Global Stability for N-species Lotka-Volterra Systems with Delay, II: Reducible Cases, Sufficiency and Necessity

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Abstract. In this paper, an n-species delayed Lotka-Volterra system without delayed intraspecific competitions is considered. It is proved that the system is globally stable for all off-diagonal delays $\tau_{ij} \geq 0$ if and only if the interaction matrix $A$ of the system satisfies Condition (WDD).

1. Introduction

Consider the following n-species Lotka-Volterra discrete delay system,

$$\dot{x}_i(t) = x_i(t)[r_i + \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij})], \quad i = 1, ..., n, \quad (1)$$

with initial conditions

$$x_i(t) = \phi_i(t) \geq 0, \quad t \in [-\tau_0, 0]; \quad \phi_i(0) > 0, i = 1, ..., n. \quad (2)$$

Here $x_i$ represents the density of species $i$, and $r_i$ the reproduction rate, $\tau_{ij} \geq 0$ ($i, j = 1, ..., n$) the constant time lag, and $\tau_0 = \max\{\tau_{ij}\}$. $a_{ij}$ ($i, j = 1, ..., n$) is constant and $\phi_i(t)$ ($i = 1, ..., n$) continuous on $[-\tau_0, 0]$.

Suppose that system (1) has a unique positive equilibrium $x^* = (x_1^*, ..., x_n^*)$, then it can be written in the form

$$\dot{x}_i(t) = x_i(t)[\sum_{j=1}^{n} a_{ij}(x_j(t - \tau_{ij}) - x^*)], \quad i = 1, ..., n, \quad (3)$$

The global stability of system (1) has been studied by many authors[? , ? , ?]. The main result in [?] implies the following,

THEOREM A. Suppose the following conditions hold:

1) System (1) is ultimately, uniformly bounded, i.e., there is a constant $M > 0$ such that any solution $x(t) = (x_1(t), ..., x_n(t))$ of (3) and (2) satisfies

$$\lim_{t \to +\infty} x_i(t) \leq M \quad \text{for} \quad i = 1, ..., n;$$

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2) the diagonal delays (i.e., $\tau_{ii}$) are small enough;
3) the interaction matrix $A = (a_{ij})_{n \times n}$ satisfies condition (DD), i.e., $A$ is diagonally dominant,
then the positive equilibrium $x^*$ is globally stable.

Two examples are given in [?] to show that delays can make a cooperative system possessing an unbounded solution, although the undelayed (i.e., ODE) one is globally stable. This indicates that for the global stability of system (3), the boundedness assumption for the solutions is necessary. In [?], a numerical example is given to show that if, for a special case of system (3), the diagonal delays are large enough, the dynamical behavior of the solutions can be chaotic. This means that we need the smallness of $\tau_{ii}$ to ensure the global stability of $x^*$ of system (3).

The above results indicate that under the assumptions of the boundedness of solutions and the smallness of the diagonal delays, the global stability of system (3) depends largely upon the structure of the interaction matrix $A$.

In another aspect, condition 3) in theorem A is proved to be almost necessary for $n = 2[?]$. In fact, it is shown that, for $n = 2$, if the diagonal delays $\tau_{ii}(i = 1, 2)$ are zero, then system (3) is globally stable if and only if matrix $A$ is weakly diagonally dominant.

In [?], the sufficiency of [?] is extended to general $n$, when matrix $A$ is irreducible.

THEOREM B. Suppose $\tau_{ii} = 0$ for $i = 1, ..., n$. If matrix $A$ is weakly diagonally dominant (definition 2) and irreducible, then the positive equilibrium $x^*$ of system (3) is globally stable for all $\tau_{ij}$.

In this paper, we focus on the relationship of the global stability of system (3) and the structure of its interaction matrix $A$ and extend the two-species result of [?] and the sufficiency one of [?] to get the sufficient and necessary conditions for general $n$-species delay system to be globally stable. In the sequel, we always make the following assumption:

ASSUMPTION (A): $\tau_{ii} = 0$ for $i = 1, ..., n$.

The main results of the present paper are as follows.

THEOREM 1. If matrix $A$ is weakly diagonally dominant, then the positive equilibrium $x^*$ of system (3) is globally stable for all $\tau_{ij}$.

