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Robust Control for Two-Time-Scale Discrete Interval Systems

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Abstract. The problem of designing robust controller for discrete two-time-scale interval systems, conveniently represented using interval matrix notion, is considered. The original full order two-time-scale interval system is decomposed into slow and fast subsystems using interval arithmetic. The controllers designed independently to stabilize these two subsystems are combined to get a composite controller which also stabilizes the original full order two-time-scale interval system. It is shown that a state and output feedback control law designed to stabilize the slow interval subsystem stabilizes the original full order system provided the fast interval subsystem is asymptotically stable. The proposed design procedure is illustrated using numerical examples for establishing the efficacy of the proposed method.

1. Introduction

The interval systems are those whose parameters are known to lie within a range rather than having an exact value. They are said to have parametric uncertainty. Various analysis and design techniques available for such systems are essentially meant for application to a "nominal" model. The resulting design is said to be robust if the system performs within acceptable limits in the face of significant parameter variations and model uncertainties. The need to incorporate robustness in design is necessitated by the fact that for most practical systems, the model is known only approximately. For example, in the aircraft industry, the aircraft model is constructed using the data obtained from the wind-tunnel experiments on the aircraft body. As a consequence, the parameters of the model would not have a specific value, rather they are known to lie within an interval. Since the actual flight data are not available the controller should be able to account for the unmodeled parameters that can be obtained only when the aircraft is airborne. The other examples include robotic manipulators, nuclear reactors, electrical machines and

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large power networks etc., which have parametric uncertainties for the entire range of operation. The knowledge about the actual physical system may be approximate and representation of such systems by fixed parameters may become inadequate. Moreover, it is inevitable that the system matrices will have perturbations due to component variations, ageing, operating conditions, etc. Simulation and control of such systems can be attempted by developing their approximants.

A discrete time interval system possessing two-time-scale nature for the entire range of parameter variation is called as two-time-scale interval system. Such class of two-time-scale interval systems can be conveniently described using the notion of interval matrix. It is assumed that, such two-time-scale interval system possesses two-time-scale property for the entire range of parameter variation, that is, it has a cluster of n_1 eigenvalues distributed near the unit circle and a cluster of n_2 eigenvalues centered around the origin in the complex plane. The results presented here are essentially the extension of the results in [11], [12] for fixed case of discrete two-time-scale systems. The slow and fast interval subsystems are obtained using interval arithmetic [1], [3]–[5], [9], [13]–[15]. It is shown that a state and output feedback control law designed to stabilize the slow interval subsystem stabilizes the original full order system provided the fast interval subsystem is asymptotically stable. It is also shown that a composite controller constructed from the slow and fast controllers designed from the respective models stabilizes the original full order interval system when applied to it.

2. Interval Analysis Preliminaries

An interval number $[a, b]$ can be defined by the set of $x \in \Re$ (the reals) such that $a \leq x \leq b$. For $a = b$, the interval number becomes $[a, a]$ which can be described as a degenerate interval. The arithmetic operations on intervals are defined as follows [1], [3], [5], [8], [13]–[15]:

1. $[a, b] + [c, d] = [a + c,\ b + d]$,
2. $[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$,
3. $[a, b] - [c, d] = [a - d,\ b - c]$,
4. $[a, b] \div [c, d] = [a, b] \times \left[\frac{1}{d}, \frac{1}{c}\right]$ provided that, $0 \notin [c, d]$.

Alternatively, the interval number x^I can be represented as $x^I = [a, b] = \{x \in \Re \mid a \leq x \leq b\} = [x_0 - \Delta x, x_0 + \Delta x]$, where $x_0 = (a + b) / 2$ (the nominal value) and $\Delta x = (b - a) / 2$ (the uncertainty).

An interval matrix by definition [3] is a real matrix in which all the elements are known only to the extent that each element belongs to a specified interval. For all $n \times n$ interval real matrices, $F^I = \{f_{ij}^I\} \in \Re^{n\times n}$ with interval elements f_{ij}^I, and $G^I = \{g_{ij}^I\} \in \Re^{n\times n}$ with interval elements g_{ij}^I, for all i and j, the addition, subtraction and multiplication operations can be written as follows:

1. $F^I \pm G^I = \{f^I_{ij} \pm g^I_{ij}\} \in \Re^{n\times n}$,
2. $F^I G^I = \{f^I_{ij}\}\{g^I_{ij}\} = \left\{\sum_{k=1}^{n} f^I_{ik} \times g^I_{kj}\right\} \in \Re^{n\times n}$.

