Analytical approximants of time-dependent partial differential equations with tau methods

Antonio García-Olivares

LITEC (CSIC), C/María de Luna, 10.50018 Zaragoza, Spain

Abstract

Tau spectral methods and Adomian’s decomposition can be fruitfully combined to quickly approximate the analytical solution of any time-dependent partial differential equation with boundary conditions defined on the four sides of a rectangle. In this work, combinations of Legendre polynomials have been used to generate orthogonal two-dimensional polynomials on a rectangular domain. The time evolution of the solution is condensed in a set of nonlinear differential equations for the polynomial coefficients. This system can be integrated by using Adomian’s decomposition method with analytic extension or, alternatively, successive approximations, which generate a time series that can be truncated at the required precision order. The result is an analytic approximation to the final solution which can be easily obtained by using any commercial symbolic processor.

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1. Introduction

One of the most frequent problems in the physical sciences is to obtain the time solution of a (linear or nonlinear) partial differential equation which satisfies a set of boundary conditions on a rectangular boundary. For instance, let us consider the following problem:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - Ru \frac{\partial u}{\partial x} + f(x, y)
\]

(1)

with the following boundary conditions defined on a rectangle:

\[u(x, y, t = 0) = \epsilon(x, y)\]
This kind of partial differential equations (PDE) appears frequently coupled with others. For instance, in the incompressible fluid flow problem, the equation above would be slightly completed to become the first component of the Navier–Stokes equations which should be solved in parallel with a Poisson equation for the pressure.

Some methods to obtain an analytical solution of PDE with boundary conditions by means of power series have been explored by the author in [11,12]. Those works, based on the method proposed by George Adomian with the name of decomposition method [5–8], use analytic functions to approximate the problem solution.

These two previous works showed two alternative ways to avoid the strong consequences of the Adomian and Rach “equality of partial solutions theorem” [9] in order to obtain analytic solutions in a initial-boundary value PDE problem: (i) to perturbationally “weaken” the lateral boundary conditions and (ii) to add a specific operator that imposes the boundary conditions.

A third alternative to these approaches is to use “weak” boundary conditions in every boundary by the utilization of spectral methods. This alternative is studied in this article.

In spectral methods, \( u(x, y, t) \) is assumed to be an element of a Hilbert space \( H \) with inner product and norm defined. For each \( t > 0 \), the solution \( u(t) \) belongs to the subspace \( B \) of \( H \) consisting of all \( u \in H \) satisfying the boundary conditions at the edges of the spatial domain.

Let us use the following notation for the problem (1):

\[
\frac{\partial u(x, t)}{\partial t} = L(x, t)u(x, t) + f(x, y) \tag{2}
\]

where \( L \) is the whole spatial operator that includes the double derivative space operators and the nonlinear advective operator \( u \partial u / \partial x \).

Spectral methods approximate \( u(x, t) \) in (2) by a new function \( u_N(x, t) \) belonging to an \( N \)-dimensional subspace \( B_N \) of \( B \) and approximate the operator \( L \) in (2) by a new operator \( L_N \) from \( H \) to \( B_N \) of the form \( L_N = P_N LP_N \), where \( P_N \) is the projection operator of \( H \) onto \( B_N \) and \( f_N = P_N f \). \( B_N \) is called the approximation space.

These methods are in the base of many numerical methods as finite elements, collocation and finite volume methods. In addition, they can be used combined with numerical solvers of ordinary differential equations (ODE) to obtain semi-analytic approximations to the solution of problem (1).

In many cases, specially in control theory and applications, numerical and semi-analytical solutions are not sufficient. This can be the case when solution stability or the solution sensitivity to a set of parameters is being studied. Analytical solutions are then unavoidable and the methods able to approximate these solutions are of the maximum interest.

In this work, we approximate the analytical solution of the nonlinear ODE system with Adomian’s decomposition or, alternatively, successive approximations. This allows the straightforward construction
of efficient computer solvers based on symbolic processors, as Mathematica\textsuperscript{1} or Maple\textsuperscript{2}, able to obtain quick analytical approximations for the whole problem solution.

2. Tau spectral methods

The tau method was invented by Lanczos in 1938 (see for instance \[13\]).

