphi}_n(\psi(t)) - \tilde{\varphi}(\psi(t))]|^2 &= \frac{1}{n} \sum_{j=1}^n \left| e^{-i\psi(t)' X_j^*} - \tilde{\varphi}(\psi(t)) \right|^2 \\
&\quad + \sum_{j \neq \ell} \sum \left(e^{-i\psi(t)' X_j^*} - \tilde{\varphi}(\psi(t)) \right) \left(\overline{e^{-i\psi(t)' X_\ell^*} - \tilde{\varphi}(\psi(t))} \right).
\end{aligned}$$

Since the expectations of the cross terms are nil one has from above that

$$E \left| \sqrt{n} [\tilde{\varphi}_n(\psi(t)) - \tilde{\varphi}(\psi(t))] \right|^2 = \frac{1}{n} \sum_{j=1}^n E |e^{-i\psi(t)' X_j^*} - \tilde{\varphi}(\psi(t))|^2 \leq 4.$$

By Tonelli's theorem,

$$E(U_n^\psi) = \int_{\mathbb{R}^p} E \left| \sqrt{n} [\tilde{\varphi}_n(\psi(t)) - \varphi(\psi(t))] \right|^2 w(t) dt \leq 4C.$$

The application of Markov's inequality to U_n^ψ yields the tightness of $(U_n^\psi)_{n \geq 1}$. The tightness of $(V_n^\psi)_{n \geq 1}$ follows immediately. Indeed, by Jensen's inequality, one has $[E(V_n^\psi)]^2 \leq E(U_n^\psi)$, from which it can be seen that $E(V_n^\psi) \leq \sqrt{E(U_n^\psi)} \leq 2C$.

For the tightness of $(Z_n^\psi)_{n \geq 1}$, the triangle inequality and the inequality (7.1) give, for all $n \geq 1$,

$$\begin{aligned} Z_n^\psi &\leq 9a^2 \int_{\mathbb{R}^p} |\sqrt{n} [\tilde{\varphi}_n(\psi(t)) - \tilde{\varphi}(\psi(t))]|^2 w(t) dt \\ &\quad + 9 \int_{\mathbb{R}^p} |\sqrt{n} [\tilde{\varphi}_n(a^{1/\alpha}\psi(t)) - \tilde{\varphi}(a^{1/\alpha}\psi(t))]|^2 w(t) dt. \\ &\quad + 9 \int_{\mathbb{R}^p} |\psi(t)' \sqrt{n} (\hat{\delta}_n - \delta)|^2 |\Psi(\vartheta; \psi(t))|^2 w(t) dt. \end{aligned}$$

The tightness of $(Z_n^\psi)_{n \geq 1}$ then follows from those of $(T_n^\psi)_{n \geq 1}$ and $(U_n^\psi)_{n \geq 1}$. ■

Lemma 7.2. *Assume that (3.6)-(3.8) hold. Then, under \mathcal{H}_0 ,*

$$\Delta_{n,w}(\vartheta_1) = |\hat{\Sigma}_n^{1/2}| \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \hat{\Sigma}_n^{1/2}) dt + o_P(1).$$

Proof: Note that for all $a > 0$, $\alpha \in (0, 2]$ and $t \in \mathbb{R}^p$,

$$\phi_n(t) = e^{-it' \hat{\Sigma}_n^{-1/2} \hat{\delta}_n} \varphi_n(t' \hat{\Sigma}_n^{-1/2}) \quad \text{and} \quad \phi_n(a^{1/\alpha} t) = e^{-a^{1/\alpha} it' \hat{\Sigma}_n^{-1/2} \hat{\delta}_n} \varphi_n(a^{1/\alpha} t' \hat{\Sigma}_n^{-1/2}).$$

From this, for all $a > 0$, $\alpha \in (0, 2]$ and $t \in \mathbb{R}^p$, one can write

$$\phi_n^a(t) - \phi_n(a^{1/\alpha} t) = e^{-ait' \hat{\Sigma}_n^{-1/2} \hat{\delta}_n} \varphi_n^a(t' \hat{\Sigma}_n^{-1/2}) - e^{-a^{1/\alpha} it' \hat{\Sigma}_n^{-1/2} \hat{\delta}_n} \varphi_n(a^{1/\alpha} t' \hat{\Sigma}_n^{-1/2}).$$