THEOREM 2. If the positive equilibrium $x^*$ of system (3) is locally stable for all $\tau_{ij}$, then matrix $A$ is weakly diagonally dominant.

2. Definitions and Lemmas

For a given matrix $A = (a_{ij})_{n \times n}$, we denote $\bar{A} = (\bar{a}_{ij})_{n \times n}$ with $\bar{a}_{ii} = a_{ii}$ and $\bar{a}_{ij} = |a_{ij}|$ for $i \neq j; \ i, j = 1, ..., n$.

Definition 1. Matrix $A$ is said to satisfy the diagonally dominant condition (DD) in column (resp., in row), if there exists a positive constant $\alpha$ (resp., $\beta$) such that $\alpha^T \bar{A} < 0$, (resp., $\bar{A} \beta < 0$).

REMARK. Since condition (DD) in column is equivalent to which in row[?], so in this case we simply call matrix $A$ satisfy condition (DD).
Without loss of generality, we suppose that the interaction matrix \( A \) has the following lower-triangle form

\[
A = (a_{ij})_{n \times n} = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
\times & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\times & \times & \cdots & A_k
\end{pmatrix},
\]

where each \( A_i \) \((i = 1, \ldots, k)\) is a \( r_i \times r_i \) matrix with \( \sum_{i=1}^{k} r_i = n \) and irreducible (i.e., the linear transformation \( A_i \) does not map into itself any nonzero proper linear subspace spanned by a subset of the standard basis vectors), and all elements in the upper-right blocks of \( A \) are zero, and all matrices in the left-lower have any elements.

**Definition 2.** Matrix \( A \) is said to satisfy the weakly diagonally dominant condition \((WDD)\), if

1) \( A \) is irreducible, then there is a positive vector \( \alpha \) such that \( \alpha^T \bar{A} \leq 0 \);

2) \( A \) is reducible (i.e., it has form (4)), then each submatrix \( A_i \) satisfies condition 1).

The following lemma is proved in [?].

**Lemma 1.** If matrix \( A \) is irreducible and satisfies condition \((WDD)\), then there is a positive vector \( \beta \) such that \( \bar{A} \beta \leq 0 \).

**Lemma 2.** If matrix \( A \) is irreducible, then the following statements are equivalent:

1) \( A \) is weakly diagonally dominant;

2) \((-1)^k \det(\bar{A}_k) > 0\), for \( k \leq n - 1 \), and \((-1)^n \det(\bar{A}) \geq 0\).

Here \( \bar{A}_k = (a_{ij})_{k \times k} \) is the \( k \)-th leading submatrix of \( A \).

**Proof.** That 1) implies 2) is a direct result of theorem 21.1 of [?].

Denote

\[
A = \begin{pmatrix}
A_n^{-1} & a \\
b^T & x
\end{pmatrix} \quad \text{and} \quad A_\epsilon = \begin{pmatrix}
A_n^{-1} & a \\
b^T & x - \epsilon
\end{pmatrix}.
\]

Clearly, for any \( \epsilon > 0 \), we have

\[
(-1)^n \left| \begin{array}{cc}
\bar{A}_n^{-1} & a \\
b^T & x - \epsilon
\end{array} \right| = (-1)^n \left| \begin{array}{cc}
\bar{A}_n^{-1} & a \\
b^T & x
\end{array} \right| - \left| \begin{array}{cc}
\bar{A}_n^{-1} & 0 \\
b^T & \epsilon
\end{array} \right| \geq (-1)^n (-\epsilon) \det(\bar{A}_{n-1}) > 0.
\]

It follows from theorem 21.1 of [?] that \( A_\epsilon \) is diagonally dominant. Therefore, there is a \( \alpha_\epsilon > 0 \) such that \( \alpha_\epsilon^T \bar{A}_\epsilon < 0 \). Let \( a_\epsilon = \max_{1 \leq i \leq n} \| (A_\epsilon)_{ii} \| \), then \( \bar{A}_\epsilon + a_\epsilon \mathbf{I} \) is an irreducible \( M \)-matrix (see [?] for the definition), where \( \mathbf{I} \) is the identity matrix.