In it, the approximate solution \( u_N(t) \) is assumed to be expanded in terms of a set of expansion functions, \( \phi_n \), belonging to a complete set of orthogonal functions:

\[
    u_N(x, t) = \sum_{n=1}^{N+k} a_n(t) \phi_n(x, t)
\]

Here \( k \) is the number of independent boundary constraints \( Bu_N = 0 \) that must be applied. These \( k \) boundary constraints:

\[
    \sum_{n=1}^{N+k} a_n B \phi_n = 0
\]

are imposed as part of the conditions determining the expansion coefficients \( a_n \). The other \( N \) equations are:

\[
    \frac{da_n}{dt} = (\phi_n, Lu_N) + (\phi_n, f), \quad n = 1, \ldots, N.
\]

The origin of the name “tau method” is that the resulting approximation \( u_N \) is the exact solution to the modified problem:

\[
    \frac{\partial u_N}{\partial t} = Lu_N + f + \sum_{p=1}^{\infty} \tau_p(t) \phi_{N+p}(x)
\]

For each initial value problem and choice of orthogonal basis \( \phi_n \) and associated inner product, there is a choice of tau coefficients such that \( u_N \in B_N \), namely:

\[
    \tau_p(t) = -\left( \phi_{N+p}, Lu_N + f \right), \quad p = k+1, k+2, \ldots
\]

and the remaining tau coefficients are determined by the \( k \) boundary constraints \( B\partial u_N/\partial t = 0 \) (\[14\], p. 12).

In order to solve Eq. (1) we have chosen the following two-dimensional orthogonal basis on the rectangle \(-x_l \leq x \leq x_l, -y_l \leq y \leq y_l\):

\[
    \phi_{ij}(x,y) = \phi_{ij} = \sqrt{\frac{(2i+1)(2j+1)}{4y_l^2}} L_i \left( \frac{x}{x_l} \right) L_j \left( \frac{y}{y_l} \right), \quad 0 \leq i \leq \infty, 0 \leq j \leq \infty
\]

where \( L_i, L_j \) are Legendre polynomials. In this work, the first five Legendre polynomials will be used to generate 25 two-dimensional polynomials.

The first step is to expand \( u(x, y, t) \) in these 25 polynomials and to substitute then in Eq. (1). Then, the resulting equation is multiplied by the polynomial \( \phi_{ij} \) for \( 0 \leq i \leq 2, 0 \leq j \leq 2 \) with the following inner product:

\[
    (f(x,y), \phi_{ij}(x,y)) = \int_{-1}^{1} \int_{-1}^{1} f(x,y) \phi_{ij}(x,y) \, dx \, dy
\]

\textsuperscript{1} Mathematica is a trademark of Wolfram Research.
This generates a system of nine nonlinear ODE for the coefficients $a_{ij}(t)$, $i = 0, 1, 2$, $j = 0, 1, 2$. The scalar product $(\varphi_n, L_{kn})$ in (4) has the following form:

$$e_{ij} = \int_{-\infty}^{\infty} \frac{2i+1}{2\alpha} L_i \left( \frac{x}{\alpha} \right) \sum_{n=0}^{4} a_{ni} \frac{2n+1}{2\alpha} L_n \left( \frac{x}{\alpha} \right) dx$$

$$+ \int_{-\infty}^{\infty} \frac{2j+1}{2\beta} L_j \left( \frac{y}{\beta} \right) \sum_{m=0}^{4} a_{mj} \frac{2m+1}{2\beta} L_m \left( \frac{y}{\beta} \right) dy$$

where $e_{ij}$ is the term corresponding to the $a_{ij}$ time variation $da_{ij}/dt$. To obtain the explicit final form of the previous expression is useful to use the following property of the Legendre polynomials expansion:

If $u(x) = \sum_{n=1}^{\infty} a_n P_n(x)$ where $P_n$ are Legendre polynomials, then $\frac{d^2 u(x)}{dx^2} = \sum_{n=1}^{\infty} b_n P_n(x)$ where:

$$b_n = \left( n + \frac{1}{2} \right) \sum_{p=n+2}^{\infty} \frac{[p(p+1) - n(n+1)] a_p}{p+\alpha(p+\alpha)}$$

The nonlinearity comes from the advective term in Eq. (1).

