Bin Packing and Cutting Stock Problems: Mathematical Models and Exact Algorithms

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AIRO 2014

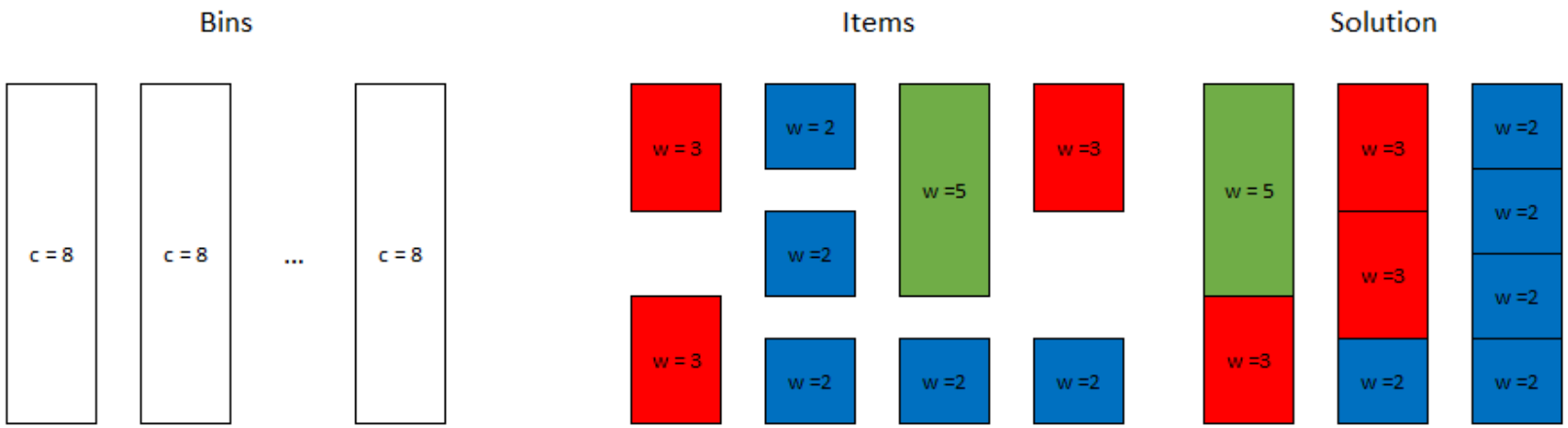
Outline

1. Introduction
2. Most common exact methods for solving the BPP and the CSP
3. Computational Results
4. Conclusion & Future Research

The Bin Packing Problem (BPP)

Classical Bin Packing Problem

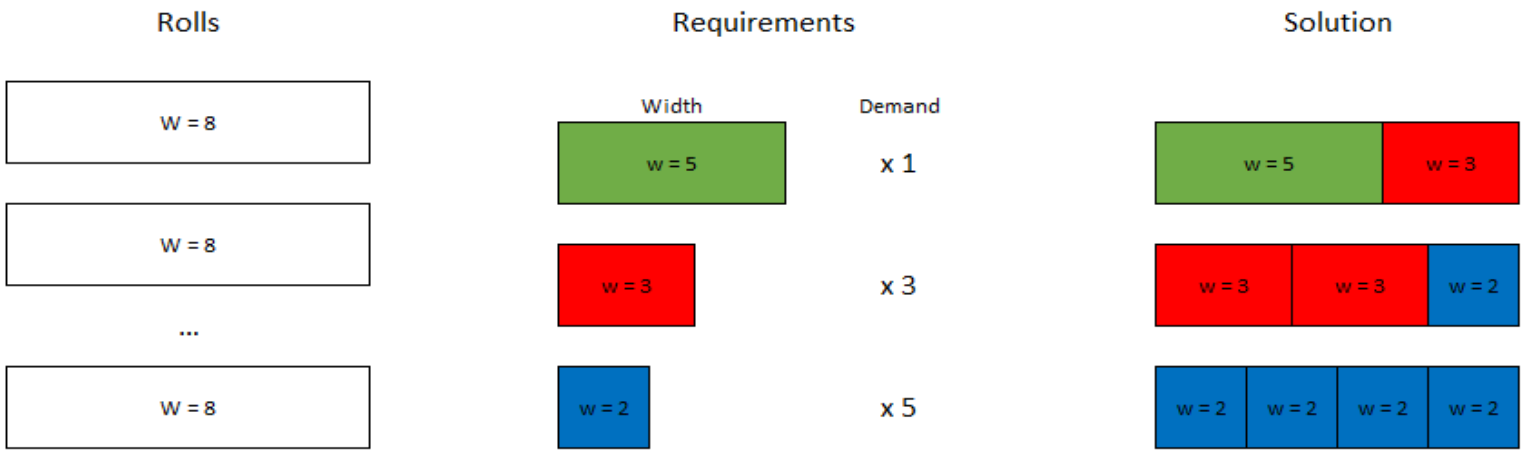
Given a set of weighted items and an unlimited number of identical capacitated bins, the Bin Packing Problem consists in packing all the items into the minimum number of bins.



