From Uniform Continuity to Absolute Continuity

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The notion of uniform continuity emerged slowly in the lectures of Dirichlet(1854) and of Weierstrass(1861) [1]. Then in 1905, Vitali established the absolute continuity for a class of functions in the paper "Sulle funzioni integrali" [2]. Sooner or later, several equivalent and sufficient conditions for a function to be absolutely continuous were derived (see [3]), which depend on several results of measure theory and integration. For example, Banach-Zarecki Criterion[4] states that, "f is absolutely continuous on $[a, b]$ if and only if f is continuous and of bounded variation on $[a, b]$ and maps null sets to null sets. ", and in particular, every Lipschitz continuous function is absolutely continuous.

Absolute continuity implies uniform continuity, but generally not vice versa. However, under certain conditions (piecewise convexity), uniform continuity will also imply absolute continuity. In this short note, we will present a sufficient condition for a uniformly continuous function to be absolutely continuous.

Definition 1 (piecewise convex function) *A function f defined on an interval $I_{a,b}$ of the real line is piecewise convex, if there exits a finite partition $P = \{a_i\}_{i=0}^N$ such that on each subinterval $[a_i, a_{i+1}]$ $(i = 0, \dots, N-1)$, f is concave or convex.*

Remark: Here $I_{a,b}$ is an interval of $\mathbb{R}$ with a, b as the endpoints and it can be open, closed or half-open; moreover, a, b can also take the value $\pm\infty$ when a or b is not contained in $I_{a,b}$ (in this case, we use $(a_0, a_1]$ or $[a_{N-1}, a_N)$ instead of the closed subinterval).

Theorem 1 *For a uniformly continuous function f defined on $I_{a,b}$ of $\mathbb{R}$, if it is piecewise convex, then it is also absolutely continuous on $I_{a,b}$.*

Remark: We can verify that if $I_{a,b} = [a, b]$, then the conditions in Theorem 1 satisfy the Banach-Zarecki Criterion. However, our attempt is based on some elementary properties of uniform continuity and convexity. In particular, one simple example is that $f(x) = \sqrt{x}$ is absolutely continuous (not Lipschitz continuous) on $[0, c]$. Moreover, the converse statement of this theorem is false. One proper counterexample is $f(x) = x^2 \sin(1/x)$ on $[0, 1]$ (define $f(0) = 0$). In addition, the cantor function[6] is uniformly continuous but not absolutely continuous.

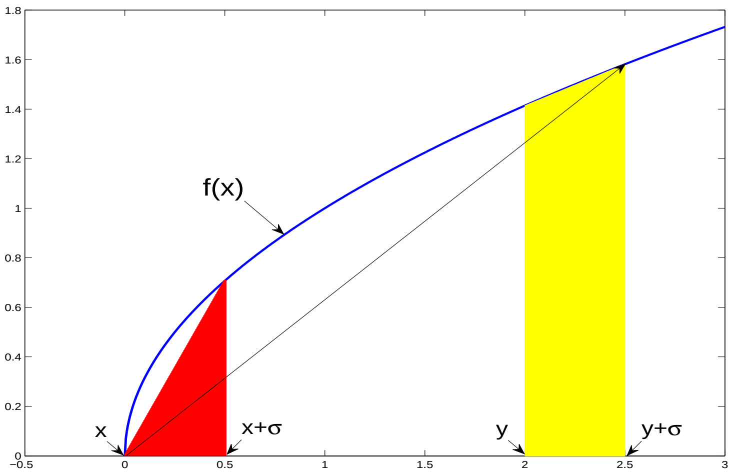

Figure 1: graph of $f(x) = \sqrt{x}$ when $\sigma = 0.5$

This picture gives us the idea of Lemma 1 that the slope of an increasing concave function of two points with the same distance of the x coordinate will decrease. In addition, the monotonicity of the change of slopes will still be valid for other monotone convex or concave functions.