3. Representation of Two-Time-Scale Discrete Interval System

A discrete time interval system possessing two-time-scale nature for the entire range of parameter variation can be described by the following equations

$$x_1(k+1) = A^I_{11}x_1(k) + A^I_{12}x_2(k) + B^I_1 u(k), \tag{3.1}$$

$$x_2(k+1) = A^I_{21}x_1(k) + A^I_{22}x_2(k) + B^I_2 u(k), \tag{3.2}$$

where the state $x(k) \in \Re^n$ is formed by the n_1 and n_2 dimensional vectors $x_1(k)$ and $x_2(k)$ at the discrete instant k and the control $u(k)$ is an m dimensional vector. The matrices A^I_{ij} $(i,j = 1,2)$ are interval matrices. It is assumed that the system (3.1), (3.2) possesses two-time-scale property for the entire range of parameter variation, that is, it has a cluster of n_1 eigenvalues distributed near the unit circle and a cluster of n_2 eigenvalues centered around the origin in the complex plane. Clearly, the n_1 eigenvalues have large magnitudes compared with small magnitudes of the n_2 eigenvalues over the period $[0,T]$. The system behaviour, therefore, can be approximately decomposed into a slow subsystem with n_1 eigenvalues and a fast subsystem with n_2 eigenvalues. In an asymptotically stable system the fast modes corresponding to the eigenvalues of small magnitudes are important only during a short initial period $[0,T_f]$. After that period they are negligible and the behaviour of the system can be described by its slow modes.

Neglecting the effects of the fast modes is equivalent to letting $x_2(k+1) = x_2(k)$ in (3.2). Without the fast modes system (3.1), (3.2) reduces to

$$\bar{x}_1(k+1) = A^I_{11}\bar{x}_1(k) + A^I_{12}\bar{x}_2(k) + B^I_1\bar{u}(k) \qquad \bar{x}_1(0) = x_{10}, \tag{3.3}$$

$$\bar{x}_2(k) = A^I_{21}\bar{x}_1(k) + A^I_{22}\bar{x}_2(k) + B^I_2\bar{u}(k), \tag{3.4}$$

where a bar indicates a discrete quasi-steady state [12]. Assuming that $\left[I_2 - A^I_{22}\right]^{-1}$ exists, where I_2 is the identity matrix with degenerate interval of dimension $n_2 \times n_2$, we can express $\bar{x}_2(k)$ as

$$\bar{x}_2(k) = [I_2 - A^I_{22}]^{-1}\{A^I_{21}\bar{x}_1(k) + B^I_2\bar{u}(k)\}, \tag{3.5}$$

and, substituting it into (3.3), the slow subsystem of (3.1), (3.2) is given by

$$x_s(k+1) = A^I_s x_s(k) + B^I_s u(k), \tag{3.6}$$

where

$$A^I_s = A^I_{11} + A^I_{12}[I_2 - A^I_{22}]^{-1}A^I_{21},$$
$$B^I_s = B^I_1 + A^I_{12}[I_2 - A^I_{22}]^{-1}B^I_2. \tag{3.7}$$

Hence $\bar{x}_1(k) = x_s(k)$, $\bar{x}_2(k)$ and $\bar{u}(k) = u_s(k)$ are the slow components of the corresponding variables in system (3.1), (3.2). The fast subsystem is derived by making the assumptions that $\bar{x}_1(k) = x_s(k) = \text{constant}$ and $\bar{x}_2(k+1) = \bar{x}_2(k)$. From (3.2) and (3.5) we get

$$x_2(k+1) - \bar{x}_2(k+1) = A_{22}^I\{x_2(k) - \bar{x}_2(k)\} + B_2^I\{u(k) - u_s(k)\}. \tag{3.8}$$

Defining $x_f(k) = x_2(k) - \bar{x}_2(k)$ and $u_f(k) = u(k) - u_s(k)$, the fast subsystem of (3.1), (3.2) can be expressed as

$$x_f(k+1) = A_{22}^I x_f(k) + B_2^I u_f(k); \qquad x_f(0) = x_{20} - \bar{x}_2(0). \tag{3.9}$$

The assumptions used in deriving the fast subsystem are justified by noting that the slow modes of system (3.1), (3.2) have magnitudes which are close to unity and during transients, they are changing very slowly with respect to the fast modes.

