Beat phenomenon at the arrival of a guided mode in a semi-infinite acoustic duct

P. Gatignol[a], M. Bruneau[b], P. Lanceleur[a] and C. Potel[b]

[a]Université de Technologie de Compiègne, Laboratoire Roberval, BP 20529, 60205 Compiegne, France
[b]Laboratoire d'acoustique de l'université du Maine, Bât. IAM - UFR Sciences Avenue Olivier Messiaen 72085 Le Mans Cedex 9
philippe.gatignol@utc.fr

A guided mode, generated at an initial time in the terminal section of a semi-infinite acoustic duct, will undergo dispersion effects during its propagation. It is well-known that at an observation point in the duct, after the arrival of a small precursor that moves with the speed of sound, the main part of the signal does arrive with the group velocity relative to the driving frequency. The axial propagation of the mode is governed by a Klein-Gordon equation and the problem may be solved by a Laplace transform. A careful numerical evaluation of the inverse Laplace integral in the complex plane shows that the amplitude of the mode yields oscillations at the first stage of its arrival. A steepest descent evaluation of the Laplace integral allows to explain that phenomenon in terms of beats between the main wave and the modes with neighboring frequencies.

1 Introduction

In this paper, we study the generation of acoustic modes in a semi-infinite fluid-filled pipe initially at rest (no acoustic movement), when convenient boundary conditions are given over the normal section at the entrance of the pipe, for positive values of time. These boundary conditions are specific of the particular mode which is expected to be generated, and the driving frequency is greater than the cut off frequency of that mode. This problem has been extensively studied in the literature (see for example [1,2]). Since the guided mode is dispersive, the main signal will arrive at a given location in the pipe with a delay compared to the time of flight which is associated with the sound speed. It is well known that the group velocity, calculated for the driving frequency, is that velocity with which the mode does propagate. However, the detailed description of what happens when the wave mode reaches a given point in the pipe does not seem to have retained major attention in previous publications. It is the aim of the present paper to show that after a period of increase, the amplitude of the mode will undergo a number of oscillations before it reaches its final value, and to show what physical process explains these oscillations.

2 The problem and its solution

Acoustic modes in cylindrical pipes with normal section of any shape (see Figure 1), governed by the following wave equation for the velocity potential ϕ,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0\,, \qquad (1)$$

may be described analytically by separating the space and time variables, namely (see [3]) :

$$\phi\big(x,y,z,t\big)=\psi(x,y)\,f(z,t)\,. \qquad (2)$$

The modal function $\psi(x,y)$ satisfies the reduced wave equation of the form

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\kappa^2\psi=0\,, \qquad (3)$$

and is subjected to conditions on the frontier Γ that express the physical behaviour of the pipe boundary.

This eigen value problem has an infinite discrete number of solutions $\psi_N(x,y)$ with the modal wave numbers κ_N. These wave numbers κ_N are real if the pipe boundary is purely reactive, which will be assumed henceforth.

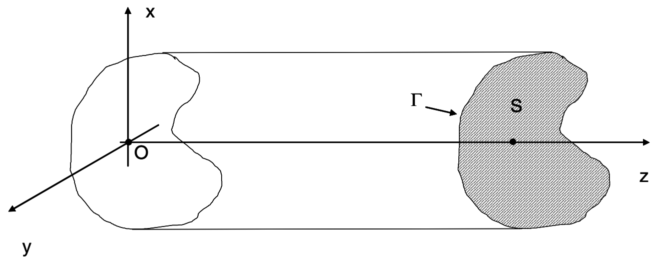

Figure 1: Acoustic pipe with normal section of any shape.

2.1 The equation of modal propagation

The propagation of the modal wave labeled N along the pipe is now governed by the Klein-Gordon equation for the function $f(z,t)$

$$\frac{\partial^2 f}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}-\kappa_N^2 f=0\,, \qquad (4)$$

where c is the sound speed in the fluid.