By the Perron-Frobenius Theorem [?], it is known that there is a positive vector \( \beta_\epsilon \) such that

\[
(\bar{A}_\epsilon + a_\epsilon \mathbf{I})\beta_\epsilon = \lambda_\epsilon \beta_\epsilon,
\]

where \( \lambda_\epsilon \) is the largest eigenvalue of \( (\bar{A}_\epsilon + a_\epsilon \mathbf{I}) \). By Lemma 1, \((\lambda_\epsilon - a_\epsilon)\beta_\epsilon = \bar{A}_\epsilon \beta_\epsilon < 0 \). Since \( \lim_{\epsilon \to 0} (\lambda_\epsilon - a_\epsilon) = \lambda_0 - a_0 \) and \( A_0 \beta_0 = (\lambda_0 - a_0) \beta_0 \). Therefore, \( A_0 \beta_0 \leq 0 \), namely, \( A \) is weakly diagonally dominant.
This proves the lemma.

Denote $A(y) = (a_{ij}(y))_{n \times n}$ with $a_{ij}(y) = a_{ij}$ and $a_{ii}(y) = -\sqrt{a_{ii}^2 + y^2}(i \neq j; i, j = 1, ..., n)$.

By using Perron-Frobenius theorem, we can prove the following lemma similarly to the above one.

**LEMMA 3.** If $A = (a_{ij})_{n \times n}$ is irreducible, $a_{ii} \leq 0$ and $A(y)$ is diagonally dominant for $y \in (0, \delta_0)$ ($\delta_0 > 0$), then $A$ is weakly diagonally dominant.

### 3. The Proof of Theorem 1

**LEMMA 4.** Suppose $A$ is weakly diagonally dominant, then system (3) is ultimately, uniformly bounded, i.e., there is a $M_A > 0$ such that

$$\lim_{t \to +\infty} x_i(t) \leq M_A, \quad i = 1, ..., n.$$

**Proof.** Without loss of generality, we can suppose that the interaction matrix $A$ in system (3) takes the form

$$A = \begin{pmatrix} A_1 & 0 \\ \times & A_2 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \\ a_{r+1,1} & \cdots & a_{r+1,r} \\ \vdots & \ddots & \vdots \\ a_{r+m,1} & \cdots & a_{r+m,r} \end{pmatrix}.$$

(7)

Here $A_1$ and $A_2$ are irreducible, 0 denotes the $r \times m$ right-upper submatrix with all zero elements and $\times$ denotes the left-lower submatrix with any elements. Since $A_2$ satisfies condition (WDD) and irreducible, there is a positive vector $\beta$ such that $A_2\beta \leq 0$. We assume $\beta = 1$, otherwise the transformation $x_i = \beta \bar{x}_i$ will work.

Define $y(t) = \max_{k \in \mathbb{I}_2} \{x_k(t) - x_k^+\}$ and denote $\delta = \max_k \{|a_{kk}| - \sum_{j \in \mathbb{I}_2 \setminus \{k\}} |a_{kj}|\}$, where $(f(t))^+ = \max\{f(t), 0\}$, $I_1 = \{1, ..., r\}$ and $I_2 = \{r + 1, ..., r + m\}$. Then for almost every $t$, there is a $k \in I_2$ such that

$$\dot{y}(t) = (y(t) + x_k^+)h_k(t) + \sum_{j \in \mathbb{I}_2 \setminus \{k\}} a_{kj}(\bar{x}_{kj}(t) - x_j^*) + a_{kk}y(t),$$

(8)

where $h_k(t) = \sum_{j \in I_1} a_{kj}(\bar{x}_{kj}(t) - x_j^*)$ and $\bar{x}_{kj}(t) = x_{kj}(t - \tau_{kj})$. For any given $T_n \to \infty$ (with $T_{n+1} > T_n$), define sequence $\{t_n\}$ as follows,

$$y(t_n) = \sup_{t \in [0, T_n]} y(t).$$

Clearly, $t_n$ is increasing. By the definition of $y(t)$, we have that $\dot{y}(t_n) \geq 0$, $y(t_n) \geq y(t)$ for $t \leq t_n$ and $y(t_n) = (x_k(t_n) - x_k^*)^+$ for some $k \in I_2$.

