A NOTE ON FUZZY $R$-SUBGROUPS OF NEAR-RINGS

BY

KYUNG HO KIM AND YOUNG BAE JUN

Abstract. We formulate fuzzy characteristic right (resp. left) $R$-subgroups and the fuzzy same type right (resp. left) $R$-subgroups of a near-ring $R$, and give its characterization.

1. Introduction

Fuzzy ideals of a ring are studied by W. Liu [13], and several concepts are investigated in [4, 9, 11, 16]. The fuzzification of a subnear-ring and a left (resp. right) ideal in a near-ring is discussed in [1, 5, 6, 7, 9, 10]. The concept of $R$-subgroups of a near-ring is introduced by S. Abou-Zaid [1], and present authors [8] investigated further properties of fuzzy right (resp. left) $R$-subgroups of a near-ring $R$. The aim of this paper is to formulate fuzzy characteristic right (resp. left) $R$-subgroups and the fuzzy same type right (resp. left) $R$-subgroups of a near-ring $R$, and to give its characterization.

2. Preliminaries

A near-ring ([15]) is defined to be a non-empty set $R$ with two binary operations “+” and “.” satisfying the following axioms:

(i) $(R, +)$ is a group,

(ii) $(R, \cdot)$ is a semigroup,

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

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Because of the condition (iii), it is called also a left near-ring. In this paper, we will use the word “near-ring” in stead of “left near-ring”. We denote $xy$ instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in R$. A two sided $R$-subgroup of a near-ring $R$ is a subset $H$ of $R$ such that

(i) $(H, +)$ is a subgroup of $(R, +)$,
(ii) $RH \subset H$,
(iii) $HR \subset H$.

If $H$ satisfies (i) and (ii) then it is called a left $R$-subgroup of $R$. If $H$ satisfies (i) and (iii) then it is called a right $R$-subgroup of $R$.

We now review some fuzzy logic concepts (see [2], [16] and [17] for details).

A fuzzy set $\mu$ in a set $R$ is a function $\mu : R \rightarrow [0, 1]$.

Let $\text{Im}(\mu)$ denote the image set of $\mu$. Let $\mu$ be a fuzzy set in a set $R$. For $t \in [0, 1]$, the set $R^t_\mu := \{x \in R | \mu(x) \geq t\}$ is called a level subset of $\mu$. In what follows the letter $R$ denotes a near-ring unless otherwise specified. Let $\mu$ be a fuzzy set in $R$. We say that $\mu$ is a fuzzy subnear-ring of $R$ if, for all $x, y \in R$,

(F1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
(F2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

If a fuzzy set $\mu$ in $R$ satisfies the property (F1) then $\mu(0) \geq \mu(x)$ for all $x \in R$ (see [9, Lemma 2.3]).

Let $f$ be a mapping from a set $R$ to a set $S$ and let $\mu$ be a fuzzy set in $R$. Then $f(\mu)$, the image of $\mu$, is a fuzzy set in $S$ defined by

$$f(\mu)(y) := \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise}, \end{cases}$$

for all $y \in S$.

3. Fuzzy Characteristic and Fuzzy Same Type $R$-Subgroups

Definition 3.1. ([1]) A fuzzy set $\mu$ in $R$ is called a fuzzy right (resp. left) $R$-subgroup of $R$ if

(F3) $\mu$ is a fuzzy subgroup of $(R, +)$,
(F4) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$), for all $r, x \in R$. 


Recall that a fuzzy set $\mu$ in $R$ is a fuzzy right (resp. left) $R$-subgroup of $R$ if and only if the level subset $R^\mu_t$ is a right (resp. left) $R$-subgroup of $R$, which is called a level right (resp. left) $R$-subgroup of $R$, where $t \in \text{Im}(\mu)$ (see [8]).

**Example 3.2.** ([8]) Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

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Then $(R, +, \cdot)$ is a near-ring. We define a fuzzy subset $\mu : R \to [0, 1]$ by $\mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then $\mu$ is a fuzzy subgroup of $(R, +)$, and we have that $\mu(xr) \geq \mu(x)$ for all $r, x \in R$. Hence $\mu$ is a fuzzy right $R$-subgroup of $R$.

**Example 3.3.** ([8]) Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

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Then $(R, +, \cdot)$ is a near-ring. We define a fuzzy subset $\mu : R \to [0, 1]$ by $\mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then $\mu$ is also a fuzzy right $R$-subgroup of $R$.

For an endomorphism $f$ of $R$ and a fuzzy set $\mu$ in $R$, we define a new fuzzy set $\mu^f$ in $R$ by $\mu^f(x) = \mu(f(x))$ for all $x \in R$.