The Cutting Stock Problem (CSP)

Classical Cutting Stock Problem

Given a set of order requirements, each requirement consisting in a demand and a width, and an unlimited number of identical rolls, the classical cutting stock problem consists of determining the smallest number of rolls that have to be cut in order to satisfy all the demands.

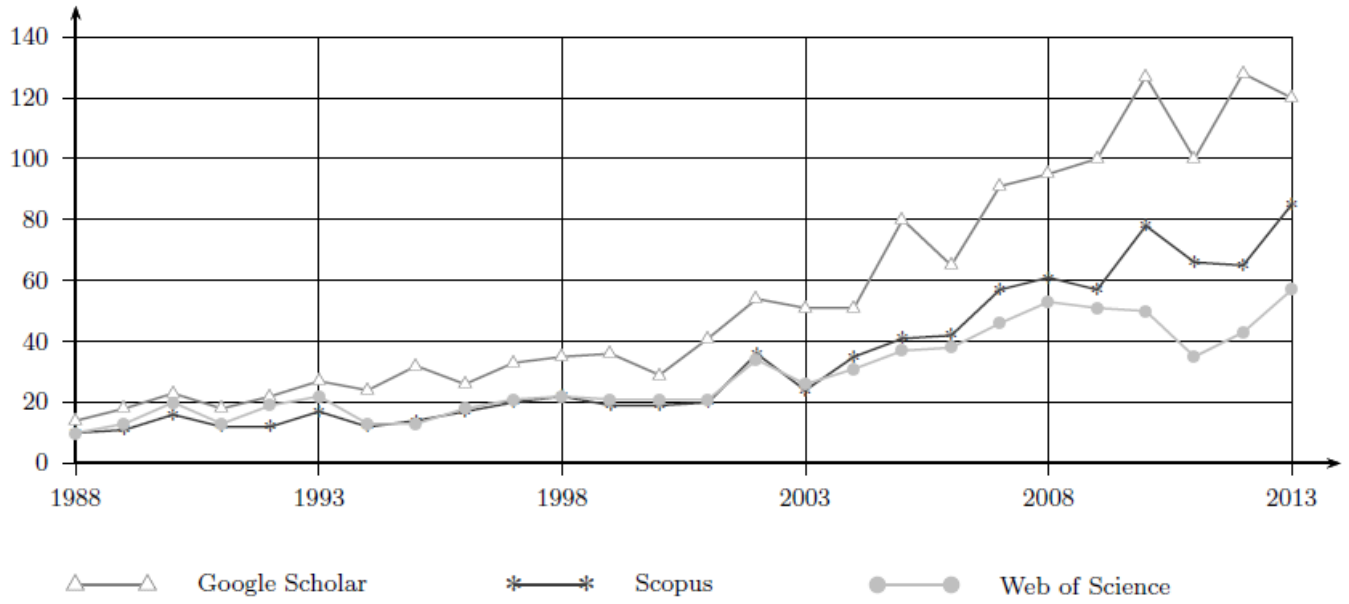


Some applications for the BPP and the CSP

- Cutting materials in industry (wood, paper, aluminium ...)
  - Kantorovich, Mathematical methods of organizing and planning production, *Management Science*, 1960 (originally from 1939)
  - Stadler, A one-dimensional cutting stock problem in the aluminium industry and its solution, *European Journal of Operational Research*, 1990
- Loading when 1 dimension is considered (file storage on computers, container loading ...)
- Solving more difficult problems (vehicle routing, multi-dimensional BPP, ...)

Context of our work

- BPP and CSP are more and more studied by researchers
  The subject is of interest
- Many techniques can be used to solve the BPP and the CSP
  A survey is relevant
- All the literature instances are solved
  A will to create a new computational challenge



Objectives of our work

- Gathering in a survey the most important articles
- Implementing some of the most common methods (Kantorovich model, Branch-and-Bound, Branch-and-Price, Constraint Programming, Pseudo-Polynomial model ...)
- Study the behaviour of those methods when the parameters of the test instances change
- Try to create new instances that are difficult to solve in practice

Kantorovich Mathematical Formulation

Kantorovich, Mathematical methods of organizing and planning production, *Management Science*, 1960 (originally from 1939)

$$\min \quad \sum_{i=1}^{m} y_i \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} w_j x_{ij} \leq c y_i \quad (i = 1, \ldots, m), \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad (j = 1, \ldots, n), \tag{3}$$