Set $M_A = 2 \max_{i \in \mathbb{I}_2} \{x_i^+\}$, we now show that $\lim_{t \to +\infty} y(t) \leq M_A.$
If $\lim_{t \to \infty} y(t) > M_A$, then
\[
\lim_{t_n \to \infty} (|\bar{x}_{kj}(t_n) - x_j^*| - y(t_n)) \leq 0 \quad \text{for } k, j \in I_2.
\]

If $A$ is diagonally dominant, i.e., $\delta > 0$, then from (8), we get
\[
\lim_{t_n \to \infty} [h_k(t_n) + \sum_{j \in I_2 - \{k\}} |a_{kj}|(|\bar{x}_{kj}(t_n) - x_j^*| - y(t_n)) - \delta y(t_n)] \geq 0.
\]

Hence $\lim_{n \to \infty}(-\delta y(t_n)) \geq 0$. That is $\lim_{n \to \infty}y(t_n) \leq 0$ which contradicts to the assumption.

Now we consider the case of $\delta = 0$.

Clearly, for large $n$,
\[
y(t_n) \leq (y(t_n) + x_k^*)[h_k(t_n) + \sum_{j \in I_2 - \{k\}} |a_{kj}|(|\bar{x}_{kj}(t_n) - x_j^*| + a_{kk}y(t_n)). \tag{9}
\]

Now we only consider these $k$'s such that there are infinitely many $t_n$ satisfying (9).

If $|a_{kj}| \neq 0$, since $\dot{y}(t_n) \geq 0$ and $\lim_{t \to \infty} (|\bar{x}_{kj}(t_n) - x_j^*| - y(t_n)) \leq 0$, then from (9), we have
\[
\lim_{n \to \infty} (|\bar{x}_{kj}(t_n) - x_j^*| - y(t_n)) = 0.
\]

Since $A_2$ is irreducible, we assert that for each $s \in I_2$, there is a sequence $t_n^{(s)} \to \infty \ (n \to \infty)$ denoted by $t_n$ again such that
\[
\dot{x}_s(t_n) \geq 0, \quad x_s(t_n) - x_s^* \geq y(t) - \epsilon_n, \quad \epsilon_n > 0, \quad \epsilon_n \to 0 \quad \text{for } t \leq t_n,
\]

and
\[
y(t_n) - \bar{x}_s \to 0, \quad n \to \infty.
\]

Since $\delta = 0$, there exist $l, m$ such that $a_{lm} < 0$.

By letting $t_n \to \infty$, we obtain
\[
0 \leq \lim_{n \to \infty} \dot{x}_l(t_n)
= \lim_{n \to \infty} x_l(t_n) \sum_{j \in I_2 - \{l\}} a_{lj}(\bar{x}_l(t_n) - x_j^*) + a_{ll}(x_l(t_n) - x_l^*)
\leq \lim_{n \to \infty} x_l(t_n) \sum_{j \in I_2 - \{l\}} |a_{lj}|(\bar{x}_l(t_n) - x_j^*) + a_{ll}(x_l(t_n) - x_l^*)
\leq \lim_{n \to \infty} x_l(t_n) \sum_{j \in I_2 - \{l\}} 2a_{lm}(\bar{x}_l(t_n) - x_j^*) \tag{10}
\]

Namely,
\[
\lim_{n \to \infty} 2a_{lm}(\bar{x}_l(t_n) - x_j^*) \geq 0. \tag{11}
\]

This implies that
\[
\lim_{t \to \infty} y(t) \leq x_l^*,
\]

which contradicts to $\lim_{t \to \infty} y(t) \geq M_A$.

This completes the proof of the lemma.
The proof of theorem 1. We suppose again matrix $A$ takes the form (7). Write system (3) with the interaction matrix (7) into the following form

$$
\dot{x}_i(t) = x_i(t)\left[\sum_{j=1}^{r} a_{ij}(x_j(t - \tau_{ij}) - x_j^*)\right], \quad i = 1, \ldots, r, \quad (12)
$$

$$
\dot{x}_{r+k}(t) = x_{r+k}(t)\left[\sum_{j=1}^{m} a_{r+k,r+j}(x_{r+j}(t - \tau_{r+k,r+j}) - x_{r+j}^*)\right] + \phi_{r+k}(t), \quad k = 1, \ldots, m, \quad (13)
$$

where $\phi_{r+k}(t) = x_{r+k}(t)\left[\sum_{j=1}^{r} a_{r+k,j}(x_j(t - \tau_{r+k,j}) - x_j^*)\right]$, for $k = 1, \ldots, m$.