**Proposition 3.4.** Let $f$ be an endomorphism of $R$. If $\mu$ is a fuzzy right (resp. left) $R$-subgroup of $R$, then so is $\mu^f$.

**Proof.** Let $x, y \in R$. Then

\[
\mu^f(x - y) = \mu(f(x - y)) = \mu(f(x) - f(y)) \\
\geq \min\{\mu(f(x)), \mu(f(y))\} \\
= \min\{\mu^f(x), \mu^f(y)\},
\]
and for every \( r, x \in R \) we have
\[
\mu^f(xr) = \mu(f(xr)) = \mu(f(x)f(r)) \geq \mu(f(x)) = \mu^f(x),
\]
resp.
\[
\mu^f(rx) = \mu(f(rx)) = \mu(f(r)f(x)) \geq \mu(f(x)) = \mu^f(x).
\]
Hence \( \mu^f \) is a fuzzy right (resp. left) \( R \)-subgroup of \( R \).

**Definition 3.5.** A right (resp. left) \( R \)-subgroup \( H \) of \( R \) is called a characteristic right (resp. left) \( R \)-subgroup of \( R \) if \( f(H) = H \) for all \( f \in \text{Aut}(R) \).

**Definition 3.6.** A fuzzy right (resp. left) \( R \)-subgroup \( H \) of \( R \) is said to be fuzzy characteristic if \( \mu^f(x) = \mu(x) \) for all \( x \in R \) and all \( f \in \text{Aut}(R) \).

For a family of fuzzy sets \( \{\mu_i : i \in \Lambda\} \), the intersection \( \bigwedge_{i \in \Lambda} \mu_i \) is defined by
\[
(\bigwedge_{i \in \Lambda} \mu_i)(x) := \inf \{\mu_i(x) : i \in \Lambda\}.
\]

**Proposition 3.7.** If \( \{\mu_i : i \in \Lambda\} \) is a family of fuzzy characteristic right (resp. left) \( R \)-subgroups of a near-ring \( R \), then \( \bigwedge_{i \in \Lambda} \mu_i \) is a fuzzy characteristic right (resp. left) \( R \)-subgroup of \( R \).

**Proof.** Recall that \( \bigwedge_{i \in \Lambda} \mu_i \) is a fuzzy right (resp. left) \( R \)-subgroup of \( R \) (see [8, Proposition 3.4]). Let \( x \in R \) and \( f \in \text{Aut}(R) \). Then
\[
(\bigwedge_{i \in \Lambda} \mu_i)^f(x) = (\bigwedge_{i \in \Lambda} \mu_i)(f(x))
= \inf \{\mu_i(f(x)) : i \in \Lambda\}
= \inf \{\mu_i^f(x) : i \in \Lambda\}
= \inf \{\mu_i(x) : i \in \Lambda\}
= (\bigwedge_{i \in \Lambda} \mu_i)(x).
\]
Hence \( (\bigwedge_{i \in \Lambda} \mu_i)^f \) is a fuzzy characteristic right (resp. left) \( R \)-subgroup of \( R \).

**Lemma 3.8.** Let \( \mu \) be a fuzzy right (resp. left) \( R \)-subgroup of \( R \) and let \( x \in R \). Then \( \mu(x) = t \) if and only if \( x \in R^t_\mu \) and \( x \notin R^s_\mu \) for all \( s > t \) in \([0, 1]\).

**Proof.** Straightforward.

**Theorem 3.9.** For a fuzzy right (resp. left) \( R \)-subgroup of \( R \), the following statements are equivalent:
(i) \( R \) is fuzzy characteristic.

(ii) Each level right (resp. left) \( R \)-subgroup of \( \mu \) is characteristic.

**Proof.** (i) \( \Rightarrow \) (ii): Let \( t \in \text{Im}(\mu), f \in \text{Aut}(R) \) and \( x \in R^t_\mu \). Then \( \mu^f(x) = \mu(x) \geq t \), i.e., \( \mu(f(x)) \geq t \) and so \( f(x) \in R^t_\mu \) which shows that \( f(R^t_\mu) \subseteq R^t_\mu \).

Now let \( x \in R^t_\mu \) and let \( y \in R \) be such that \( f(y) = x \). Then \( \mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t \), showing that \( y \in R^t_\mu \). Thus \( x = f(y) \in f(R^t_\mu) \), proving that \( R^t_\mu \subseteq f(R^t_\mu) \). Therefore \( f(R^t_\mu) = R^t_\mu \) and \( R^t_\mu \) is characteristic.