By the change of variable $\tau = t' \hat{\Sigma}_n^{-1/2}$, one has for all $\vartheta_1 = (a, \alpha) \in (0, \infty) \times (0, 2]$,

$$\Delta_{n,w}(\vartheta_1) = \int_{\mathbb{R}^p} |Q_n(\vartheta_1; \tau)|^2 w(\tau' \hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| d\tau,$$

where for all $t \in \mathbb{R}^p$,

$$Q_n(\vartheta_1; t) = \sqrt{n} \left[e^{-ait' \hat{\delta}_n} \varphi_n^a(t) - e^{-a^{1/\alpha} it' \hat{\delta}_n} \varphi_n(a^{1/\alpha} t) \right].$$

Adding and subtracting appropriate terms, for all $\vartheta_1 = (a, \alpha) \in (0, \infty) \times (0, 2]$ and for all $t \in \mathbb{R}^p$, one can write

$$\begin{aligned} Q_n(\vartheta_1; t) &= \sqrt{n} \left\{ e^{-ait' \hat{\delta}_n} [\varphi_n^a(t) - \varphi^a(t)] - e^{-a^{1/\alpha} it' \hat{\delta}_n} [\varphi_n(a^{1/\alpha} t) - \varphi(a^{1/\alpha} t)] \right. \\ &\quad \left. + \left[e^{-ait' \hat{\delta}_n} \varphi^a(t) - e^{-a^{1/\alpha} it' \hat{\delta}_n} \varphi(a^{1/\alpha} t) \right] \right\}. \end{aligned}$$

Under \mathcal{H}_0 , by first-order Taylor expansions of the complex-valued functions $z \mapsto z^a$ and $x \mapsto e^{ix}$, one can see that there exist a complex-valued function $\varphi_{0,n}(t)$ and a p -dimensional

random vector $\tilde{\delta}_n$ such that for all $t \in \mathbb{R}^p$, $|\varphi_{0,n}(t) - \varphi(t)| \leq |\varphi_n(t) - \varphi(t)|$, $|\tilde{\delta}_n - \delta| \leq |\hat{\delta}_n - \delta|$ and

$$Q_n(\vartheta; t) = R_n(\vartheta; t) + \varepsilon_n(\vartheta; t), \quad (7.2)$$

where for all $\vartheta \in (0, \infty) \times (0, 2] \times \mathbb{R}^p \times \mathcal{M}^p$, $\varepsilon_n(\vartheta; \cdot)$ is the complex-valued function defined for all $t \in \mathbb{R}^p$ by

$$\begin{aligned} \varepsilon_n(\vartheta; t) &= a(\varphi_{0,n}^{a-1}(t) - \varphi^{a-1}(t))e^{-ait'\tilde{\delta}_n} \sqrt{n} [\varphi_n(t) - \varphi(t)] \\ &+ a\varphi^{a-1}(t) \left(e^{-ait'\tilde{\delta}_n} - e^{-ait'\delta} \right) \sqrt{n} [\varphi_n(t) - \varphi(t)] \\ &+ \left(e^{-a^{1/\alpha}it'\tilde{\delta}_n} - e^{-a^{1/\alpha}it'\delta} \right) \sqrt{n} [\varphi_n(a^{1/\alpha}t) - \varphi(a^{1/\alpha}t)] \\ &- t' \sqrt{n} \left(\hat{\delta}_n - \delta \right) i \left[a \left(e^{-iat'\tilde{\delta}_n} - e^{-iat'\delta} \right) \varphi^a(t) - a^{1/\alpha} \left(e^{-ia^{1/\alpha}t'\tilde{\delta}_n} - e^{-ia^{1/\alpha}t'\delta} \right) \varphi(a^{1/\alpha}t) \right]. \end{aligned}$$

Now, one has easily that

$$\begin{aligned} &\int_{\mathbb{R}^p} |Q_n(\vartheta; t)|^2 w(\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt \\ &= \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt + \int_{\mathbb{R}^p} |\varepsilon_n(\vartheta; t)|^2 w(\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt \\ &+ \int_{\mathbb{R}^p} \varepsilon_n(\vartheta; t) \overline{R_n(\vartheta; t)} w(\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt + \int_{\mathbb{R}^p} R_n(\vartheta; t) \overline{\varepsilon_n(\vartheta; t)} w(\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt \\ &= \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt + \varpi_{1,n} + \varpi_{2,n} + \varpi_{3,n}. \end{aligned}$$