Theorem B implies that $\lim_{t\to\infty} x_i(t) = x_i^*$ for $i = 1, \ldots, r$ which in turn implies that $\lim_{t\to\infty} \phi_{r+k}(t) = 0$ for $k = 1, \ldots, m$, since $x_{r+k}(t) (k = 1, \ldots, m)$ is ultimately uniformly bounded.

Consider the limiting system of system (13) as follows,

$$
\dot{x}_{r+k}(t) = \ddot{x}_{r+k}(t)\left[\sum_{j=1}^{m} a_{r+k,r+j}(\ddot{x}_{r+j}(t - \tau_{r+k,r+j}) - x_{r+j}^*)\right], \quad k = 1, \ldots, m. \quad (14)
$$

Since system (14) is globally stable (Theorem B) and system (13) is ultimately, uniformly bounded (Lemma 4) with $\phi_{r+k}(t) \to 0$ as $t \to \infty$, the Markus-Thieme theorem in [?], in turn implies that $\lim_{t\to\infty} x_{r+k}(t) = x_{r+k}^*$ for $k = 1, \ldots, m$, since $x_{r+k}(t) (k = 1, \ldots, m)$ is ultimately uniformly bounded.

Hence

$$
\lim_{t\to\infty} x = x^*. \quad (16)
$$

Considering the linearization of system (3) and using the following Liapunov functional

$$
\mathcal{V} = \sum_{i=1}^{n} \alpha_i(x_i(t) - x_i^*)^2 + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \alpha_i|a_{ij}| \int_{t-\tau_{ij}}^{t} (x_i(s) - x_j^*)^2 ds,
$$

we can show similarly to the above procedure that when $A$ satisfies condition (WDD), the linear system of (3) is asymptotically stable. Since it is autonomous, the asymptotical stability implies uniform asymptotic stability[?], which in turn implies [?] the asymptotic stability of the original nonlinear system (3).

Combining this result with (16) completes the proof of the theorem.

4. The Proof of Theorem 2

LEMMA 5. $\det(\bar{A}(y)) = 0$ has at most finitely many real roots.

Proof. Denote $F_0(y_1, \ldots, y_n) = \det(\bar{A}(y_1, \ldots, y_n))$ with $y_i = \sqrt{a_{ii}^2 + y^2}$ for $i = 1, \ldots, n$, where

$$
\bar{A}(y_1, \ldots, y_n) = \begin{pmatrix}
-\bar{y}_1 & |a_{12}| & \cdots & |a_{1n}| \\
|a_{21}| & -\bar{y}_2 & \cdots & |a_{2n}| \\
\vdots & \vdots & \ddots & \vdots \\
|a_{n1}| & |a_{n2}| & \cdots & -\bar{y}_n
\end{pmatrix}.
$$
Then we obtain a set of polynomials of \( y, y_1, \ldots, y_n \) as follows,
\[
\begin{align*}
&y^2 - y_1^2 + a_{11}^1 = 0, \quad (PS_1) \\
&y^2 - y_2^2 + a_{22}^2 = 0, \quad (PS_2) \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
&y^2 - y_n^2 + a_{nn}^n = 0, \quad (PS_n) \\
&F(y_1, \ldots, y_n) = 0. \quad (PS_{n+1})
\end{align*}
\]

Clearly, if \( y \) is a root of \( \det(\hat{A}(y)) = 0 \), then there must be \( y_1, \ldots, y_n \) such that \( (y, y_1, \ldots, y_n) \) is a root of \( (PS) \). Therefore, to show the finiteness of real roots of \( \det(\hat{A}) = 0 \), it suffices to prove that \( (PS) \) has only finitely many roots (zeros). We designate the totality of zeros of polynomials of \( (PS) \) by \( \text{zero}(PS) \). The degree of a polynomial \( f \) is denoted by \( \deg(f) \) and the monomial with highest degree (principal term) in \( f \) by \( \text{pri}(f) \). We follow Wu’s well ordering principle [7] to simplify \( (PS) \) and obtain its corresponding characteristic set \( (CS) \). Then by the relation between the zeros of \( (PS) \) and \( (CS) \), we can obtain the finiteness result for zeros of \( (PS) \).