On the emitting normal section $z=0$, it is assumed that, after an initial time value $t=0$, the axial component of the acoustical velocity is given with an amplitude repartition on the section that fits the modal pattern $\psi_N(x,y)$, and with a sinusoïdal time dependence with angular frequency ω. This frequency ω is chosen greater than the cut off frequency of the modal wave N: $\omega_N=c\,\kappa_N$. The fluid is at rest until the instant $t=0$ (no acoustic signal). The axial velocity component may be written

$$w(x,y,z;t)=\psi(x,y)\,\frac{\partial f}{\partial z}(z,t)=\psi(x,y)\,g(z,t)\,, \qquad (5)$$

where the function $g(z,t)$ is also a solution of equation (4), which must satisfy the following conditions :

- for $t=0$ (fluid at rest in the semi-infinite pipe)

$$g(z,0)=0\,,\ \ \frac{\partial g}{\partial t}(z,0)=0\quad(\forall z\geq 0)\ \ ; \qquad (6)$$

- for $z=0$ (emitting section)

$$g(0,t)=\sin(\omega t)\,,\quad \omega>\omega_N=c\,\kappa_N\quad(\forall t\geq 0)\,, \qquad (7)$$

where the amplitude of the emitted signal has been normalized to unity.

2.2 Analytical solution of the problem

The function $g(z,t)$, as solution of equation (4) with conditions (6) and (7), may be found by Laplace transform (see [4])

$$G(z,s) = \mathscr{L}\{g(z,t)\} = \int_0^{+\infty} g(z,t)\, e^{-st}\, dt\,. \qquad (8)$$

The transformed function $G(z,s)$ is then obtained in the following form

$$G(z,s) = \frac{\omega}{s^2+\omega^2}\, e^{-z\sqrt{\frac{s^2}{c^2}+\kappa_N^2}}\,. \qquad (9)$$

By the use of theinverse Laplace transform, the solution for $g(z,t)$ may be expressed by means of an integral representation in the plane of the complex variable $s = a + ib$:

$$g(z,t) = \frac{\omega}{2\pi i}\int_{Br} \frac{e^{st - z\sqrt{\frac{s^2}{c^2}+\kappa_N^2}}}{s^2+\omega^2}\, ds\,, \qquad (10)$$

where, at this stage, the integration path Br is the Bromwich straight line parallel to the imaginary axis, as shown in Figure 2.

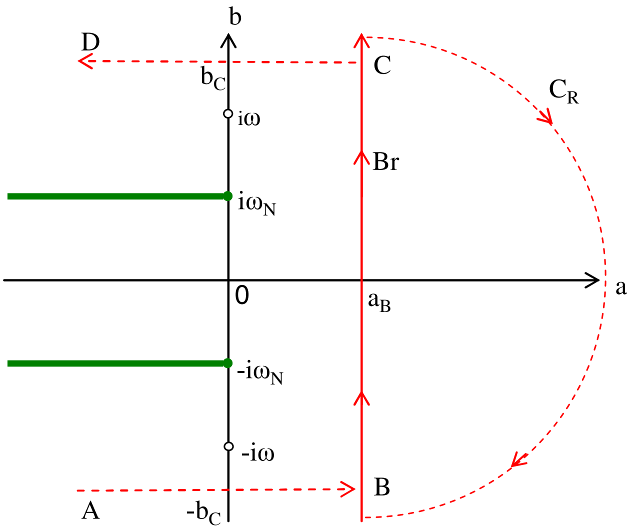

Figure 2: The complex plane for Laplace transform.

It must be noticed that the function to be integrated in (10) has two branch points in $b = \pm i\omega_N$. The function is one-valued in the plane limited to the two cut half lines that are chosen as parallel to the negative real axis. Also, the function has two poles $b = \pm i\omega$, which are outside the interval $\left[-i\omega_N, i\omega_N\right]$ due to the assumption on the value of the driving frequency.

3 Numerical evaluation of the inverse Laplace integral

First we get a numerical evaluation of the integral (10) for a given value $z > 0$ of the observation location and for any time $t \geq 0$.

3.1 Choice of the integration path

The following argument is quite usual. For $0 \leq t < z/c$, the function in the integral (10) is infinitely small on the half circle C_R (see Figure 2) with the radius R as large as desired. Since the function is analytic in the half plane $a > 0$, it may be deduced, by Cauchy theorem, that the integral (10) is equal to zero. No signal may arrive at the point z before the characteristic time $t_c = z/c$.

For $t \geq t_c$, this integral is no longer equal to zero. However, its direct numerical evaluation along the straight line Br would need to keep a very large part of this path, due to the slow decreasing of the function in the direction of the imaginary axis. But again Cauchy theorem allows to replace this straight line by the path (ABCD) shown in Figure 2. Strictly speaking, the points A and D should be at infinity. But in the direction of the negative real axis, with the hypothesis $t \geq z/c$, the function in the integral decreases very fastly. So these half lines may be drastically truncated, which makes the numerical evaluation very fast.

However, the geometric parameters a_B and b_C, as well as the discretization step, need to be carefully chosen in order to obtain results with enough good precision.