By logically equating the spectral components, system (3) for problem (1) becomes then:

$$\sum_{n=0}^{4} a_{ni} \varphi_{n-1} = c^1_i, \quad i = 0, \ldots, 4$$

$$\sum_{n=0}^{4} a_{ni} \varphi_{n+1} = c^2_i, \quad i = 0, \ldots, 4$$

$$\sum_{n=0}^{4} a_{nj} \varphi_{n-1} = c^3_j, \quad j = 0, \ldots, 4$$

$$\sum_{n=0}^{4} a_{nj} \varphi_{n+1} = c^4_j, \quad j = 0, \ldots, 4$$

(7)

where the $c^1-c^4$ are the spectral components of the function imposed on boundaries 1–4, respectively.

Not all the 20 boundary conditions are linearly independent; there exists four linear relations among them, namely

$$\sum_{n=0}^{4} \sum_{m=0}^{4} (\pm 1)^n (\pm 1)^m a_{mn} = 0.$$

The previous linear system can be solved for the last 16 coefficients, $a_0$ to $a_{44}$, and the result used in Eq. (4) to obtain a system of ODEs for the remaining nine coefficients.
For instance, if we have \( R = 1 \) and \( f(x, y) = 0 \) in (1) and we take the following initial and boundary conditions:

\[
e(x, y) = \frac{(x^2 - y^2)(x_l - x)}{y_l^2 x_l} \\
f_1(y, t) = \frac{y^2 - y_l^2}{y_l^2} \\
f_2(y, t) = 0 \\
g_1(x, t) = 0 \\
g_2(x, t) = 0
\]

then a system of ODE’s is obtained giving the time variation of every \( a_{ij} \) coefficient as a nonlinear function of the set of coefficients.

The specific expression obtained for the problem (1) can be examined in the author’s web page: www.litec.csic.es/olivares.html.

The initial conditions for the system obtained are the spectral coefficients of \( u(x, y, t = 0) \):

\[
a_{ij}(t = 0) = \int_{x_l}^{x_l} \int_{y_l}^{y_l} u(x, y, t = 0) \phi_{ij}(\frac{x}{x_l}, \frac{y}{y_l}) \, dx \, dy
\]

The system of ODEs obtained can be solved either numerically or by analytical approximation. In the following section, both approaches are used and their respective results are compared.

3. Numerical and analytical integration

The system of ODEs can be numerically integrated by a variety of methods which include predictor–corrector, Adams, Gear and Runge-Kutta finite difference methods. A 10-digit accurate solution has been obtained by a combination of Gear and Runge-Kutta methods with the Mathematica processor.

However, if analytical solutions are needed, Adomian’s decomposition and Picard successive approximations can be very useful choices.

Let us write the system of ODEs obtained in the following way:

\[
L_t a = Ma + Na + g
\]

where \( a \) is the array-variable formed by the nine components \( a_{00}, a_{01}, \ldots, a_{22} \), \( L_t \) is the time-derivative operator, \( M \) is a linear multiplicative operator operating on the set of \( a \) components \( a_{ij}, ij = 00, 01, \ldots, 22 \), \( N \) is a nonlinear multiplicative operator containing products between the \( a \) components and \( g \) is an array of constant terms.

This system can be written formally as:

\[
a = a(t = 0) + L_t^{-1} Ma + L_t^{-1} Na + L_t^{-1} g
\]

Now \( a \) is written as a set of contributions: \( a = a^0 + a^1 + a^2 + a^3 + \cdots \)
By substitution in (9) and making a one-to-one correspondence between the contributions on the left side and the terms on the right side we obtain the following iterative scheme:

\[ a_0 = a(t = 0) + L^{-1}_t q(x, y) \]
\[ a_1 = L^{-1}_t Ma^0 + L^{-1}_t A^1 \]
\[ a_n = L^{-1}_t Ma^{n-1} + L^{-1}_t A^n, \quad 0 < n < \infty \]

(10a)

Here \( v \) has been consistently expanded in Adomian polynomials. For instance, for any product nonlinearity in the form \( a^l a_m \) in any equation component:

\[ a^l a_m = (a^l_0 + a^l_1 + a^l_2 + \cdots)(a^m_0 + a^m_1 + a^m_2 + \cdots) \]
\[ = \sum_{n=1}^{\infty} A_n, \]

(10b)

where in our case:

\[ A_n = \sum_{i=0}^{n-1} a^l_i a^m_{n-i-1} \]

even though other choices are also possible for the expression of \( A_n \).

In this way, the decomposition has been systematically arranged to be sure that all the combinations appearing sooner or later are included in the operator. Every nonlinear product included in the operator \( Na \) will be expanded in this way, \( \sum_{n=0}^{\infty} A_n \) is equal to the sum of a generalized Taylor series about \( u_0 \) for the nonlinear contribution [7].