4. State Feedback Control

4.1. Lower Order Control

Assume that the fast system is stable and the pair (A_s^I, B_s^I) is controllable [8], [17]–[19], [21]. Neglecting the fast subsystem, consider the lower-order feedback control

$$u(k) = K_0 x_1(k). \tag{4.1}$$

If this $u(k)$ is applied to (3.6), closed loop slow model becomes

$$\begin{aligned} x_s(k+1) &= [A_s^I + B_s^I K_0] x_s(k) \\ &= H_0^I x_s(k). \end{aligned} \tag{4.2}$$

If the pair (A_s^I, B_s^I) is controllable then the closed loop system matrix can be stabilized by the appropriate selection of the gain matrix K_0. We now apply control (4.1) to the discrete system (3.1), (3.2) to obtain

$$x_1(k+1) = [A_{11}^I + B_1^I K_0] x_1(k) + A_{12}^I x_2(k), \tag{4.3}$$

$$x_2(k+1) = [A_{21}^I + B_2^I K_0] x_1(k) + A_{22}^I x_2(k). \tag{4.4}$$

LEMMA 4.1. *If the pair (A_s^I, B_s^I) is controllable, the matrix A_{22}^I is stable, and $(I_2 - A_{22}^I)^{-1}$ exists, then the closed loop discrete system (4.3), (4.4) is asymptotically stable.*

Proof. Since matrix A_{22}^I is stable, we can neglect the fast dynamics over the period $[0, T_f]$. The slow subsystem of (4.3), (4.4) is given by

$$\begin{aligned} x_s(k+1) &= \{[A_{11}^I + B_1^I K_0] + A_{12}^I (I_2 - A_{22}^I)^{-1} [A_{21}^I + B_2^I K_0]\} x_s(k) \\ &= R_0^I x_s(k). \end{aligned} \tag{4.5}$$

To prove the lemma it is only necessary to show that $R_0^I = H_0^I$. Consider the matrix H_0^I:

$$\begin{aligned} H_0^I &= A_s^I + B_s^I K_0 \\ &= [A_{11}^I + A_{12}^I (I_2 - A_{22}^I)^{-1} A_{21}^I] + [B_1^I + A_{12}^I (I_2 - A_{22}^I)^{-1} B_2^I] K_0 \\ &= [A_{11}^I + B_1^I K_0] + A_{12}^I (I_2 - A_{22}^I)^{-1} [A_{21}^I + B_2^I K_0] \\ &= R_0^I. \end{aligned}$$

□

The significance of the above lemma lies in the fact that, it reduces the design of stabilizing feedback controllers for discrete interval systems of dimensions $n_1 + n_2$ to the reduced system of order n_1 when the pair (A_s^I, B_s^I) is controllable.

4.2. Composite Control

Consider a discrete time uncertain system possessing two-time-scale nature for the entire range of parameter variation described by

$$x_1(k+1) = A_{11}^I x_1(k) + A_{12}^I x_2(k) + B_1^I u(k), \quad (4.6)$$

$$x_2(k+1) = A_{21} x_1(k) + A_{22} x_2(k) + B_2 u(k). \quad (4.7)$$

It is assumed that A_{21}, A_{22}, and B_2 are constant matrices. The problem is to find a linear state feedback which stabilizes the uncertain system (4.6), (4.7).

The slow uncertain subsystem can be obtained as in (3.6) with

$$\begin{aligned} A_s^I &= A_{11}^I + A_{12}^I [I_2 - A_{22}]^{-1} A_{21}, \\ B_s^I &= B_1^I + A_{12}^I [I_2 - A_{22}]^{-1} B_2. \end{aligned} \quad (4.8)$$

The fast subsystem can be obtained as

$$x_f(k+1) = A_{22} x_f(k) + B_2 u_f(k). \quad (4.9)$$

Assume that the pair (A_{22}, B_2) and the pair (A_s^I, B_s^I) is controllable. Let $u_s(k) = K_0 x_s(k)$ and $u_f(k) = K_f x_f(k)$ be designed to stabilize the slow and fast subsystem in (4.8) and (4.9) respectively.