3.2 The numerical result

The result obtained through this numerical evaluation is shown in Figure 3 where the time evolution of the signal, for a given value z of the observation point, is presented.

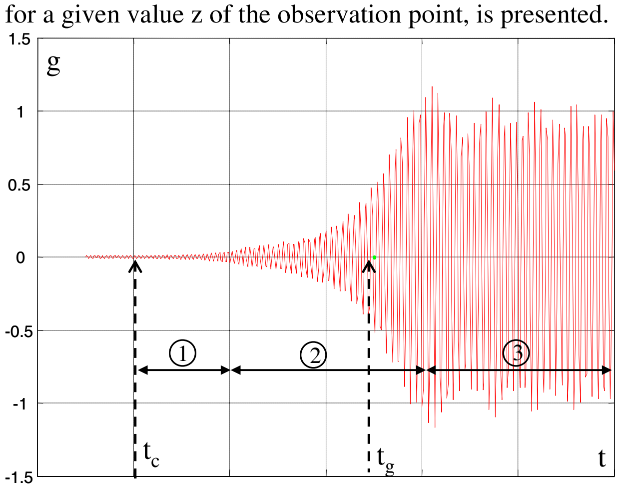

Figure 3: The time signal $g(z,t)$ for a given value z.

After the characteristic time t_c, during the phase ①, the signal appears with a very small amplitude ; this part of the signal is usually called the "precursor". We will see that it corresponds to the high frequency part of the wave train. This part is difficult to describe numerically, due to comparable numerical noise, as it may be seen in Figure 3 where the computation has been initialized before the characteristic time, when the solution should be equal to zero.

Then the phase ② corresponds to a relatively fast growth of the signal amplitude. This phase extends on both sides of the "group time" t_g, i.e. the time of flight, from the emitting section to the point z, calculated with the group velocity of the N mode at the driving frequency.

During the phase ③, the signal has reached its nominal amplitude 1 but this amplitude oscillates about this value. These oscillations gradually decrease until they become negligible in the final phase ④ of the signal (not shown in Figure 3) when the mode propagates with the constant emitted amplitude. This temporary oscillation phenomenon will be discussed in the next section.

4 Asymptotic evaluation of Laplace integral

The previous description of the arrival of the modal wave at the point z may be well understood, from the physical point of view, by an asymptotic evaluation of the Laplace integral (10).

4.1 The steepest descent method

Indeed, the Laplace integral may be evaluated by the so-called "steepest descent" method in the complex plane of the variable s (see [5]) if we assume that the observation point is far enough from the emitting section, namely if the condition $\kappa_N z \gg 1$ is fullfilled. By Cauchy theorem, the integration path Br may be replaced by that path along which the real part of the argument function of the exponential in the integral (10)

$$h(s;z,t) = st - z\sqrt{\frac{s^2}{c^2} + \kappa_N^2} \ , \qquad (11)$$

increases from minus infinity and then decreases to minus infinity as fast as possible. When the real part reaches its maximum value, this particular path goes across a saddle point. For $\kappa_N z \gg 1$, the main contribution to the value of the integral (10) comes from the integration along the path in the neighborhood of the saddle point.

The steepest descent path in the case of the integral (10), for values of time greater than the characteristic time t_c, is shown in Figure 4. It is made of two branches.

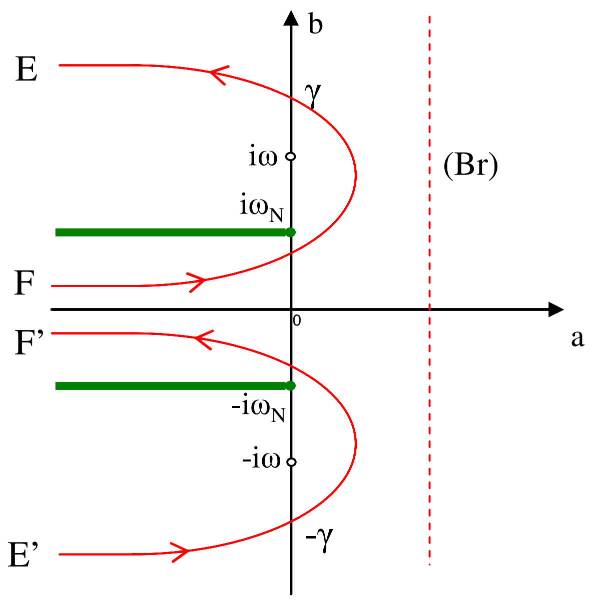

Figure 4: The steepest descent path in the complex plane s.