Lemma 1 *If f is monotone and concave or convex on an interval I of $\mathbb{R}$, then $G_\sigma(x) = |f(x+\sigma) - f(x)|$ is monotone with respect to x, where σ is any positive real number.*

Proof. We only demonstrate the case that $f(x)$ is monotone increasing and concave, other situations can be proved similarly. Obviously, in this case, $G_\sigma(x) = f(x+\sigma) - f(x)$ for $\sigma > 0$. We will show that $G_\sigma(x)$ is decreasing on I with respect to x.

Suppose $x, y+\sigma \in I$, and $x < y$. Since $f(x)$ is concave, for any $\theta \in (0,1)$, we get

$$f(\theta a + (1-\theta)b) \geq \theta f(a) + (1-\theta)f(b), \quad \forall a, b \in I. \tag{1}$$

1) Set $a = x$, $b = y+\sigma$, $\theta = \frac{y-x}{y-x+\sigma}$: By (1) , we have

$$f(x+\sigma) \geq \frac{y-x}{y-x+\sigma} f(x) + \frac{\sigma}{y-x+\sigma} f(y+\sigma),$$

which is equivalent to

$$\frac{f(y+\sigma) - f(x)}{y-x+\sigma} \leq \frac{f(x+\sigma) - f(x)}{\sigma}. \tag{2}$$

2) Set $a = x$, $b = y+\sigma$, $\theta = \frac{\sigma}{y-x+\sigma}$: Similarly, we can get

$$\frac{f(y+\sigma) - f(y)}{\sigma} \leq \frac{f(y+\sigma) - f(x)}{y-x+\sigma}. \tag{3}$$

By (2) and (3) , we obtain

$$\frac{f(y+\sigma) - f(y)}{\sigma} \leq \frac{f(x+\sigma) - f(x)}{\sigma}.$$

Thus, $G_\sigma(y) \leq G_\sigma(x)$, which implies that $G_\sigma(x)$ is a monotone decreasing function.

Remark: Lemma 1 is one special case of a classical property of convex or concave functions on an interval of $\mathbb{R}$, see [5].

Lemma 2 *For a monotone concave or convex function $f(x)$ defined on an interval I of $\mathbb{R}$, if f is uniformly continuous, then f is absolutely continuous.*

Proof. Since f is uniformly continuous on I, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in I$ and $|x-y| < \delta$, we have

$$|f(x) - f(y)| < \epsilon.$$

Now, we consider every finite collection $\{(x_i, y_i)\}_{i=1}^n$ of nonoverlapping subintervals of I with $x_i < x_{i+1}$, and

$$\sum_{i=1}^{n} (y_i - x_i) < \delta.$$

Define $\sigma_i = y_i - x_i > 0$, then

$$|f(y_i) - f(x_i)| = G_{\sigma_i}(x_i).$$

By Lemma 1, we know that $G_\sigma(x)$ is monotone.

1) $G_\sigma(x)$ is monotone deceasing. Since (x_2, y_2) and (x_1, y_1) are nonoverlapping, we get

$$G_{\sigma_2}(x_2) \leq G_{\sigma_2}(y_1) = G_{\sigma_2}(x_1 + \sigma_1).$$

Inductively, for $i \geq 2$, we will obtain

$$G_{\sigma_i}(x_i) \leq G_{\sigma_i}(x_1 + \sum_{j=1}^{i-1} \sigma_j).$$

Therefore, we have

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| = \sum_{i=1}^{n} G_{\sigma_i}(x_i) \leq G_{\sigma_1}(x_1) + \sum_{i=2}^{n} G_{\sigma_i}(x_1 + \sum_{j=1}^{i-1} \sigma_j).$$

Define $z_1 = x_1$, and $z_i = x_1 + \sum_{j=1}^{i-1} \sigma_j$ $(2 \leq i \leq n+1)$. Since f is monotone, the above inequality is equivalent to

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| \leq \sum_{i=1}^{n} |f(z_{i+1}) - f(z_i)| = |f(z_{n+1}) - f(z_1)|.$$

In addition, $|z_{n+1} - z_1| = \sum_{i=1}^{n} \sigma_i = \sum_{i=1}^{n} (y_i - x_i) < \delta$, since f is uniformly continuous, we obtain

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| \leq |f(z_{n+1}) - f(z_1)| < \epsilon.$$

Hence, f is also absolutely continuous on I.