By virtue of equation (3.5),

$$\bar{x}_2(k) = [I_2 - A_{22}]^{-1} \{A_{21} \bar{x}_1(k) + B_2 \bar{u}(k)\},$$

the composite control

$$u_c(k) = u_s(k) + u_f(k) = K_0 x_s(k) + K_f x_f(k),$$

can be written as

$$\begin{aligned} u_c(k) &= \{[I_m - K_f (I_2 - A_{22})^{-1} B_2] K_0 - K_f (I_2 - A_{22})^{-1} A_{21}\} x_s(k) \\ &\quad + K_f \{(I_2 - A_{22})^{-1} (A_{21} + B_2 K_0) x_s(k) + x_f(k)\}. \end{aligned} \quad (4.10)$$

If we replace $x_s(k)$ by $x_1(k)$ and $\bar{x}_2(k)+x_f(k)$ by $x_2(k)$, we get the composite controller in terms of the states of the higher order system as

$$\begin{aligned} u_c(k) &= \{[I_m - K_f(I_2 - A_{22})^{-1}B_2]K_0 - K_f(I_2 - A_{22})^{-1}A_{21}\}x_1(k) \\ &\quad + K_f x_2(k). \end{aligned} \tag{4.11}$$

LEMMA 4.2. *If* $[I_2 - A_{22}]^{-1}$ *exists and the pairs* (A_s^I, B_s^I) *and* (A_{22}, B_2) *are stabilizable, then the linear state feedback control*

$$\begin{aligned} u_c(k) &= \{[I_m - K_f(I_2 - A_{22})^{-1}B_2]K_0 - K_f(I_2 - A_{22})^{-1}A_{21}\}x_1(k) \\ &\quad + K_f x_2(k), \end{aligned} \tag{4.12}$$

stabilizes (4.6), (4.7) where K_0 *and* K_f *are designed to make* $(A_s^I + B_s^I K_0)$ *and* $(A_{22} + B_2 K_f)$ *stable matrices respectively [16].*

Proof. Consider any one member of the interval system in (4.6), (4.7). Let it be represented as

$$\begin{aligned} x_1(k+1) &= A_{11l}x_1(k) + A_{12l}x_2(k) + B_{1l}u(k), \\ x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k). \end{aligned} \tag{4.13}$$

The slow and fast models of this system are obtained as

$$\begin{aligned} x_s(k+1) &= A_{sl}x_s(k) + B_{sl}u_s(k), \\ x_f(k+1) &= A_{22}x_f(k) + B_2u_f(k), \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} A_{sl} &= A_{11l} + A_{12l}[I_2 - A_{22}]^{-1}A_{21}, \\ B_{sl} &= B_{1l} + A_{12l}[I_2 - A_{22}]^{-1}B_2. \end{aligned}$$

As $A_{11l} \subset A_{11}^I$ (i.e., $a_{11l} \in a_{11}^I$) and $A_{12l} \subset A_{12}^I$, $A_{sl} \subset A_s^I$. Similarly $B_{sl} \subset B_s^I$.

The feedback system, when the controller (4.12) is applied to system (4.13) is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{4.15}$$

where

$$\begin{aligned} F_1 &= A_{11l} + B_{1l}[I_m - K_f(I_2 - A_{22})^{-1}B_2]K_0 - B_{1l}K_f(I_2 - A_{22})^{-1}A_{21}, \\ F_2 &= A_{12l} + B_{1l}K_f, \\ F_3 &= (I_2 - A_{22} - B_2K_f)(I_2 - A_{22})^{-1}(A_{21} + B_2K_0), \\ F_4 &= A_{22} + B_2K_f. \end{aligned}$$

The complete separation into the slow and fast system is achieved by a transformation $\hat{x}(k) = Hx(k)$ [11],

$$H = \begin{bmatrix} I_1 + NK & N \\ K & I_2 \end{bmatrix}, \qquad H^{-1} = \begin{bmatrix} I_1 & -N \\ -K & I_2 + KN \end{bmatrix}, \tag{4.16}$$

where I_1 and I_2 are $n_1 \times n_1$ and $n_2 \times n_2$ identity matrices respectively and

$$\begin{aligned} K &= -(I_2 - A_{22})^{-1}(A_{21} + B_2 K_0) + O(\varepsilon), \\ N &= (A_{sl} + B_{sl} K_0)^{-1}(A_{12l} + B_{1l} K_f) + O(\varepsilon). \end{aligned} \tag{4.17}$$

From equations (4.15), (4.16), and (4.17) we get

$$H \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} H^{-1} = \begin{bmatrix} F_{0l} & 0 \\ 0 & F_f \end{bmatrix}, \tag{4.18}$$

where

$$\begin{aligned} F_{0l} &= A_{sl} + B_{sl} K_0 + O(\varepsilon), \\ F_f &= A_{22} + B_2 K_f + O(\varepsilon). \end{aligned}$$