One then has to show that $\varpi_{i,n}$, $i = 1, 2, 3$, vanish in probability as n grows. For the first term, using (7.1), one can write

$$\begin{aligned} \varpi_{1,n} &\leq 16a^2 \int_{\mathbb{R}^p} |\varphi_{0,n}^{a-1}(t) - \varphi^{a-1}(t)|^2 \sqrt{n} |\varphi_n(t) - \varphi(t)|^2 w(t'\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt \\ &+ 16a^2 \int_{\mathbb{R}^p} \left| e^{-ait'\tilde{\delta}_n} - e^{-ait'\delta} \right|^2 \left| \sqrt{n} [\varphi_n(t) - \varphi(t)] \right|^2 w(t'\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt \\ &+ 16 \int_{\mathbb{R}^p} \left| e^{-a^{1/\alpha}it'\tilde{\delta}_n} - e^{-a^{1/\alpha}it'\delta} \right|^2 \left| \sqrt{n} [\varphi_n(a^{1/\alpha}t) - \varphi(a^{1/\alpha}t)] \right|^2 w(t'\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt \\ &+ 16 \int_{\mathbb{R}^p} \left[t' \sqrt{n} \left(\hat{\delta}_n - \delta \right) \right]^2 \\ &\quad \times \left| a \left(e^{-iat'\tilde{\delta}_n} - e^{-iat'\delta} \right) \varphi^a(t) - a^{1/\alpha} \left(e^{-ia^{1/\alpha}t'\tilde{\delta}_n} - e^{-ia^{1/\alpha}t'\delta} \right) \varphi(a^{1/\alpha}t) \right|^2 \\ &\quad \times w(t'\hat{\Sigma}_n^{1/2}) |\hat{\Sigma}_n^{1/2}| dt \\ &= \varpi_{1,1,n} + \varpi_{1,2,n} + \varpi_{1,3,n} + \varpi_{1,4,n}. \end{aligned}$$

Since the function $t \mapsto |\varphi_n^{a-1}(t) - \varphi^{a-1}(t)|$ is bounded on \mathbb{R}^p , by the change of variable $\tau = t'\widehat{\Sigma}_n^{1/2}$, one can write the following inequality

$$\varpi_{1,1,n} \leq C \sup_{t \in \mathbb{R}^p} |\varphi_n^{a-1}(t) - \varphi^{a-1}(t)|^2 \int_{\mathbb{R}^p} \left| \sqrt{n} \left[\varphi_n(t'\widehat{\Sigma}_n^{-1/2}) - \varphi(t'\widehat{\Sigma}_n^{-1/2}) \right] \right|^2 w(t) dt.$$

Observing that, by Lemma 7.1 the random integral in the right-hand side of the last inequality is tight and recalling from Theorem 3.2.1 of Ushakov (1999) that $\sup_{t \in \mathbb{R}^p} |\varphi_n^{a-1}(t) - \varphi^{a-1}(t)|^2$ goes almost surely to 0 as n tends to infinity, one can conclude that $\varpi_{1,1,n}$ tends in probability to 0.

For the convergence of $\varpi_{1,2,n}$, observe that the function $t \mapsto |e^{-ait'\widehat{\delta}_n} - e^{-ait'\delta}|^2 = [\cos(at'\widehat{\delta}_n) - \cos(at'\delta)]^2 + [\sin(at'\widehat{\delta}_n) - \sin(at'\delta)]^2$ is bounded on \mathbb{R}^p . Hence,

$$\varpi_{1,2,n} \leq C \sup_{t \in \mathbb{R}^p} \left| e^{-ait'\widehat{\delta}_n} - e^{-ait'\delta} \right|^2 \int_{\mathbb{R}^p} \left| \sqrt{n} [\varphi_n(t) - \varphi(t)] \right|^2 w(t'\widehat{\Sigma}_n^{1/2}) |\widehat{\Sigma}_n^{1/2}| dt.$$

