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# QED Fermi-Fields as Operator Valued Distributions and Anomalies\*

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**Abstract.** The treatment of fields as operator valued distributions (OPVD) is recalled with the emphasis on the importance of causality following the work of Epstein and Glaser. Gauge invariant theories demand the extension of the usual translation operation on OPVD, thereby leading to a generalized *QED* formulation. At  $D=2$  the solvability of the Schwinger model is totally preserved. At  $D=4$  the paracompactness property of the Euclidean manifold permits using test functions which are decomposition of unity and thereby provides a natural justification and extension of the non perturbative heat kernel method (Fujikawa) for abelian anomalies. On the Minkowski manifold the specific role of causality in the restauration of gauge invariance (and mass generation for  $QED_2$ ) is exemplified in a simple way.

## 1 Introduction

The identification of fields as operator-valued distributions (OPVD) is almost as old as Quantum Field Theory (QFT) itself. To cite only two early basic texts among many just recall Bogoliubov-Shirkov's [1] and Schweber's [2] monographs. Bogoliubov and Shirkov were original in that their construction of the S-matrix involves test-functions in the range  $(0, 1)$  but different from zero only in a certain finite space-time region. The condition of causality applied to the supports of the test functions induces essential relations between the S-matrix amplitudes. This approach was later pursued and developed by Epstein and Glaser [3] leading to a well recognized causal perturbation theory which is "free of mathematically undefined quantities but hides the multiplicative structure of renormalization" [4]. Apparently for most of QFT practitioners this was a fatal disease and Epstein and Glaser's work is scarcely referenced and almost fell into oblivion, save for the important contributions of Scharf [5]. The situation is different in the world

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of mathematicians dedicated to the construction of a rigorous and mathematically well defined QFT. In this respect the necessary steps involve the extension of singular distributions to the whole space-time manifold. An often reported conjecture made by A. Connes and independently by R. Estrada [6] states that "Hadamard's finite part theory is in principle enough to deal with QFT divergences". Clearly, for outsiders, the statement called for some clarifications. They came only recently [7]. It appears that a rigorous way to get an extension of a singular distribution is a weighted Taylor series surgery, that is to throw away an appropriate jet of the test function at the singularity. Transposed to Fourier space the procedure amounts to a subtraction method which includes BPHZ renormalization [1, 8] as a special case. In Minkowskian metric this is equivalent to the implementation of causality while in the Euclidean counterpart it is a symmetry preserving prescription for subtractions. The OPVD approach we implemented in the Euclidean LCQ study of the critical properties of  $\phi_{1+1}^4$ -theory [9] is in the line of Epstein and Glaser [3]. However it has the important additional feature that the paracompactness of the Euclidean manifold allows using any partition of unity as  $C^\infty$  test functions with compact support in Fourier space. Thereby any UV- divergences in Fourier space integrations are properly regulated. It is our aim here to develop the treatment of Fermi-fields as OPVD in a gauge field environment, sketched at the LC03 Durham meeting [10].

## 2 Fermi-field as OPVD: problems with gauge invariant translation. Lessons from Schwinger's $QED_2$ .

Let  $\psi(x)$  be the Dirac massive free field, then  $(i \not{\partial} - m)\psi(x) = 0$ .  $\psi(x)$  is an OPVD which defines a functional  $\Phi(\rho)$  with respect to a test function :  $\Phi(\rho) \equiv \langle \psi, \rho \rangle = \int d^{(D)}y \psi(y) \rho(y)$ .  $\Phi(\rho)$  is an operator-valued functional with the possible interpretation of a more general functional  $\Phi(x, \rho)$  evaluated at  $x = 0$ . Indeed the translated functional is a well defined object [11] such that  $T_x \Phi(\rho) \equiv \langle T_x \psi, \rho \rangle = \langle \psi, T_{-x} \rho \rangle = \int d^{(D)}y \psi(y) \rho(x - y)$ . Due to the properties of  $\rho$ ,  $\Psi(x) \equiv T_x \Phi(\rho)$  obeys also Dirac's equation and is taken as the physical field which is now and most importantly an analytic function of the  $x$ -variable [11]. The possible singular behaviour of the original field  $\psi(x)$  is now transferred to the class of test functions necessary to define a *bona-fide* tempered extension of the original OPVD on the whole space-time manifold [3, 5, 7]. For  $QED$  the fermionic field obeys  $(i \not{\partial} - e \not{A} - m)\psi(x) = 0$ , and it is clear that the translation in  $\psi(x)$  must be done in a way compatible with gauge transformations. The immediate candidate would be the well known Dirac string from  $x$  to  $y$ . But now  $\Psi(x)$  depends on the path  $\gamma(s)$  from  $x$  to  $y$  and does not obey exactly  $QED$ 's equation. It induces an alteration of the path integral formulation. Moreover for  $QED_2$   $\Psi(x)$  does not exhibit the bosonization features of the known solution. However, following Dirac's early analysis [12], the most general writing for  $\Psi(x)$  is

$$\Psi(x) = \int d^{(D)}y \rho(y - x) \exp[ie \int d^{(D)}z \mathcal{C}^\mu(x, y, z) A_\mu(z)] \psi(y). \quad (1)$$