It is seen that eigenvalues of the transformed closed-loop system (4.18) are formed by eigenvalues of $(A_{sl} + B_{sl}K_0)$ and $(A_{22} + B_2K_f)$ to a first-order approximation. Hence the stabilization of system (4.13) amounts to the selection of the matrices K_0 and K_f such that the pairs $(A_{sl} + B_{sl}K_0)$ and $(A_{22} + B_2K_f)$ are stable. As $A_{sl} \subset A_s^I$, $B_{sl} \subset B_s^I$, and, $(A_s^I + B_s^I K_0)$ and $(A_{22} + B_2K_f)$ are stable by design, the closed loop system represented by (4.18) is a stable system. A similar analysis for another member A_{sq} of interval system will give same result since $A_{sq} \subset A_s^I$ and $B_{sq} \subset B_s^I$. This argument applies to all the members of the interval system (4.6). Hence the state feedback controller (4.12) will give a stable closed loop system. □

5. Output Feedback Control

5.1. Output Feedback Control Using Slow Model

Here we consider the problem of output feedback control design for interval discrete systems possessing the fast slow separation property [16]. The interval system is assumed to have been decomposed into slow and fast interval subsystem. It is shown that output feedback control obtained from the slow uncertain model stabilizes the full order uncertain system, provided that fast uncertain subsystem is stable.

Consider the linear shift invariant interval system possessing two-time-scale structure for the entire range of parameter variation. Let such system be described by (3.1), (3.2) with

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{5.1}$$

where $y_1(k)$ and $y_2(k)$ have dimensions p_1 and p_2 respectively. Let vector $y(k) \in \Re^p$ be composed of vectors $y_1(k)$ and $y_2(k)$. It is assumed that C_1 and C_2 are constant matrices. In terms of discrete quasi-steady state the output becomes

$$\begin{aligned} \bar{y}_1(k) &= C_1 \bar{x}_1(k), \\ \bar{y}_2(k) &= C_2 \bar{x}_2(k). \end{aligned} \tag{5.2}$$

The slow uncertain system is given by (3.6). By substituting the value of $\bar{x}_2(k)$ of (3.5) in (5.2), the output equation of the slow uncertain subsystem becomes

$$\begin{aligned} y_s(k) &= C_s^I x_s(k) + D_s^I u_s(k) \\ &= [y_{1s}^t(k)\, y_{2s}^t(k)]^t, \end{aligned} \tag{5.3}$$

where

$$C_s = \begin{bmatrix} C_1 \\ C_2(I_2 - A_{22}^I)^{-1} A_{21}^I \end{bmatrix} \quad \text{and} \quad D_s = \begin{bmatrix} 0 \\ C_2(I_2 - A_{22}^I)^{-1} B_2^I \end{bmatrix}.$$

The fast uncertain subsystem is given by (3.9) and output equation in this case is

$$\begin{aligned} y_f(k) &= \begin{bmatrix} 0 \\ C_2 \end{bmatrix} x_f(k), \\ &= [y_{1f}^t(k)\;\; y_{2f}^t(k)]^t. \end{aligned} \tag{5.4}$$

Assume that fast subsystem is stable and the pair (A_s^I, B_s^I) is controllable.

LEMMA 5.1. *If the pair (A_s^I, B_s^I) is controllable and $u_s(k) = K_0 y_{1s}$ is designed to stabilize the slow subsystem (3.6), then this control stabilizes the system (3.1), (3.2) provided that matrix A_{22}^I is stable and $(I_2 - A_{22}^I)^{-1}$ exists.*

Proof. The reduced control in terms of the states of the original system becomes $u_s(k) = K_0 C_1 x_1(k)$. When this reduced control is applied to the system (3.6), the closed loop system becomes

$$x_s(k+1) = [A_s^I + B_s^I K_0 C_1] x_s(k). \tag{5.5}$$

Now if we apply the control $u_s(k) = K_0 C_1 x_1(k)$ to the system in (3.1) and (3.2), we get

$$x_1(k+1) = [A_{11}^I + B_1^I K_0 C_1] x_1(k) + A_{12}^I x_2(k), \tag{5.6}$$

$$x_2(k+1) = [A_{21}^I + B_2^I K_0 C_1] x_1(k) + A_{22}^I x_2(k). \tag{5.7}$$

Since matrix A_{22}^I is stable, we can neglect the fast dynamics over the period $[0, T_f]$. The slow subsystem of (5.6), (5.7) is given by

$$x_s(k+1) = \Big\{\{[A_{11}^I + B_1^I K_0 C_1] + A_{12}^I (I_2 - A_{22}^I)^{-1} [A_{21}^I + B_2^I K_0 C_1]\} x_s(k)\Big\}. \tag{5.8}$$