By a change of variable, it is easy to see from Lemma 7.1 that the above random integral is tight. Since $\sup_{t \in \mathbb{R}^p} |e^{-ait'\widehat{\delta}_n} - e^{-ait'\delta}|^2$ tends in probability to 0 as n tends to infinity, one can conclude, as for $\varpi_{1,1,n}$, that $\varpi_{1,2,n}$ tends in probability to 0. The convergence in probability of $\varpi_{1,3,n}$ to zero can be handled in the same way. For the last term, using (7.1), write

$$\begin{aligned} \varpi_{1,4,n} &\leq C \left| \sqrt{n} (\widehat{\delta}_n - \delta) \right|^2 \left[a^2 \int_{\mathbb{R}^p} \left| e^{-iat'\widehat{\delta}_n} - e^{-iat'\delta} \right|^2 |t|^2 w(t'\widehat{\Sigma}_n^{1/2}) |\widehat{\Sigma}_n^{1/2}| dt \right. \\ &\quad \left. + a^{2/\alpha} \int_{\mathbb{R}^p} \left| e^{-ia^{1/\alpha}t'\widehat{\delta}_n} - e^{-ia^{1/\alpha}t'\delta} \right|^2 |t|^2 w(t'\widehat{\Sigma}_n^{1/2}) |\widehat{\Sigma}_n^{1/2}| dt \right]. \end{aligned} \quad (7.3)$$

Making once again the change of variable $\tau = t'\widehat{\Sigma}_n^{1/2}$, the first integral in the right-hand side of (7.3) can be bounded by

$$|\widehat{\Sigma}_n^{-1/2}|^2 \int_{\mathbb{R}^p} \left| e^{-iat'\Sigma_n^{-1/2}\widehat{\delta}_n} - e^{-iat'\Sigma_n^{-1/2}\delta} \right|^2 |t|^2 w(t) dt.$$

Since $\widehat{\delta}_n$ and $\Sigma_n^{-1/2}\widehat{\delta}_n$ tends in probability to δ and $\Sigma^{-1/2}\delta$ respectively, the term $|e^{-iat'\Sigma_n^{-1/2}\widehat{\delta}_n} - e^{-iat'\Sigma_n^{-1/2}\delta}|^2$, which is bounded by 4, tends in probability to 0. It is now easy to see by the Lebesgue convergence theorem, that the first integral in (7.3) tends in probability to 0. The convergence in probability to 0 of the second integral in (7.3) can be proved in the same way. By Remark 3.2, $\sqrt{n}(\widehat{\delta}_n - \delta)$ tends in distribution to a mean-zero p -dimensional Gaussian random vector. Whence, $\varpi_{1,4,n}$ tends in probability to 0 and so does $\varpi_{1,n}$.

To handle the convergence in probability of $\varpi_{2,n}$ and $\varpi_{3,n}$, write, by the Cauchy–Schwarz inequality

$$\varpi_{2,n} \leq C \varpi_{1,n}^{1/2} \left(\int_{\mathbb{R}^p} \left| \sqrt{n} [\varphi_n(t) - \varphi(t)] \right|^2 w(t'\widehat{\Sigma}_n^{1/2}) |\widehat{\Sigma}_n^{1/2}| dt \right)^{1/2}.$$

By Lemma 7.1, the second term in the right-hand side of the above inequality is tight. As $\varpi_{1,n}$ tends in probability to 0, so do $\varpi_{2,n}$ and $\varpi_{3,n} = \overline{\varpi}_{2,n}$. ■

Lemma 7.3. *Assume that (3.6)-(3.8) hold. Then, under \mathcal{H}_0 , for all $\vartheta_1 = (a, \alpha) \in (0, \infty) \times (0, 2]$,*

$$\Delta_{n,w}(\vartheta_1) = |\Sigma^{1/2}| \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \Sigma^{1/2}) dt + o_P(1).$$

Proof: Adding and subtracting appropriate terms, one obtains

$$\begin{aligned} & |\widehat{\Sigma}_n^{1/2}| \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \widehat{\Sigma}_n^{1/2}) dt \\ = & |\Sigma^{1/2}| \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \Sigma^{1/2}) dt + \left(|\widehat{\Sigma}_n^{1/2}| - |\Sigma^{1/2}| \right) \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \widehat{\Sigma}_n^{1/2}) dt \\ + & \left(|\widehat{\Sigma}_n^{1/2}| - |\Sigma^{1/2}| \right) \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 \left[w(t' \widehat{\Sigma}_n^{1/2}) - w(t' \Sigma^{1/2}) \right] dt \\ = & |\Sigma^{1/2}| \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \Sigma^{1/2}) dt + \theta_{1,n} + \theta_{2,n}. \end{aligned}$$

One has to show that $\theta_{1,n}$ and $\theta_{2,n}$ are $o_P(1)$'s. For the first, one can write the following inequality

$$|\theta_{1,n}| \leq C \left| |\widehat{\Sigma}_n^{1/2}| - |\Sigma^{1/2}| \right| \int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \widehat{\Sigma}_n^{1/2}) dt.$$

Hence $\theta_{1,n}$ tends in probability to 0, as the random integral is tight (apply Lemma 7.1) and $\left| |\widehat{\Sigma}_n^{1/2}| - |\Sigma^{1/2}| \right|$ tends in probability to 0.