Now it is possible to define the \( n \)-order approximant to the solution as \( \psi^n = \sum_{j=0}^{n} u_j \). If we have a single equation, the series terms obtained by Adomian’s method approach zero as \( 1/(mn)! \) if \( m \) is the order of the highest linear differential operator and \( \lim_{n \to \infty} \psi^n = u \) [10]. An alternative to Adomian’s decomposition is Picard’s successive approximations. In this method, the zero-order has to be redefined and the iteration re-started whenever a new order-1 approximant \( \psi^1 \) has been obtained. Therefore, the procedure is the following:

\[ a^0 = a(t = 0) + L^{-1}_t q(x, y) \]
\[ a^1 = a(t = 0) + L^{-1}_t q(x, y) + L^{-1}_t Ma^0 + L^{-1}_t Na^0 \]
\[ a^2 = a(t = 0) + L^{-1}_t q(x, y) + L^{-1}_t Ma^1 + L^{-1}_t Na^1 \]
\[ \vdots \]
\[ a^n = a(t = 0) + L^{-1}_t q(x, y) + L^{-1}_t Ma^{n-1} + L^{-1}_t Na^{n-1} \]
\[ 0 < n < \infty \]

(11)

Both Adomian’s integration and Picard’s successive approximations can be used in principle to find the analytical solution of the problem. Picard’s method is something easier to program. Adomian’s decomposition shows better convergence rates than Picard’s method in some cases [2,4]. In addition, the convergence of the Adomian’s method has been deeply studied in the last years by Abbaoui, Cherruault...
and their collaborators [1,10,15] and some sufficient conditions for the convergence of the method are currently available.

4. Integration of the problem

The two methods discussed in the previous section were employed to solve the system of ODEs obtained.

Both methods generate time series whose length increases exponentially with the approximation order. This is due to the multiplicative effect of the nonlinear terms. Given that most of the large exponent terms in the series do not contribute with significant digits to the solution, the series obtained can be systematically truncated at a given accuracy order, for instance, $O(t)^{17}$. It simplifies the iterative calculation process.

In order to obtain quickly convergent series for the nonlinear problem obtained it is necessary to have some care in the choice of the Adomian’s polynomials for the nonlinear terms in the right-hand side of the system of ODEs. In this work, the nonlinear terms of the type $a_i a_j$ in the right-hand side of the system were expanded by means of simple products in the form (10b). In our particular nonlinearity, this simple expression is identical to the one obtained by means of the general formula given by Abbaoui et al. [3] (pp. 91–92).

With the choice (10b) and the iterative process (10a) Adomian’s decomposition method can be applied. According to [15], and generalizing the result of Corollary 3.5 to functions of several variables, we have the following result: if any $k$-derivative of the left-hand side of (9) with respect to $a_{ij}$ has an absolute value smaller than a positive constant $M$, then the series obtained should be absolutely convergent within a time window $(0,T)$ where the order of magnitude of $T$ is $1/M$. Given that the partial derivatives of the left-hand side of the system with respect to $a_{ij}$ is between 80 and 280, the series obtained is expected to converge in a window of size 0.0035–0.01.

However, the method generates a solution that converges extremely slow for the component $a_{22}$ in the system of ODEs obtained. Incidentally, this component is the only one to have a local maximum at $t = 0$.

This behaviour could have been probably improved by choosing a more flexible truncation method instead of simply neglecting the terms of order larger than $O(t)^{17}$.

In our case, the introduction of a single Picard’s iteration after the seventh-order Adomian’s approximant in (10a) and (10b) increased largely the convergence, generating a solution with similar accuracy in the $a_{22}$ component than that obtained by using eight Picard’s successive approximations. This last analytical solution is shown in dashed line in Fig. 1.

Our conclusion was that the influence of the truncation strategy on the convergence of the Adomian’s method would demand additional study. Given that the accuracy of the Picard’s method showed a much lower sensitivity to the truncation strategy employed, this method was used to obtain the final solution of the problem.

In the following, only Picard’s successive approximations calculations are discussed.