Consider the system (5.5),

$$\begin{aligned} x_s(k+1) &= \Big[A_s^I + B_s^I K_0 C_1\Big] x_s(k) \\ &= \{[A_{11}^I + A_{12}^I (I_2 - A_{22}^I)^{-1} A_{21}^I] \\ &\qquad + [B_1^I + A_{12}^I (I_2 - A_{22}^I)^{-1} B_2^I] K_0 C_1\} x_s(k) \\ &= \{[A_{11}^I + B_1^I K_0 C_1] + A_{12}^I (I_2 - A_{22}^I)^{-1} [A_{21}^I + B_2^I K_0 C_1]\} x_s(k). \end{aligned} \tag{5.9}$$

Comparing (5.8) and (5.9) the lemma is proved. □

5.2. Composite Output Feedback

Consider the linear shift invariant uncertain system possessing two-time-scale structure for the entire range of parameter variation. Let such system be described by (4.6), (4.7). It is assumed that C_1 and C_2 are constant matrices and have rank p_1 and p_2 respectively. It is also assumed that $A_{21} = PC_1$ i.e., the effect of slow part on fast part is linearly related to the output of the slow part. Given A_{21} and C_1, the matrix P is computed by [12] $P = A_{21}C_1^T(C_1C_1^T)^{-1}$ which is an approximate result equivalent to a least squares estimate.

The problem is to find an output feedback control design for (4.6), (4.7). The slow uncertain system can be obtained as in (3.6) with

$$\begin{aligned} A_s^I &= A_{11}^I + A_{12}^I[I_2 - A_{22}]^{-1}PC_1, \\ B_s^I &= B_1^I + A_{12}^I[I_2 - A_{22}]^{-1}B_2. \end{aligned} \tag{5.10}$$

The output equation of the slow uncertain subsystem is given by

$$\begin{aligned} y_s(k) &= C_sx_s(k) + D_su_s(k), \\ &= [y_{1s}^t(k)\ y_{2s}^t(k)]^t, \end{aligned} \tag{5.11}$$

where

$$C_s = \begin{bmatrix} C_1 \\ C_2(I_2 - A_{22})^{-1}PC_1 \end{bmatrix} \quad \text{and} \quad D_s = \begin{bmatrix} 0 \\ C_2(I_2 - A_{22})^{-1}B_2 \end{bmatrix}.$$

The fast subsystem is (4.9) and output equation in this case is by (5.4). Assume that the pair (A_{22}, B_2) and the pair (A_s^I, B_s^I) is controllable. Suppose $u_s(k) = K_0y_{1s}(k)$ and $u_f(k) = K_fy_{2f}(k)$ are designed to stabilize the slow and fast subsystem in (5.10), (4.8), and (4.9) respectively.

From (3.5) and (5.10) we get

$$\bar{x}_2(k) = [I_2 - A_{22}]^{-1}(P + B_2K_0)y_{1s}(k), \tag{5.12}$$

the composite control

$$u_c(k) = u_s(k) + u_f(k) = K_0y_{1s}(k) + K_fy_{2f}(k),$$

can be written as

$$\begin{aligned} u_c(k) = {} & \{[I_m - K_fC_2(I_2 - A_{22})^{-1}B_2]K_0 - K_fC_2(I_2 - A_{22})^{-1}P\}y_{1s}(k) \\ & + K_f\{y_{2f}(k) + C_2(I_2 - A_{22})^{-1}(P + B_2K_0)y_{1s}(k)\}. \end{aligned} \tag{5.13}$$

If we replace $y_{1s}(k)$ by $y_1(k)$ and $\bar{y}_2(k) + y_{2f}(k)$ by $y_2(k)$, we get the composite controller in terms of the $y_1(k)$ and $y_2(k)$.

LEMMA 5.2. *The output feedback control*

$$u_c(k) = \{[I_m - K_f C_2(I_2 - A_{22})^{-1} B_2] K_0 - K_f C_2 (I_2 - A_{22})^{-1} P\} y_1(k) + K_f y_2(k) \tag{5.14}$$

stabilizes (4.6), (4.7), where K_0 and K_f are designed to make $(A_s^I + B_s^I K_0 C_1)$ and $(A_{22} + B_2 K_f C_2)$ stable matrices respectively.