(ii) \( \Rightarrow \) (i): Let \( x \in R, f \in \text{Aut}(R) \) and \( \mu(x) = t \). Then, by virtue of Lemma 3.8, \( x \in R^t_\mu \) and \( x \notin R^s_\mu \) for all \( s > t \) in \([0, 1]\). It follows from hypothesis that \( f(x) \in f(R^t_\mu) = R^t_\mu \) so that \( \mu^f(x) = \mu(f(x)) \geq t \). Let \( s = \mu^f(x) \) and assume that \( s > t \). Then \( f(x) \in R^s_\mu = f(R^s_\mu) \), which implies from the injectivity of \( f \) that \( x \in R^s_\mu \), a contradiction. Hence \( \mu^f(x) = \mu(f(x)) = t = \mu(x) \) showing that \( \mu \) is fuzzy characteristic.

**Lemma 3.10.** ([8, Proposition 3.5]) Let \( H \) be a non-empty subset of a near-ring \( R \) and let \( \mu \) be a fuzzy set in \( R \) defined by

\[
\mu(x) := \begin{cases} 
   t_1, & \text{if } x \in H, \\
   t_2, & \text{otherwise},
\end{cases}
\]

where \( t_1 > t_2 \) in \([0, 1]\). Then \( \mu \) is a fuzzy right (resp. left) \( R \)-subgroup if and only if \( H \) is a right (resp. left) \( R \)-subgroup.

**Proposition 3.11.** Let \( \mu \) be a fuzzy set in Lemma 3.10. If \( H \) is a characteristic right (resp. left) \( R \)-subgroup of \( R \), then \( \mu \) is a fuzzy characteristic right (resp. left) \( R \)-subgroup of \( R \).

**Proof.** Let \( x \in R \) and \( f \in \text{Aut}(R) \). If \( x \in H \), then \( f(x) \in f(H) = H \) and so \( \mu^f(x) = \mu(f(x)) = t_1 = \mu(x) \). Otherwise, we have \( \mu^f(x) = \mu(f(x)) = t_2 = \mu(x) \). Hence \( \mu \) is fuzzy characteristic.

Using a given fuzzy right (resp. left) \( R \)-subgroup, we construct a new fuzzy right (resp. left) \( R \)-subgroup. Let \( t \geq 0 \) be a real number. If \( m \in [0, 1] \), \( m^t \) shall mean the non-negative \( t \)-th root in case \( t < 1 \). We define \( \mu^f : R \to [0, 1] \) by \( \mu^f(x) = (\mu(x))^t \).
Lemma 3.12. ([8, Theorem 3.20]) If \( \mu \) is a fuzzy right (resp. left) \( R \)-subgroup of \( R \), then \( \mu^t \) is also a fuzzy right (resp. left) \( R \)-subgroup of \( R \).

Proposition 3.13. If \( \mu \) is a fuzzy characteristic right (resp. left) \( R \)-subgroup of \( R \), then so is \( \mu^t \) for all \( t > 0 \).

Proof. For any \( f \in \text{Aut}(R) \) and \( x \in R \) we have
\[
(\mu^t)^f(x) = \mu^t(f(x)) = (\mu(f(x)))^t = (\mu^f(x))^t = (\mu(x))^t = \mu^t(x),
\]
for all \( t > 0 \), ending the proof.

Definition 3.14. Let \( \mu \) and \( \nu \) be fuzzy right (resp. left) \( R \)-subgroup of \( R \). Then \( \mu \) is said to be fuzzy same type with \( \nu \) if there exists \( f \in \text{Aut}(R) \) such that \( \mu = \nu \circ f \), i.e., \( \mu(x) = \nu(f(x)) \) for all \( x \in R \).

We note that for all fuzzy right (resp. left) \( R \)-subgroups \( \mu, \nu \) and \( \lambda \) in \( R \), (i) \( \mu \) is fuzzy same type with \( \nu \) itself, (ii) if \( \mu \) is fuzzy same type with \( \nu \), then \( \nu \) is fuzzy same type with \( \mu \), and (iii) if \( \mu \) is fuzzy same type with \( \nu \) and if \( \nu \) is fuzzy same type with \( \lambda \), then \( \mu \) is fuzzy same type with \( \lambda \).

Theorem 3.15. For any right (resp. left) \( R \)-subgroups \( \mu \) and \( \nu \) in \( R \), the following are equivalent:

(i) \( \mu \) is fuzzy same type with \( \nu \),
(ii) \( \mu \circ f = \nu \) for some \( f \in \text{Aut}(R) \),
(iii) \( g(\mu) = \nu \) for some \( g \in \text{Aut}(R) \),
(iv) \( h(\nu) = \mu \) for some \( h \in \text{Aut}(R) \),
(v) \( \exists h \in \text{Aut}(R) \) such that \( R\mu^t = h(R\nu^t) \) for all \( t \in [0, 1] \).