To prove the convergence of $\theta_{2,n}$, by the triangle inequality and suitable changes of variable, one has :

$$\begin{aligned} |\theta_{2,n}| \leq & C \left| |\widehat{\Sigma}_n^{1/2}| - |\Sigma^{1/2}| \right| \\ & \times \left\{ |\widehat{\Sigma}_n^{-1/2}| \int_{\mathbb{R}^p} \left| R_n(\vartheta; t' \widehat{\Sigma}_n^{-1/2}) \right|^2 w(t) dt + |\Sigma^{-1/2}| \int_{\mathbb{R}^p} \left| R_n(\vartheta; t' \Sigma^{-1/2}) \right|^2 w(t) dt \right\}. \end{aligned}$$

By lemma 7.1, the random integrals in the brackets are tight. Since $\widehat{\Sigma}_n$ is consistent to Σ , $\left| |\widehat{\Sigma}_n^{1/2}| - |\Sigma^{1/2}| \right|$ tends in probability to 0 and so does $\theta_{2,n}$. ■

Proof of Theorem 3.3: By Lemmas 7.2 and 7.3, it suffices to prove that

$$\int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \Sigma^{1/2}) dt = \int_{\mathbb{R}^p} |S_n(\vartheta; t)|^2 w(t' \Sigma^{1/2}) dt + o_P(1).$$

For this, we first show that, for all $\vartheta = (a, \alpha) \in (0, \infty) \times (0, 2] \times \mathbb{R}^p \times \mathcal{M}^p$,

$$\int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t' \Sigma^{1/2}) dt = \int_{\mathbb{R}^p} |\widetilde{R}_n(\vartheta; t)|^2 w(t' \Sigma^{1/2}) dt + o_P(1),$$

with

$$\begin{aligned}\tilde{R}_n(\vartheta; t) &= a\tilde{\varphi}^{a-1}(t)\sqrt{n}[\tilde{\varphi}_n(t) - \tilde{\varphi}(t)] - \sqrt{n}[\tilde{\varphi}_n(a^{1/\alpha}t) - \tilde{\varphi}(a^{1/\alpha}t)] \\ &\quad - it' \frac{1}{\sqrt{n}} \sum_{j=1}^n \Pi(\vartheta_2; X_j^* + \delta) \Psi(\vartheta; t), \quad t \in \mathbb{R}^p.\end{aligned}$$

Recall that $\tilde{\varphi}_n$ and $\tilde{\varphi}$ are respectively the empirical and the characteristic functions of the X_j^* 's. Clearly, using (3.6) one has easily, for all $t \in \mathbb{R}^p$,

$$\begin{aligned}R_n(\vartheta; t) &= a\tilde{\varphi}^{a-1}(t)\sqrt{n}[\tilde{\varphi}_n(t) - \tilde{\varphi}(t)] - \sqrt{n}[\tilde{\varphi}_n(a^{1/\alpha}t) - \tilde{\varphi}(a^{1/\alpha}t)] \\ &\quad - it' \frac{1}{\sqrt{n}} \sum_{j=1}^n \Pi(\vartheta_2; X_j^* + \delta) \Psi(\vartheta; t) - it'r_n \Psi(\vartheta; t).\end{aligned}$$

Now, using this expression and integrating $|R_n(\vartheta; t)|^2$ with respect to $w(t'\Sigma^{1/2})dt$ yields

$$\begin{aligned}&\int_{\mathbb{R}^p} |R_n(\vartheta; t)|^2 w(t'\Sigma^{1/2})dt \\ &= \int_{\mathbb{R}^p} |\tilde{R}_n(\vartheta; t)|^2 w(t'\Sigma^{1/2})dt - i \int_{\mathbb{R}^p} t'r_n \tilde{R}_n(\vartheta; t) \overline{\Psi(\vartheta; t)} w(t'\Sigma^{1/2})dt \\ &\quad + i \int_{\mathbb{R}^p} t'r_n \overline{\tilde{R}_n(\vartheta; t)} \Psi(\vartheta; t) w(t'\Sigma^{1/2})dt + \int_{\mathbb{R}^p} (t'r_n)^2 |\Psi(\vartheta; t)|^2 w(t'\Sigma^{1/2})dt.\end{aligned}\quad (7.4)$$