The maximum time step that can be used to integrate the expression is given by the convergence radius of the series obtained, which is given by the Cauchy test [16]:

$$ R = \lim_{n \to \infty} \frac{\alpha_n}{\alpha_{n+1}} $$

where $\alpha_n$ is the general term in the power series obtained when the approximation order tends to infinity. This radius is very dependent on the kind of nonlinear operator $N$ solved.
In our test case, $R$ is close to 0.008. Therefore, to obtain the solution for any time, analytic continuation is generally necessary. This means, in our case, to approximate the solution in some time $t_1$ inside the convergence circle, restart the integration taking the $u(x, y, t_1)$ obtained as the initial value of a second problem, and so on in a set of time steps until the solution $u(x, y, t)$ is reached for the time required.

The error size, we are ready to accept in the equation residuals ($\epsilon$) should also be stated. The number of approximation orders that have to be used in every time step will depend on: (i) the size of this $\epsilon$ and (ii) the positive number $R - \delta$ where $\delta$ is the time step length that is being used. When these two numbers tend to zero, the number of approximation orders needed tends to infinity.

Fig. 1 shows the numerical and analytical solutions obtained for $0 < t < 0.008$ when 8 approximation orders and time step of size 0.008 is used in Picard’s method. Fig. 2 shows the solution obtained for
Eleven analytic continuations were necessary to reach this time. Fig. 3 shows a cross-section at $y = 0$ of the numerical and analytical solutions. As it can be observed, the two solutions overlap in the graphics and do not visibly differ. The relative difference between both solutions at $(x = 0, y = 0, t = 0.088)$ is $2.3 \times 10^{-4}$.

The whole algebraic expression $u(x, y, t)$ for $0 \leq t \leq 0.008$ is very large and it can be examined in the author’s web page: www.litec.csic.es/olivares.html. In the solution obtained, 10 polynomial components are null due to the symmetry of the solution with respect to the $y = 0$ axis.

Fig. 4(a) shows the time evolution of the solution residual at point $(0.25, 0)$ as a function of time. This residual has been obtained by substitution in Eqs. (1) and (8). It is an index of the solution error at the point. As it can be seen, the residual in the solution, is practically constant in time. If more accuracy were needed, the number of orthogonal polynomials used (25) should be increased.
However, good approximations are obtained even with a short number of polynomials. Fig. 5 shows
the time evolution of the solution in the $y = 0$ cross-section from the initial condition at $t = 0$ (top) to
the situation at $t = 1$, which no longer evolves because it is practically at steady state.

Rational expressions are preferred in order to maintain the accuracy of the calculation. If it is not
possible to use them, because real numbers are used from the beginning to define the $g(x, y)$ function
and the boundary conditions, then rounding and truncation errors must be controlled by establishing a
minimum digit number in the calculations.

A program as Mathematica® is able to make the calculation of the case studied in 5 min on a Pentium-II
computer with 32 Mb of RAM.

5. Final remarks

The method studied can be used to quickly find an approximation to the solution of any partial differential
equation with initial value and boundary conditions in a rectangle. The method is easily generalized
to time dependent $N$-dimensional problems.

In practice, the utilization of the method is straightforward if some symbolic processor as Mathematica®
or Maple® is used to implement the integrations and derivatives. The solution so obtained is an easily
programmable approximant to the analytic solution of the original problem with the required
accuracy.

In problems with coupled PDE’s as those taking place in fluid mechanics, some elliptic PDE must be
solved in parallel with the problem studied. Elliptic problems are well adapted to the Legendre–tau formu-
lation since Eqs. (3) and (4) become a set of algebraic set of linear and time-independent
equations.

For instance, by using the same expansion in $5 \times 5$ polynomials used earlier, it is easy to show that
the solution for the Poisson problem $\Delta p = f(x, y)$ with $f(x, y) = 100(1/16) - y^2(1/16) - x^2$ and
homogeneous boundary conditions is the following:

\[
p = -\frac{4615}{884736} + \frac{305y^2}{3072} - \frac{875y^4}{3456} + x^4 \left( \frac{875}{3456} + \frac{175y^2}{36} - \frac{350y^4}{27} \right) \\
+ x^2 \left( \frac{305}{3072} - \frac{545y^2}{288} + \frac{175y^4}{36} \right)
\]  

(12)

Thus, advective-diffusive equations coupled with Poisson equations can be solved in an iterative way and approximated in analytical form when it is necessary. Therefore, many applications seem to be accessible to the method studied in fluid mechanics problems as well as in other physical and engineering fields. Some of these matters are being studied currently and will be treated elsewhere.

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References