Denote \( Y_i = y_i y_{i+1} \cdots y_n \). The concrete procedure of well ordering \( (PS) \) is as follows.

First, we rewrite \( (PS) \) as following \( (PS') \) with equations \( (PS_1'), (PS_{n-1}') \) minus \( (PS_n) \),
\[
\begin{align*}
&y^2 - y_1^2 + a_{11}^1 = 0, \quad (PS_0') \\
&y_1^2 - y_2^2 + b_1 = 0, \quad (PS_1') \\
&y_2^2 - y_n^2 + b_2 = 0, \quad (PS_n') \\
&y_{n-1}^2 - y_n^2 + b_{n-1} = 0, \quad (PS_{n-1}') \\
&F_0(y_1, \ldots, y_n) = 0. \quad (PS_n)
\end{align*}
\]

where \( b_i = a_{nn} - a_{ni}^2 \). Since \( F_0 \) is linear with respect to \( y_1 \), it can be written as
\[
F_0(Y_1) = y_1 f_{01}(Y_2) + f_{02}(Y_2) = 0, \quad (17)
\]
where \( \text{pri}(f_{01}) = (-1)^n Y_2 \), \( \deg(f_{01}) = n - 1 \) and \( \deg(f_{02}) = n - 2 \). Deleting \( y_1 \) from \( (PS_1') \) and \( (17) \), we obtain
\[
\begin{align*}
F_0(Y_2) &= f_{02}^2(Y_2) - (y_n^2 - b_1) f_{01}^2(Y_2) = 0. \quad (18)
\end{align*}
\]

It is easy to check that \( \text{pri}(F_0) = -y_n^2 (\text{pri}(f_{01}))^2 = -y_n^2 y_2 \cdots y_n^2 \). Substituting \( (PS_n') \) into \( (18) \), we have
\[
F_1(Y_2) = y_2 f_{11}(Y_3) + f_{12}(Y_3) = 0. \quad (19)
\]

Here \( \deg(f_{11}(Y_3)) \leq 2n - 2 \), \( \deg(f_{12}(Y_3)) = 2n \) and \( \text{pri}(F_1) = \text{pri}(F_0) = 2n \) with coefficient 1 or -1. Repeating the above procedure, we have
\[
F_n(Y_n) = f_{n-1,1}(Y_{n}) - (x_n^2 - b_{n-1}) f_{n-1,2}(Y_n) = 0. \quad (20)
\]

Since \( Y_n = y_n \), \( (20) \) is a polynomial of \( y_n \) whose principal term has the form \( \text{pri}(F_n) = \delta y_n^s \) with \( s = 2^n n \) and \( \delta = 1 \) or -1, i.e., nonzero. Finally, we obtain a polynomial set with three polynomials,
\[
\begin{align*}
&y^2 - y_1^2 + a_{11}^1 = 0, \quad (CS) \\
&y_1^2 - y_2^2 + b_1 = 0, \\
&F_n(y_n) = 0. \quad (21)
\end{align*}
\]
Because of the inclusion relation \( \text{zero}(PS) = \text{zero}(PS') \subset \text{zero}(CS) \), the finiteness of zeros of \( F_n(y_n) \) implies that of \( \text{zero}(CS) \) which in turn implies that of \( (PS) \).

This proves the lemma.

Consider the characteristic equation of the linear part of system (3) at the equilibrium \( x^* = (x_1^*, ..., x_n^*) \), without loss of generality, we suppose that \( x_i^* = 1 \) for \( i = 1, ..., n \),

\[
f_\tau(\lambda, n) = \begin{vmatrix}
\lambda - a_{11} & -a_{12}e^{-\lambda \tau_{12}} & \cdots & -a_{1n}e^{-\lambda \tau_{1n}} \\
-a_{21}e^{-\lambda \tau_{21}} & \lambda - a_{22} & \cdots & -a_{2n}e^{-\lambda \tau_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1}e^{-\lambda \tau_{n1}} & -a_{n2}e^{-\lambda \tau_{n2}} & \cdots & \lambda - a_{nn}
\end{vmatrix}.
\] (22)

We say that system (3) satisfies condition (LS) if \( x^* \) is locally stable for all \( \tau_{ij}(i \neq j; i, j = 1, ..., n) \).