We have to show that the second, third and fourth terms in the right-hand side of (7.4) are all asymptotically negligible. To handle the last one, observe that

$$\int_{\mathbb{R}^p} (t'r_n)^2 |\Psi(\vartheta_1; t)|^2 w(t'\Sigma^{1/2})dt \leq C|r_n|^2 \int_{\mathbb{R}^p} |t|^2 w(t)dt.$$

Since r_n tends in probability to 0 and the integral in the right-hand side of the above inequality is finite, the last term in the right-hand side of (7.4) tends in probability to 0. The change of variable $\tau = t'\Sigma^{1/2}$ in the second and third terms and the fact that $|\Psi(\vartheta; t)|^2 \leq C$ allow to see, after applying the Cauchy–Schwarz inequality to each of them, that they can be bounded by

$$C \left(\int_{\mathbb{R}^p} |\tilde{R}_n(\vartheta; t'\Sigma^{-1/2})|^2 w(t)dt \right)^{1/2} \left(|r_n|^2 \int_{\mathbb{R}^p} |t|^2 w(t)dt \right)^{1/2},$$

which tends in probability to 0, since the first factor is tight by Lemma 7.1 and the second tends in probability to 0.

Note that as the X_j^* 's are symmetric around 0, $\tilde{\varphi}$ is a real-valued function. In the present setting, it has the form

$$\tilde{\varphi}(t) = e^{-(t'\Sigma t)^{\alpha/2}}, \quad t \in \mathbb{R}^p.\quad (7.5)$$

It is easy to check that expanding $|\tilde{R}_n(\vartheta; t)|^2$ and integrating with respect to $w(t'\Sigma^{1/2})dt$, the use of the assumption (3.9) and the equality (7.5) yields :

$$\begin{aligned}
& \int_{\mathbb{R}^p} |\tilde{R}_n(\vartheta; t)|^2 w(t'\Sigma^{1/2}) dt = \\
& n^{-1} \left\{ \int_{\mathbb{R}^p} a^2 \tilde{\varphi}^{2(a-1)}(t) \left[\sum_{j,k} \cos[t'(X_j^* - X_k^*)] - 2n\tilde{\varphi}(t) \sum_j \cos(t'X_j^*) + n^2 \tilde{\varphi}^2(t) \right] \right. \\
& \quad - a\tilde{\varphi}^{a-1}(t) \left[2 \sum_{j,k} \cos[t'(X_j^* - a^{1/\alpha} X_k^*)] - 2n\tilde{\varphi}(t) \sum_j \cos(a^{1/\alpha} t' X_j^*) \right] \\
& \quad \left. - 2n\tilde{\varphi}(a^{1/\alpha} t) \sum_j \cos(t' X_j^*) + 2n^2 \tilde{\varphi}(t) \tilde{\varphi}(a^{1/\alpha} t) \right. \\
& \quad + \sum_{j,k} \cos[a^{1/\alpha} t'(X_j^* - X_k^*)] - 2n\tilde{\varphi}(a^{1/\alpha} t) \sum_j \cos(a^{1/\alpha} t' X_j^*) + n^2 \tilde{\varphi}^2(a^{1/\alpha} t) \\
& \quad \left. - 2a\tilde{\varphi}^{a-1}(t) \sum_{j,k} \sin(t' X_j^*) t' \Pi(\vartheta_2; X_k^* + \delta) \Psi(\vartheta; t) \right. \\
& \quad \left. - 2 \sum_{j,k} \sin(a^{1/\alpha} t' X_j^*) t' \Pi(\vartheta_2; X_k^* + \delta; \delta) \Psi(\vartheta; t) \right. \\
& \quad \left. + \sum_{j,k} t' \Pi(\vartheta_2; X_j^* + \delta) t' \Pi(\vartheta_2; X_k^* + \delta) |\Psi(\vartheta; t)|^2 \right\} w(t'\Sigma^{1/2}) dt. \tag{7.6}
\end{aligned}$$

Now, expanding $S_n^2(\vartheta; t)$ (see the equation below (3.10)) and using $\cos(c-d) = \cos(c)\cos(d) + \sin(c)\sin(d)$, even and odd functions appear in the resulting expression. Integrating this expression with respect to $w(t'\Sigma^{1/2})dt$ under the condition (3.9), integrals with odd integrand vanish and one obtains the right-hand side of (7.6). This ends the proof of Theorem 3.3. ■