Proof. Consider any one member of the interval system in (4.6), (4.7). Let it be represented as in (4.13). The slow and fast models of the system (4.13) are given by (4.14), with

$$A_{sl} = A_{11l} + A_{12l}[I_2 - A_{22}]^{-1} P C_1,$$
$$B_{sl} = B_{1l} + A_{12l}[I_2 - A_{22}]^{-1} B_2.$$

As $A_{11l} \subset A_{11}^I$ and $A_{12l} \subset A_{12}^I$, $A_{sl} \subset A_s^I$. Similarly $B_{sl} \subset B_s^I$.

The feedback system, when the controller (5.14) is applied to system (4.13) is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{5.15}$$

where

$$F_{11} = A_{11l} + B_{1l}[I_m - K_f C_2 (I_2 - A_{22})^{-1} B_2] K_0 C_1 - B_{1k} K_f C_2 (I_2 - A_{22})^{-1} P C_1,$$
$$F_{12} = A_{12l} + B_{1l} K_f C_2,$$
$$F_{21} = (I_2 - A_{22} - B_2 K_f C_2)(I_2 - A_{22})^{-1}(P + B_2 K_0) C_1,$$
$$F_{22} = A_{22} + B_2 K_f C_2.$$

The complete separation into the slow and fast system is achieved by a transformation $\hat{x}(k) = Jx(k)$ [12]

$$J = \begin{bmatrix} I_1 + RQ & R \\ Q & I_2 \end{bmatrix}, \qquad J^{-1} = \begin{bmatrix} I_1 & -R \\ -Q & I_2 + QR \end{bmatrix}, \tag{5.16}$$

where I_1 and I_2 are $n_1 \times n_1$ and $n_2 \times n_2$ identity matrices respectively and

$$\begin{aligned} Q &= Q_0 + O(\varepsilon), \\ Q &= -(I_2 - A_{22})^{-1}(P + B_2 K_0) C_1 + O(\varepsilon), \\ R &= R_0 + O(\varepsilon), \\ R &= (A_{sl} + B_{sl} K_0 C_1)^{-1}(A_{12l} + B_{1l} K_f C_2) + O(\varepsilon). \end{aligned} \tag{5.17}$$

The Q_0 and R_0 are of the "order of ε" approximation to Q and R, respectively. Retaining the first order term we get

$$J \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} J^{-1} = \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}, \tag{5.18}$$

Table 1. Open loop eigenvalues of the two-time-scale discrete interval system.

Eigenvalue number	Lower bound	Upper bound
1	0.9051	0.9131
2	0.7756	0.8250
3	0.0909	0.0936
4	0.1528	0.2026

where

$$\begin{aligned} D_1 &= (A_{sl} + B_{sl}K_0C_1) + O(\varepsilon), \\ D_4 &= (A_{22} + B_2K_fC_2) + O(\varepsilon). \end{aligned}$$

It is seen that eigenvalues of the transformed closed-loop system (5.18) are formed by eigenvalues of $(A_{sl}+B_{sl}K_0C_1)$ and $(A_{22}+B_2K_fC_2)$ to a first-order approximation. Hence the stabilization of system (4.13) amounts to the selection of the matrices K_0 and K_f so that the pairs $(A_{sl} + B_{sl}K_0C_1)$ and $(A_{22} + B_2K_fC_2)$ are stable. As $A_{sl} \subset A_s^I$, $B_{sl} \subset B_s^I$, and, $(A_s^I + B_s^I K_0C_1)$ and $(A_{22} + B_2K_fC_2)$ are stable by design, the closed loop system represented by (5.15) is a stable system. A similar analysis for another member A_{sq} of interval system will give same result since $A_{sq} \subset A_s^I$ and $B_{sq} \subset B_s^I$. This argument applies to all the members of the interval system (4.6). Hence the controller (5.14) will give a stable closed loop system. □

6. Numerical Examples

EXAMPLE 6.1. Consider the following uncertain two-time-scale system of the form (3.1) and (3.2) with

$$\begin{aligned} A_{11}^I &= \begin{bmatrix} [0.9, 0.91] & [0, 0] \\ [0.1, 0.1] & [0.8, 0.85] \end{bmatrix}, \\ A_{12}^I &= \begin{bmatrix} [0, 0] & [0.1, 0.12] \\ [0.05, 0.05] & [-0.1, -0.1] \end{bmatrix}, \\ A_{21}^I &= \begin{bmatrix} [-0.1, -0.1] & [0, 0] \\ [0.12, 0.13] & [0.003, 0.003] \end{bmatrix}, \\ A_{22}^I &= \begin{bmatrix} [0.15, 0.2] & [0, 0] \\ [0, 0] & [0.1, 0.1] \end{bmatrix}, \\ B_1 &= [\,0 \;\; 1\,]^T, \qquad B_2 = [\,1 \;\; 1\,]^T. \end{aligned}$$

The bound on eigenvalues of the interval matrix can be obtained using the method suggested in [2], [6], [7], and [10]. The bounds on the open loop eigenvalues obtained using the method described in [2] are given in Table 1.