Proof. (i) \( \Rightarrow \) (ii). Let \( \mu \) be fuzzy same type with \( \nu \). Then \( \nu \) is fuzzy same type with \( \mu \). Hence (ii) follows from the Definition 3.14.

(ii) \( \Rightarrow \) (iii). Assume that \( \mu \circ f = \nu \) for some \( f \in \text{Aut}(R) \). Then \( \mu(f(x)) = \nu(x) \) for all \( x \in R \). It follows that
\[
f^{-1}(\mu)(x) = \sup_{y \in f(x)} \mu(y) = \mu(f(x)) = \nu(x),
\]
for all \( x \in R \). Taking \( g = f^{-1} \), then \( g \in \text{Aut}(R) \) and \( g(\mu) = \nu \).
(iii) \(\Rightarrow\) (iv). Assume that (iii) holds. Then

\[
\nu(x) = g(\mu)(x) = \sup_{y \in g^{-1}(x)} \mu(y) = \mu(g^{-1}(x)),
\]

for all \(x \in R\). Hence

\[
g^{-1}(\nu)(x) = \sup_{y \in g(x)} \nu(y) = \nu(g(x)) = \mu(g^{-1}(g(x))) = \mu(x),
\]

for all \(x \in R\). If we take \(h = g^{-1}\), then \(h \in \text{Aut}(R)\) and \(h(\nu) = \mu\).

(iv) \(\Rightarrow\) (v). If there exists \(h \in \text{Aut}(R)\) such that \(h(\nu) = \mu\), then

\[
\mu(x) = h(\nu)(x) = \sup_{y \in h^{-1}(x)} \nu(y) = \nu(h^{-1}(x)),
\]

for all \(x \in R\). Let \(t \in [0, 1]\). We need to show that \(R^t_\mu = h(R^t_\nu)\). If \(x \in R^t_\mu\), then \(\nu(h^{-1}(x)) = \mu(x) \geq t\) which implies that \(h^{-1}(x) \in R^t_\nu\), i.e., \(x \in h(R^t_\nu)\). This shows that \(R^t_\mu \subseteq h(R^t_\nu)\). Now let \(x \in h(R^t_\nu)\). Then \(h^{-1}(x) \in R^t_\nu\), and so \(\mu(x) = \nu(h^{-1}(x)) \geq t\). It follows that \(x \in R^t_\mu\). Hence \(h(R^t_\mu) \subseteq R^t_\mu\) and (v) holds.

(v) \(\Rightarrow\) (i). Suppose that there exists \(h \in \text{Aut}(R)\) such that \(R^t_\mu = h(R^t_\nu)\) for all \(t \in [0, 1]\). Let \(x \in R\) and \(\mu(x) = t\). Then \(h^{-1}(x) \in h^{-1}(R^t_\mu) = R^t_\nu\) and so \(\nu(h^{-1}(x)) \geq t = \mu(x)\). Putting \(\nu(h^{-1}(x)) = s\), then \(h^{-1}(x) \in R^s_\nu\) and hence \(x \in h(R^s_\nu) = R^s_\mu\). It follows that \(\mu(x) \geq s = \nu(h^{-1}(x))\), Hence \(\mu(x) = \nu(h^{-1}(x))\) for all \(x \in R\). Noticing that \(h^{-1} \in \text{Aut}(R)\), then \(\mu\) is fuzzy same type with \(\nu\). This completes the proof.

**Theorem 3.16.** Let \(\mu\) and \(\nu\) be fuzzy right (resp. left) \(R\)-subgroups of \(R\) such that \(\mu\) is fuzzy same type with \(\nu\). Then \(\mu\) is isomorphic to \(\nu\).

**Proof.** Since \(\mu\) is fuzzy same type with \(\nu\), there exists \(\phi \in \text{Aut}(R)\) such that \(\mu(x) = \nu(\phi(x))\) for all \(x \in R\). Let \(f : \mu(R) \to \nu(R)\) such that \(f(\mu(x)) = \nu(\phi(x))\) for all \(x \in R\). Then for every \(x, y \in R\), we have

\[
f(\mu(x + y)) = \nu(\phi(x + y)) = \nu(\phi(x) + \phi(y)),
\]

and

\[
f(\mu(xy)) = \nu(\phi(xy)) = \nu(\phi(x)\phi(y)).
\]

If \(f(\mu(x)) = f(\mu(y))\) for all \(x, y \in R\), then \(\nu(\phi(x)) = \nu(\phi(y))\) and hence \(\mu(x) = \mu(y)\), showing that \(f\) is injective. This completes the proof.
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