Proof of Theorem 3.4: As in Gürtler & Henze (2000) or Matsui & Takemura (2008), we first work in $C(\Theta \rightarrow \mathbb{R})$, the space of real-valued continuous function defined on a compact subset Θ of \mathbb{R}^p endowed with the supremum norm $\|u\|_\infty = \sup_{t \in \Theta} |u(t)|$. For all $(x, t) \in \mathbb{R}^p \times \Theta$, define the real-valued function

$$k(x; t) = a\tilde{\varphi}^{a-1}(t) [\cos(t'x) + \sin(t'x)] - [\cos(a^{1/\alpha} t'x) + \sin(a^{1/\alpha} t'x)] - t' \Pi(\vartheta_2; x + \delta) \Psi(\vartheta; t).$$

Recall that \tilde{F} is the cumulative distribution function of $X_j^* = X_j - \delta$, $j = 1, 2, \dots, n$. It can be checked easily that the function $(s, t) \mapsto k(s, t)$ satisfies the requirements of Csörgö (1983):

- the function $x \mapsto k(x; t)$ is Borel measurable on \mathbb{R}^p for any $t \in \Theta$;

- the function $t \mapsto k(x; t)$ is continuous on Θ for almost all x with respect to $d\tilde{F}$;
- the transformation $t \mapsto \int_{\mathbb{R}^p} k(x; t) d\tilde{F}(x) = a\tilde{\varphi}^a(t) - \tilde{\varphi}(a^{1/\alpha}t)$ is a continuous function on the compact set Θ .

Denote by $\tilde{F}_n(x)$, the empirical distribution function of the X_j^* 's. It is a trivial matter that $S_n(\vartheta; t)$ has the representation

$$S_n(\vartheta; t) = \int_{\mathbb{R}^p} k(x; t) d \left\{ \sqrt{n} \left[\tilde{F}_n(x) - \tilde{F}(x) \right] \right\}. \quad (7.7)$$

Hence, $S_n(\vartheta; \cdot)$ can be seen as a random element of $C(\Theta \rightarrow \mathbb{R})$. The study of its weak convergence can be obtained using the results of Csörgö (1983) after checking conditions $(i)^*$ and $(ii)^*$ in that paper. The condition $(i)^*$ immediately holds. Indeed, for every $\epsilon > 0$, by (7.1), the moment assumption (3.7) and the continuity of the function $t \mapsto t\Psi(\vartheta; t)$ on the compact set $\Theta \subset \mathbb{R}^p$, one has that

$$\int_{\mathbb{R}^p} \sup_{t \in \Theta} |k(x; t)|^{2+\epsilon} d\tilde{F}(x) < \infty.$$

For checking the second condition $(ii)^*$, adding and subtracting appropriate terms, one has, for all $s, t \in \Theta$,

$$\begin{aligned} k(x; s) - k(x; t) &= a \left[\tilde{\varphi}^{a-1}(s) - \tilde{\varphi}^{a-1}(t) \right] \left[\cos(t'x) + \sin(t'x) \right] \\ &\quad + a\tilde{\varphi}^{a-1}(s) \left\{ [\cos(t'x) - \cos(s'x)] + [\sin(t'x) - \sin(s'x)] \right\} \\ &\quad - \left\{ [\cos(a^{1/\alpha}t'x) - \cos(a^{1/\alpha}s'x)] + [\sin(a^{1/\alpha}t'x) - \sin(a^{1/\alpha}s'x)] \right\} \\ &\quad - \left\{ s'\Pi(\vartheta_2; x + \delta) \right\} \Psi(\vartheta; s) - t'\Pi(\vartheta_2; x + \delta) \Psi(\vartheta; t). \end{aligned} \quad (7.8)$$

By a first-order Taylor expansion of the complex-valued function $z \mapsto z^{a-1}$, and by the fact that $\tilde{\varphi}$ is the characteristic function of a random vector symmetric around 0, for some positive constant C_1 , the first term in (7.8) can be bounded by

$$2a(a-1) |E[\cos(t'X_1^*) - \cos(s'X_1^*)]| \leq C_1 |t - s|^{\gamma/2}, \quad s, t \in \Theta.$$