Table 2. Closed loop eigenvalues of the discrete two-time-scale interval system.

	Closed loop eigenvalues			
lower bounds	0.9029	0.7752	0.0935	0.1528
upper bounds	0.9152	0.8247	0.0920	0.2025

Using interval arithmetic, the slow uncertain subsystem is given by

$$A_s^I = \begin{bmatrix} [0.9126, 0.9284] & [0.0003, 0.0004] \\ [0.0784, 0.0816] & [0.7996, 0.8497] \end{bmatrix}, \qquad B_s^I = \begin{bmatrix} [0.1046, 0.1417] \\ [0.9407, 0.9579] \end{bmatrix}.$$

Let $K_0 = [k_1, k_2]$ be the stabilizing controller for the slow uncertain system obtained using LMI toolbox [20]. In LMI toolbox the function *psys* specifies the state space models where the state space matrices may be uncertain or parameter dependent. The function *pvec* is used in conjunction with *psys* to specify parameter dependent systems. The *pvec* uses the type "*box*" which corresponds to independent parameter ranging in interval $\underline{P}_j \leq P_j \leq \overline{P}_j$. Finally the function *msfsyn* computes a state feedback control $u = Kx$ that places the closed loop poles inside the LMI region specified by *region*. The composite state feedback controller, $u_s = [-0.0990\ \ -0.0256\ \ 0\ \ 0]x(k)$, when applied to actual higher order discrete two-time-scale system, results in a stable closed loop system. The closed loop eigenvalues of discrete two-time-scale interval are given in Table 2.

EXAMPLE 6.2. Consider the following uncertain two-time-scale system of the form (4.6) and (4.7) [16]

$$A_{11}^I = \begin{bmatrix} [0.9, 0.91] & [0, 0] \\ [0.1, 0.1] & [0.8, 0.85] \end{bmatrix}, \quad A_{12}^I = \begin{bmatrix} [0, 0] & [0.1, 0.12] \\ [0.05, 0.05] & [-0.1, -0.1] \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -0.1 & 0 \\ 0.12 & 0.003 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using interval arithmetic the slow uncertain subsystem is given by

$$A_s^I = \begin{bmatrix} [0.9126, 0.9284] & [0.0003, 0.0004] \\ [0.0784, 0.0816] & [0.7996, 0.8497] \end{bmatrix}, \qquad B_s^I = \begin{bmatrix} [0.1046, 0.1417] \\ [0.9407, 0.9579] \end{bmatrix}.$$

Using LMI toolbox [20] the controller obtained for slow subsystem is

$$K_0 = [-0.0990\ \ -0.0256].$$

The stabilizing controller for fast system (A_{22}, B_2) of (4.9) is

$$K_f = [-0.0240\ \ -0.0040].$$

Table 3. Closed loop eigenvalues of discrete two-time-scale interval system.

	Closed loop eigenvalues		
lower bounds	0.8283	0.7170	0.1793±0.0390i
upper bounds	0.8599	0.7172	0.2183±0.0642i

The composite state feedback controller (4.12)

$$u_s = [-1.0169 \;\; -0.0262 \;\; -0.0240 \;\; 0.0040]x(k)$$

when applied to actual higher order discrete two-time-scale system, results in a stable closed loop system. The closed loop eigenvalues of discrete two-time-scale interval system are given in Table 3.

7. Conclusion

A method for designing controller for discrete two-time-scale interval system is presented. The discrete two-time-scale interval system is decomposed into slow and fast interval subsystems. It is shown that a state feedback control law designed to stabilize the slow model stabilizes the actual full order system provided the fast modes are asymptotically stable. It is also shown that the composite controller formed from the subsystem controller stabilizes the original discrete two-time-scale interval system when applied to it. The problem of designing output feedback controller for discrete two-time-scale uncertain system using slow model and both the slow and fast subsystem is also considered. The output feedback controller so designed stabilizes the discrete two-time-scale interval system.

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