$$y_i \in \{0, 1\} \quad (i = 1, \ldots, m), \tag{4}$$

$$x_{ij} \in \{0, 1\} \quad (i = 1, \ldots, m; j = 1, \ldots, n). \tag{5}$$

Easy to implement but very bad in practice

Branch & Bound algorithms

Martello, Toth, Knapsack Problems: Algorithms and Computer Implementations, *Wiley*, 1990
Scholl, Klein, Jürgens, BISON: a fast hybrid procedure for exactly solving the one-dimensional BPP, *Computers & Operations Research*, 1997

Main idea: Explore (in a "smart" way) through an enumeration tree all the feasible packings
Smart because:

- Use of reduction procedures
- Use of a set of specially designed Lower Bounds (L1, L2, L3 ... L6)
- Use of fast and good heuristics to get Upper Bounds

Usually very good for small instances ($n \leq 100$).

Branch & Price algorithms

Gilmore, Gomory, A linear programming approach to the CSP, *Operations research*, 1963
Vance, Barnhart, Johnson, Nemhauser, Solving binary CSP by column generation and branch-and-bound, *Computational optimization and applications*, 1994
Belov, Scheithauer, A branch-and-cut-and-price algorithm for one-dimensional stock cutting and two-dimensional two-stage cutting, *European Journal of Operational Research*, 2006

Main idea: Consider the Bin Packing Problem as a Set Covering Problem

Set Covering Problem (SCP)

Given a set of elements $\{1, 2, ..., n\}$ (called the universe) and a family S of m sets whose union equals the universe, the set covering problem is to identify the smallest subfamily of S whose union equals the universe.

- The elements $\{1, 2, ..., n\}$ are the objects
- The family S is composed by every feasible bin (column generation used to generate bins)

Very good method because it has a very tight lower bound: 99% of the time equal to the optimum, conjectured to be at most one bin away from the optimum (called non IRUP).