**LEMMA 6.** If condition (LS) holds true for system (3), then \( a_{ii} \leq 0 \) for \( i = 1, ..., n \).

**Proof.** Firstly, we show that condition (LS) implies the following inequality

\[
f_\tau(\lambda, n) \geq 0 \quad \text{for} \quad \lambda \geq 0.
\] (23)

If there is a positive \( \lambda_0 \) such that \( f_\tau(\lambda_0, n) < 0 \), since \( \lim_{\lambda \to +\infty} f_\tau(\lambda, n) = +\infty \), there exists a positive \( \lambda^+ \) such that \( f_\tau(\lambda^+, n) = 0 \). This contradicts to (LS).

Clearly, there is at least one \( a_{ii}, i \in \{1, ..., n\} \) such that \( a_{ii} \leq 0 \), say \( a_{nn} \leq 0 \). Otherwise, the corresponding ODE system of (3) (i.e., \( \tau_{ij} = 0 \) for all \( i \neq j; i, j = 1, ..., n \)) must be unstable.

The following proof proceeds by induction on the order \( n \) of \( f_\tau(\lambda, n) \). The case of \( n = 2 \) is a direct result of [?]. Suppose that (23) is valid for \( n = k \), then in the case of \( n = k + 1 \), we have

\[
\lim_{\tau_{ij}(i = 1, ..., n - 1) \to \infty} f_\tau(\lambda, n) = (\lambda - a_{nn})f_\tau(\lambda, n - 1).
\]

Both \( f_\tau(\lambda, n) \geq 0 \) and \( \lambda - a_{nn} \geq 0 \) for \( \lambda \geq 0 \) imply that

\[
f_\tau(\lambda, n - 1) \geq 0 \quad \text{for} \quad \tau \geq 0.
\] (24)

By induction assumption, (24) implies that \( a_{ii} \leq 0 \) for \( i = 1, ..., n - 1 \).

This completes the proof.

**LEMMA 7.** If \( a_{ii} \leq 0 \) for \( i = 1, ..., n \), \( A \) is irreducible and system (3) is locally stable for all \( \tau_{ij} \geq 0 \), \( (i \neq j; i, j = 1, ..., n) \), then there exists a positive \( \delta_0 > 0 \) such that

\[
(-1)^k \det(\bar{A}(y)) > 0,
\]

for \( 1 \leq k \leq n \) and \( y \in (0, \delta_0) \). Namely, \( A(y) \) satisfies condition (DD) for \( y \in (0, \delta_0) \). Here, \( \bar{A}(y) \) is the \( k \)-th leading principal submatrix of \( A(y) \).

**Proof.** Let \( B_\delta(0) = \{(x, y)|0 < \sqrt{x^2 + y^2} < \delta\} \). Define map \( \theta(x, y) : B_\delta(0) \to [\pi, \pi] \) as follows,

\[
\cos \theta_i = \frac{-\delta_{ii}x - a_{ii}}{\sqrt{(x - a_{ii})^2 + y^2}},
\sin \theta_i = \frac{-\delta_{ii}y}{\sqrt{(x - a_{ii})^2 + y^2}}, \quad i = 1, ..., n.
\] (25)
Set
\[ \tau_{ij} = \frac{k_{ij} \pi - \theta_i}{y}, \quad y > 0. \]

Here \( K_{ij} \) is the smallest positive integer satisfying \( \cos K_{ij} \pi = \delta_{ij} \) and \( K_{ij} \pi - \theta \geq 0 \).

Let \( \lambda = x + iy \), then \( f_\tau(\lambda, n) \) in (22) takes the form
\[ f_\tau(\lambda, n) = f_\tau(x + iy, n) = F(x, y) \exp(i\theta), \]
where \( \theta = \sum_{i=1}^{n} \theta_i \) and \( F(x, y) = \det(\hat{A}(x, y)) \). Here \( \hat{A}(x, y) = (\hat{a}_{ij}(x, y))_{n \times n} \) with \( \hat{a}_{ii}(x, y) = -\sqrt{(x - a_{ii})^2 + y^2} \) and \( \hat{a}_{ij}(x, y) = -|a_{ij}| \exp(-x\tau_{ij}) \). Now denote
\[ \hat{F}(x, y) = \det(\hat{A}(x, y)), \]
where \( \hat{A}(x, y) = (\hat{a}_{ij}(x, y))_{n \times n} \) with \( \hat{a}_{ii}(x, y) = \hat{a}_{ii}(x, y) \) and \( \hat{a}_{ij}(x, y) = -|a_{ij}| \).