2) $G_\sigma(x)$ is monotone increasing. The strategy is quite similar to the previous one; however, we just fix (x_n, y_n) first and define $z_{n+1} = y_n$, $z_i = y_n - \sum_{j=i}^{n} \sigma_j$ for $1 \leq i \leq n$. Similarly, we have

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| \leq \sum_{i=1}^{n} |f(z_{i+1}) - f(z_i)| = |f(z_{n+1}) - f(z_1)| < \epsilon.$$

Therefore, f is absolutely continuous on I.

Remark: The idea of this lemma is to glue disjoint subintervals (x_i, y_i) together as one subinterval then apply the property of uniform continuity. By Lemma 2, it is clear that $f(x) = \sqrt{x}$ is absolutely continuous on $[0, c]$. In addition, we can consider more general functions that oscillate finite times, and on each monotone subinterval, they also admit convexity. If these functions are uniformly continuous, then they are also absolutely continuous by utilizing the same strategy in the proof of Lemma 2 on each subinterval.

Proof of Theorem 1. If f is not monotone on $[a_i, a_{i+1}]$, then we can split $[a_i, a_{i+1}]$ into two subintervals such that on each of them, f is monotone, since f assumes convexity. If we relabel them, then on each $[a_i, a_{i+1}]$, f is monotone and concave or convex. The following proof is based on this situation.

Since f is uniformly continuous on I, for any $\epsilon/N > 0$, there exists $\delta > 0$ such that for any $x, y \in I$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \epsilon/N.$$

Choose $\delta_1 < \min\{a_1 - a_0, \ldots, a_N - a_{N-1}, \delta\}$, then for every finite collection $\{(x_i, y_i)\}_{i=1}^{n}$ of nonoverlapping subintervals of I with $x_i < x_{i+1}$ and $\sum_{i=1}^{n} (y_i - x_i) < \delta_1$, each subinterval (x_i, y_i) can contain at most one a_j. For such interval (x_i, y_i) containing a_j, by triangle inequality, we have

$$|f(y_i) - f(x_i)| \leq |f(y_i) - f(a_j)| + |f(a_j) - f(x_i)|. \tag{4}$$

Now, we still treat (x_i, a_j) and (a_j, y_i) as two "nonoverlapping subintervals". Then, we relabel all the subintervals as $\{(x_j, y_j)\}_{j=1}^{m}$ $(n \leq m \leq 2n)$. Through the above strategy, each new (x_j, y_j) will lie exactly in one $[a_i, a_{i+1}]$ and

$$\sum_{j=1}^{m} (y_j - x_j) = \sum_{i=1}^{n} (y_i - x_i) < \delta_1.$$

For all the new subintervals that lie in $[a_i, a_{i+1}]$ $(i = 0, \ldots, N-1)$, since f is also monotone and concave or convex and

$$\sum_{(x_j,y_j)\subset[a_i,a_{i+1}]} (y_j - x_j) < \delta_1 < \delta,$$

by the method in Lemma 2, we obtain that

$$\sum_{(x_j,y_j)\subset[a_i,a_{i+1}]} |f(y_j) - f(x_j)| < \epsilon/N.$$

Therefore,

$$\sum_{j=1}^{m} |f(y_j) - f(x_j)| = \sum_{i=0}^{N-1} \sum_{(x_j,y_j)\subset[a_i,a_{i+1}]} |f(y_j) - f(x_j)| < \epsilon. \tag{5}$$

By (4) and (5) , we get

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| \leq \sum_{j=1}^{m} |f(y_j) - f(x_j)| < \epsilon.$$

Hence, f is also absolutely continuous on $I_{a,b}$.

Question: We consider the case on the real line; however, one can think about the situation for the multidimensional Euclidean space.

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