$\Psi(x)$  transforms as the original  $\psi(x)$  under a gauge transformation provided that

$$\partial_z^\mu \mathcal{C}_\mu(x, y, z) = \delta(z - x) - \delta(z - y). \quad (2)$$

There  $\mathcal{C}^\mu$  may have a matrix structure. To clarify the importance of this fact consider again  $QED_2$ . In this case the basic matrices are  $\mathcal{I}, \gamma_\mu, \gamma_5$ . The contributions due to  $(\mathcal{I} \pm \gamma_5)$  can be eliminated in the Lorentz gauge, for then one may look for  $\mathcal{C}^\mu$  under the form  $\partial_z^\mu C \rightarrow \int d^{(D)}z (\partial_z^\mu C) A_\mu = - \int d^{(D)}z C (\partial_z^\mu A_\mu) = 0$ . Hence the relevant term involves only  $\gamma_\mu$  and  $\mathcal{C}^\mu(x, y, z) = C(x, y, z) \gamma^\mu$ . But at  $D = 2$  the longitudinal part of  $A_\mu(z)$  can be gauged away, only the transverse part matters and one has then  $\int d^{(2)}z C(x, y, z) \gamma^\mu A_\mu(z) = \int d^{(2)}z C(x, y, z) \gamma^\mu \epsilon_{\mu\nu} \partial_z^\nu \phi(z) = -\gamma_5 \int d^{(2)}z C(x, y, z) \gamma_\nu \partial_z^\nu \phi(z) = \gamma_5 \int d^{(2)}z (\partial_z^\nu C(x, y, z) \gamma_\nu) \phi(z) = \gamma_5 [\phi(x) - \phi(y)]$ , where Eq.(2) and the linearity of  $C(x, y, z)$  in  $\gamma$  matrices has been used.  $\Psi(x)$  writes now  $\Psi(x) = \exp[ie\gamma_5\phi(x)] \int d^{(2)}y \rho(y - x) \exp[-ie\gamma_5\phi(y)] \psi(y) = \exp[ie\gamma_5\phi(x)] \chi(x)$ . This is just the bosonization ansatz of the conventional theory and it is checked that  $\Psi(x)$  still obeys  $(i\partial - eA)\Psi(x) = 0$ , giving the usual anomaly and mass generation (cf below). At  $D = 4$   $\mathcal{C}^\mu(x, y, z)$  can be a  $(4 \otimes 4)$  matrix built from  $\mathcal{I}, \gamma^\mu, \gamma_5, \gamma_5\gamma^\mu, \sigma^{\mu\nu}$ . As for  $D = 2$  contributions from  $\mathcal{I}, \gamma_5$  can be disregarded. Clearly we don't want  $\mathcal{C}^\mu$  to mix the chiralities of the original  $\psi(y)$  while preserving its equation of motion and gauge transformation property. This imposes a unique writing for  $\Psi(x)$  as

$$\Psi(x) = \int d^{(4)}y \rho(y - x) [H_+(x, y) + H_-(x, y)] \quad (3)$$

with

$$H_\pm(x, y) = \frac{1}{2} \exp[i\frac{e}{2} \int d^{(4)}z C(x, y, z) \gamma^\mu (1 \pm \gamma_5) A_\mu(z)] (1 \pm \gamma_5) \psi(y). \quad (4)$$

The solution of Eq.(2) is now path independant and gives  $C(x, y, z) = -i \int \frac{d^{(4)}k}{(2\pi)^4} \frac{\gamma \cdot k}{k^2} [\exp(ik(z - x)) - \exp(ik(z - y))]$ , which cannot be gauged away [10]. It is verified that  $\Psi(x)$  obeys the classical  $QED$  original equation<sup>1</sup> thereby permitting a path integral formulation in terms of this "smeared" field. The question of charged particles and asymptotic states in gauge theories [13] is now to be adressed with respect to this field.

### 3 Anomalies: Fujikawa's method revisited

Using the properties of  $\gamma$ -matrices the right- and left-handed components can be recombined. With the variable change  $y = x + \epsilon$  and  $C(x, y, z) = c(z - x) - c(z - y) = \epsilon \cdot \partial_z c(z - x) + \mathcal{O}(\epsilon^2)$ ,  $\Psi(x)$  reads now

$$\Psi(x) = \int \frac{d^{(D)}p}{(2\pi)^D} \int d^{(D)}\epsilon \rho(\epsilon) \exp[i\epsilon \cdot (p + e \int d^{(D)}z \partial_z c(z - x) A(z))] \exp[ip \cdot x] \tilde{\psi}(p).$$

<sup>1</sup>The necessary commutations result from  $[\gamma^\mu(1 \pm \gamma_5), \gamma^\nu(1 \pm \gamma_5)] = 0, \forall(\mu, \nu)$ . Whereas for  $m > 0$  the distinction between right and left-handed fermions is only technical, for  $m = 0$  the chirality will remain a conserved quantum number.