Constraints Programming

Shaw, A constraint for bin packing, 2004
Schaus, Régin, Van Schaeren, Dullaert, Raa, Cardinality reasoning for bin-packing constraint: Application to a tank allocation problem, 2012

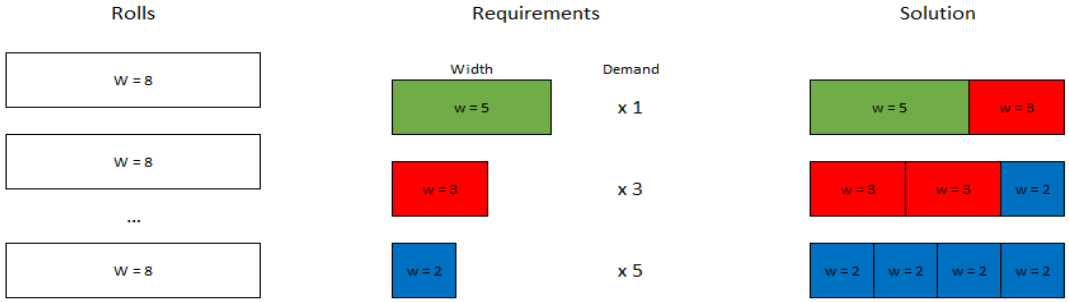
Main idea: Use properties of feasible solutions to define the search domain. Shaw did a very good job and created a constraint implemented in CPLEX.

Generally, Constraints Programming is not the most powerful method for solving the BPP and it does not provide any lower bound. But it can be useful when you add some additional constraints on the objects (For example, bins have different sizes, there are incompatibilities among items ...).

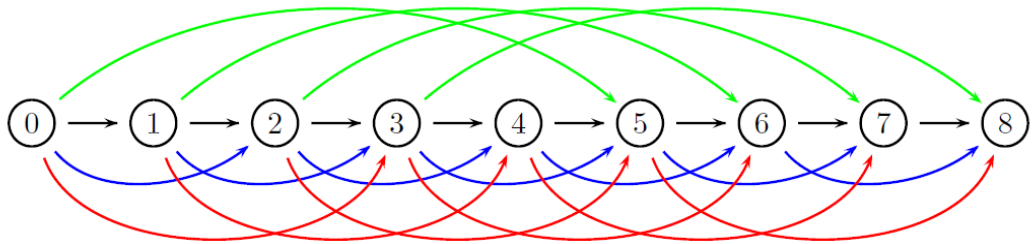
Pseudo Polynomial Models based on graphs

Cambazard, O'Sullivan, Propagating the bin packing constraint using linear programming, 2010
Valerio de Carvalho, Exact solution of bin-packing problems using column generation and branch-and-bound, *Annals of Operations Research* 1999
Brandão, Pedroso, Bin Packing and Related Problems: General Arc-flow Formulation with Graph Compression, 2013

Main idea: Consider a bin as a path in a graph where arcs are items

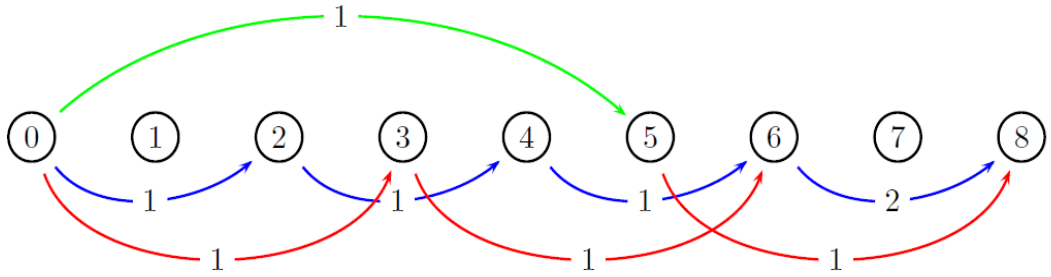


Graph of possible flows (without simplifications)