Clearly, \( \hat{F}(x, y) = F(x, y)|_{\tau=0} \). Since \( \hat{F}(x, y) \) is continuous, for any \( y_0 > 0 \), \( \hat{F}(x, y_0) \) is continuous with respect to \( x \). If there is a \( y_0 > 0 \) such that \( \hat{F}(0, y_0) < 0 \), the boundedness of \( \tau(x, y_0) \) implies that \( F(0, y_0) = \lim_{x \to 0} F(x, y_0) = \hat{F}(0, y_0) < 0 \). Since \( \lim_{x \to +\infty} F(x, y_0) = +\infty \), there is a positive \( x \) such that \( F(x, y_0) = 0 \), namely \( f_\tau(x, y_0, n) = f_\tau(x_0 + iy_0, n) = 0 \). This contradicts to the local stability of \( x^* \). By Lemma 5, \( \hat{F}(0, y) \) has finitely many real roots, therefore there exists a \( \delta_{0n} > 0 \) such that \( \hat{F}(0, y) > 0 \) for \( y \in (0, \delta_{0n}) \). Hence
\[ (-1)^n \det(\hat{A}(y)) = \hat{F}(0, y) > 0 \quad \text{for} \quad y \in (0, \delta_{0n}). \]

Similarly to the above proof, by letting \( \tau_{ij} \to +\infty \) for \( i, j \geq k \), we obtain that for \( \delta_{0k} > 0 \),
\[ (-1)^n \left( \prod_{i=k+1}^{n} \delta_{ii} \sqrt{a_{ii}^2 + y^2} \right) \det(\hat{A}_k(y)) = \hat{F}(0, y) > 0 \quad \text{for} \quad y \in (0, \delta_{0k}), \]
where \( \hat{A}_k(y) \) is the \( k \)-th leading submatrix of \( \hat{A}(y) \).

Since \( \delta_{ii} = -1 \), we have \( (-1)^k \det(\hat{A}_k(y)) > 0 \), for \( y \in (0, \delta_{0k}) \). Taking \( \delta_0 = \min_k \{\delta_{0k}\} \), we finally obtain
\[ (-1)^k \det(\hat{A}_k(y)) > 0 \quad \text{for} \quad y \in (0, \delta_0), \quad k = 1, ..., n. \]
Namely, \( \hat{A}(y) \) satisfies condition \((DD)\) for \( y \in (0, \delta_0) \).

**Proof of Theorem 2.** By Lemma 7, if \( x^* \) of system (3) is locally stable, then \( A(y) \) satisfies condition \((DD)\). By Lemma 3, \( A \) is weakly daigonally dominant.

This completes the proof of the theorem.

**5. Concluding Remarks**

In this paper, we prove that if the interaction matrix \( A \) of system (3) is weakly daigonally dominant, then the system with \( \tau_{ii} = 0 \) \( (i = 1, ..., n) \) is globally stable. Condition \((WDD)\) is also proved to be necessary for the positive equilibrium being locally stable.

When all the delays in system (3) are zero, the system becomes an ODE one. In this case, Theorem 1 ensures that condition \((WDD)\) for matrix \( A \) implies the global stability of the system and Volterra-Liapunov semistability of each irreducible diagonal submatrix \( A_i \).
of $A$. The latter one extends the classic result about the diagonal dominance implying the Volterra-Liapunov stability for a matrix.[?]  

In another aspect, when considering the following linear system,

$$
\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}), \quad i = 1, \ldots, n,
$$

(27)

under the assumption:

$$
\tau_{ii} = 0 \quad \text{for} \quad i = 1, \ldots, n,
$$

(28)

we have, from the proof of theorem 1 and theorem 2, the following corollary.

**COROLLARY 1.** The origin of system (28) is asymptotically stable for all $\tau_{ij}$ if and only if matrix $A = (a_{ij})_{n \times n}$ satisfies condition $(WDD)$.

**References**