Proceeding as above, one has, for some positive constant of the same nature as C_1 :

$$|\Psi(\vartheta; s) - \Psi(\vartheta; t)| \leq C_1 \left[|\tilde{\varphi}(t) - \tilde{\varphi}(s)| + |\tilde{\varphi}(a^{1/\alpha}t) - \tilde{\varphi}(a^{1/\alpha}s)| \right] \leq C_1 |t - s|^{\gamma/2}.$$

Using the last inequality, it is easy to see that for some positive constants C_2 and C_3 , the last term in (7.8) can be bounded by

$$C |t - s|^{\gamma/2} (C_2 |t - s|^{1-\gamma/2} + C_3) |\Pi(\vartheta; x + \delta)|.$$

Finally, it is easy to see that for some positive constant C_4 , each of the remaining terms in (7.8) can be bounded by $C_4 |t - s|^{\gamma/2} |x|^{\gamma/2}$, so that

$$|k(x; s) - k(x; t)| \leq C |t - s|^{\gamma/2} M(x, v(s, t)), \quad (7.9)$$

where for all $(s, t) \in \Theta^2$, $v(s, t) = s - t$ and for all $(x, t) \in \mathbb{R}^p \times \Theta$, $M(x, t) = C_1 + C_4|x|^{\gamma/2} + (C_2|t|^{1-\gamma/2} + C_3)|\Pi(\vartheta; x + \delta)|$ trivially satisfies

$$\int_{\mathbb{R}^p} \sup_{t \in \Theta} M^2(x, t) d\tilde{F}(x) < \infty.$$

Hence, $S_n(\vartheta; \cdot)$ satisfies the required conditions of Csörgö (1983). Thus, it converges weakly to a zero-mean Gaussian process $S(\vartheta; \cdot)$ with covariance kernel as stated in Theorem 3.3. Since the compact set Θ is arbitrary, this weak convergence also holds in the Fréchet space $C(\mathbb{R}^p \rightarrow \mathbb{R})$ endowed with the metric ρ defined earlier. As indicated in Gürtler & Henze (2000), this extension can be seen by adapting the reasoning of Karatzas & Shreve (1988), p.62. This concludes the proof of the result. ■

Proof of Theorem 3.9: We first study the convergence of $S_n(\vartheta, \cdot)$ on under \mathcal{H}_1^n , as a random element of $C(\Theta \rightarrow \mathbb{R})$, for an arbitrary compact set $\Theta \subset \mathbb{R}^p$. For this, we study its finite-dimensional distributions and its tightness under \mathcal{H}_1^n .

It easy to check that under \mathcal{H}_0 , for Λ_n given by (3.14),

$$\lim_{n \rightarrow \infty} \text{Cov}(S_n(\vartheta; t), \Lambda_n) = \int_{\mathbb{R}^p} k(x, t) f(x + \delta) h(x + \delta) dx = c(\vartheta; t), \quad t \in \mathbb{R}^p$$

and that for all $k \in \mathbb{N}$ and for $t_1, \dots, t_k \in \mathbb{R}^p$, the joint limiting distribution of the $(k + 1)$ -dimensional random vector $(S_n(\vartheta; t_1), \dots, S_n(\vartheta; t_k), \Lambda_n)'$ is Gaussian with mean $(0, \dots, 0, -\sigma^2/2)'$ and covariance matrix $\begin{pmatrix} \Phi & \varrho \\ \varrho' & \sigma^2 \end{pmatrix}$ where $\Phi = (\Gamma(\vartheta; t_\ell, t_m) : 1 \leq \ell, m \leq k)$ and $\varrho = (c(\vartheta; t_1), \dots, c(\vartheta; t_k))'$, with $\Gamma(\vartheta; \cdot)$ given by (3.11) and $c(\vartheta; \cdot)$ given by (3.15). Whence, by Le Cam's third lemma, under \mathcal{H}_1^n , the finite-dimensional distributions of $S_n(\vartheta; \cdot)$ converge to those of $\tilde{S}(\vartheta; \cdot)$. Since $S_n(\vartheta; \cdot)$ converges weakly under \mathcal{H}_0 , it is tight under \mathcal{H}_0 . Thus, by contiguity, it is also tight under \mathcal{H}_1^n . Hence, under \mathcal{H}_1^n , $S_n(\vartheta; \cdot)$ converges weakly to $\tilde{S}(\vartheta; \cdot)$. Its weak convergence in the Fréchet space $C(\mathbb{R}^p \rightarrow \mathbb{R})$ can be obtained as indicated in the proof of Theorem 3.4. ■

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