Pseudo Polynomial Models based on graphs

Graph of the solution



Model

$$\min \quad z \tag{6}$$

$$\text{s.t.} \quad -\sum_{(d,e)\in A'} x_{de} + \sum_{(e,f)\in A'} x_{ef} = \begin{cases} z & \text{if } e = 0, \\ -z & \text{for } e = W, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

$$\sum_{(d,d+w_i)\in A'} x_{d,d+w_i} \geq b_i, \quad i = 1, \ldots, m, \tag{8}$$

$$x_{de} \geq 0 \text{ and integer} \quad \forall (d,e) \in A', \tag{9}$$

Very good models because there are many simplifications, but it can become very bad when c increases (c $>$ 10000)

Instances

- New instances (3840)
  - Variable number of items: 50,...,1000
  - Variable bin capacity: 50,...,1000
  - Variable parameters for the distribution of the items
  - 10 instances for each combination
- Literature instances (2154) (available online)
  - 1D Bin-Packing Problems from Sholl, Klein and Juergens
  - The 28 VERY hard BPP instances of Schoenfield
  - 1D Bin-Packing Problems from Schwerin and Waescher (1998).
  - 1D Bin-Packing Problems from Waescher and Gau (1996).
  - 1D-bar relaxations of the two-dimensional bin packing test dataset of Lodi et al. (1999).
  - Instances taken from a real application in a paper tube industry in Japan.
  - 1D Bin-Packing Problems from Falkenauer (1996).

Results for generated instances

Number of instances for which optimal solution has been found in less than 60 seconds, on a Intel®, 1 core and 3.1 Ghz

N \ Method	Brand-and-Bound			Branch-and-Price		Pseudo-Polynomial			Other	
	CVRPSEP	MTP	BISON	VANCE	BELOV	ARCFLOW	ARCFLOWCOMP	ONECUT	ILP(KM)	CP
50	479	478	480	480	480	480	480	480	472	367
100	445	452	466	479	480	480	480	480	443	323
200	331	358	410	470	480	480	480	474	329	272
300	227	252	349	438	479	480	480	463	191	211
400	190	207	299	402	480	477	480	456	124	150
500	185	187	273	356	480	479	480	450	118	144
750	140	140	259	123	480	469	479	428	97	97
1000	109	119	234	102	480	470	467	437	73	43
Total	2106	2193	2770	2850	3839	3815	3826	3668	1847	1607

- Best methods seem to be the B&P algorithm of Belov and the Pseudo Polynomial models
- Randomly generated instances seem to be very easy to solve

Results for literature instances

Number of instances for which optimal solution has been found in less than 60 seconds, on a Intel®, 1 core and 3.1 Ghz

Instances \ Method	Total	Brand-and-Bound			Branch-and-Price		Pseudo-Polynomial			Other	
		CVRPSEP	MTP	BISON	VANCE	BELOV	ARCFLOW	ARCFLOWCOMP	ONECUT	ILP(KM)	CP
Waesher	17	0	3	10	3	17	0	4	0	2	10
Hard28	28	0	7	0	7	24	21	25	11	0	0
Falkenauer	160	28	107	91	107	124	160	160	159	24	54
Schwerin / Waesher	200	7	195	163	195	200	200	200	199	93	163
Scholl	1210	917	1115	1157	1115	1210	1042	1105	875	924	890
1D-Bar	500	497	500	500	500	500	500	500	500	498	482
Fiber	39	39	39	39	39	39	39	39	39	39	39
Total	2154	1488	1966	1960	1966	2114	1962	2033	1783	1580	1638

- the B&P algorithm of Belov seem to have more difficulties on triplets instances (optimal solution has 3 items in each bin)
- the Pseudo Polynomial algorithm seem to have more difficulties on instances with a huge capacity

Conclusion & Future Research

Conclusion

- According to our results, the best methods to solve the BPP and the CSP seem to be the pseudo-polynomial models and the B&P algorithm of Belov et al.
- The most difficult instances for these algorithms are those whith a huge capacity and the triplets

Future Research

- Try other well-known B&P algorithms from the literature
- Build non IRUP instances with triplets and a huge capacity
- Test the new instances with all the previous methods

THANK YOU