Along the branch E'F', the real part of the function $h(s;z,t)$ grows from $-\infty$ until the saddle point $-\gamma$, and then deceases to $-\infty$. Again, along the branch FE, this real part increases and then decreases after passing through the saddle point γ.

4.2 Asymptotic description of the wave train

The saddle points are the solutions of the equation

$$\frac{dh(s;z,t)}{ds} = 0 . \qquad (12)$$

The saddle point γ is then obtained as a function of time t :

$$\gamma = i\frac{\omega_N t}{\sqrt{t^2 - t_c^2}} , \qquad (13)$$

where the characteristic time $t_c = z/c$ depends on the position of the observation point.

When t increases from t_c to infinity, the saddle point γ decreases on the imaginary axis from infinity to the branch point $i\omega_N$.

Now, we must keep in mind that the global function in the integral (10) is very large in the neighborhood of the poles $\pm i\omega$. During the phase ①, the saddle point is far on the imaginary axis and hence far from the pole. Consequently, its contribution, as the major part of the value of the integral, is very small. This contribution corresponds to the high frequency part of the wave train, since $s \approx \gamma$ has its imaginary part very large.

As time t increases, the saddle point γ decreases along the imaginary axis and it moves in the direction of the pole $i\omega$. In the vicinity of that pole, the integral near the saddle point takes now more and more significant values. The crucial step is when the saddle point reaches exactly the pole. This happens for the time

$$t_g = \frac{\omega t_c}{\sqrt{\omega^2 - \omega_N^2}} , \qquad (14)$$

which may be seen as the flight time along the distance z, from the emitting section $z = 0$ to the observation point, associated with the group velocity of the mode N for the given angular frequency ω

$$V_g = \frac{\sqrt{\omega^2 - \omega_N^2}}{\omega} c . \qquad (15)$$

After the time $t = t_g$, there must be added the contribution of the pole as a residue term, to the computation of the integral along the steepest descent path. This new term corresponds to the exact value of the mode N which thus may be considered as arriving at the point z with this group velocity. However, the global signal remains perturbed by the contribution of the saddle point, as long as this point is still in the vicinity of the pole, since the amplitude of the function in the integral is yet large enough. Let us notice that when the time t goes through the value t_g, i.e. when the steepest descent path goes across the pole, the contribution of the saddle point is replaced par its opposite value and after that time, it is deduced from the value given by the residue at the pole. This explains the progressive growth of the amplitude of the global signal during this phase ② and why the mode N does not seem to arrive exactly at time t_g.

When the time increases, for some delay after t_g, the amplitude of the contribution of the saddle point is less important and the mode N appears clearly. However, the frequency of the term governed by the saddle point contribution is still near the frequency ω of the mode. That explains the phenomenon of beats which appears during the phase ③ and which produces these oscillations of the amplitude of the wave train (see Fig. 3).

Later, as the time increases more and more, the contribution of the saddle point becomes weaker and weaker, so that the value of the integral reduces to the residue term: the modal N wave is then seen at the location z with its amplitude 1. This is the final phase ④.

Notice that the same argument applies to the symmetric saddle point $-\gamma$, with the symmetric pole $-i\omega$, and that the value of the integral (10) results from the addition of these two conjugated terms, which leads to a final real value for the function $g(z,t)$.

5 Conclusion

Two approaches have been used to evaluate the Laplace integral in order to describe the transient propagation of a modal wave train in an acoustic cylindrical pipe of arbitrary shape. The numerical approach allows obtaining accurate results for the evolution of the signal at a given location in the pipe. In the case where this location point is far enough from the emitting section, an asymptotic evaluation of the integral leads to a good understanding of the physical phenomenon involved, due to the dispersion effect of the modal wave. The opinion of the authors is that, beside the physical interest of the result, the complementarity of these two methods includes an educational aspect.

References

[1] T.H. Havelock, *The propagation of disturbances in dispersive media*, Cambridge University Press (1914)

[2] L. Brillouin, *Wave Propagation and Group Velocity*, Academic Press (1960)

[3] P.M. Morse, K. U. Ingard, *Theoretical Acoustics*, McGraw-Hill Company (1968)

[4] R. Courant, D. Hilbert, *Methods of Mathematical Physics, II - Partial differential equations*, Interscience Publishers (1962)

[5] M. Roseau, *Asymptotic Wave Theory*, North-Holland (1976)