Here the neglected terms are of order  $\mathcal{O}(R^2)$ , where  $R$  is the "small" radius of the ball, support of the test function  $\rho(\epsilon)$ . Due to rotational symmetry its Fourier transform depends on  $q^2$  only and since  $\tilde{\rho}(q^2) = \tilde{\rho}((\gamma \cdot q)^2)$  it gives  $\Psi(x) = \int \frac{d^{(D)}p}{(2\pi)^D} \tilde{\rho}((\not{p} + eA(x))^2) \exp[ip \cdot x] \tilde{\psi}(p) + \mathcal{O}(R^2) = \int \frac{d^{(D)}p}{(2\pi)^D} \tilde{\rho}(-\not{D}_x^2) \exp[ip \cdot x] \tilde{\psi}(p) + \mathcal{O}(R^2)$ . Since the value of the mass is inessential in the sequel  $m = 0$  will be taken. In the path integral formalism the measure is now expressed in terms of the *regularized* complete set of eigenfunctions  $\{\Xi_n\}$  of the hermitian operator  $\not{D}$ , since  $\Xi_n(x)$  and the unregularized  $\xi_n(x)$  obey the same eigenvalue equation  $\not{D}\xi_n(x) = \lambda\xi_n(x)$  by construction. Hence  $\Psi(x) = \sum_n a_n \Xi_n(x)$ ,  $\bar{\Psi}(x) = \sum_n b_n \bar{\Xi}_n(x)$  and  $\mathcal{D}\Psi(x)\mathcal{D}\bar{\Psi}(x) = \prod_{m,n} da_m db_n$ . The analysis follows as usual [14], save for the fact that the quantity  $B(x) = \sum_n \Xi_n^+(x) \gamma_5 \Xi_n(x) = \int \frac{d^{(4)}p}{(2\pi)^4} \text{tr} \{ \gamma_5 \exp[-ik \cdot x] \tilde{\rho}^2(-\not{D}_x^2) \exp[ik \cdot x] \}$  is now finite due to the presence of test functions. The paracompactness property of the Euclidean manifold implies that  $\tilde{\rho}(p^2)$  can be taken as a decomposition of unity which introduces a scale (related to the inverse radius  $R$  of the ball, support of test function in configuration space) and in fact  $\tilde{\rho}(p^2) \rightarrow \tilde{\rho}(\frac{p^2}{\Lambda^2})$  [9, 10]. In the limit  $\Lambda \rightarrow \infty$   $B(x)$  reduces to  $B(x) = \frac{e^2}{8} \text{tr} [\gamma_5 (\sigma^{\mu\nu} F_{\mu\nu})^2] \int \frac{d^4y}{(2\pi)^4} \frac{d^2}{d^2(y^2)} [\tilde{\rho}^2(y^2)] = \frac{e^2}{16\pi^2} F^{\mu\nu}(x) F_{\mu\nu}(x)$ , since  $\tilde{\rho}^2(0) = 1$  (decomposition of unity). At  $D = 2$  only the term in  $\text{tr}[\gamma_5 \sigma^{\mu\nu} F_{\mu\nu}]$  survives giving also the known result.

#### 4 Causality and gauge invariance in $QED_2$

Whatever the importance of the non-perturbative anomaly test, it is well known that any type of regularisation in Fourier space via damping functions at large momenta violates perturbatively gauge invariance and Ward identities. This can be avoided if causality is implemented from the start in the construction of the S-matrix as advocated by Epstein and Glaser [3] and Scharf [5]. This line of thought leads to use propagators with truly causal supports satisfying dispersion relations being tantamount to subtractions dictated by analyticity. It guarantees finite results and gauge invariance. Moreover it is conceptually important to realize that the well established BPHZ renormalization scheme [1, 8] turns out to be a special case of the Epstein-Glaser prescription [7]. For the special case of  $QED_2$  (Schwinger model) the essential quantity is the causal polarization tensor whose form is [5, 15]

$$\begin{aligned} \Pi_{\mu\nu}(k) &= \int d^2x \exp[ik \cdot x] \{ \text{Tr} [\gamma_\mu S^{(+)}(x) \gamma_\nu S^{(-)}(-x)] - (\mu \leftrightarrow \nu, x \leftrightarrow -x) \} \\ &= e^2 [\hat{P}_{\mu\nu}(k) - (\mu \leftrightarrow \nu, k \leftrightarrow -k)], \end{aligned} \quad (5)$$

$S^{(\pm)}(\pm x)$  are related to the usual Feynman propagator  $S^F(\pm x)$  corrected respectively by advanced and retarded pieces  $\int \frac{d^2p}{(2\pi)^2} \frac{(\not{p}+m) \exp[-ip \cdot x]}{(p^2 - m^2 \mp ip^0 \epsilon)}$  to give propagators with truly causal supports. They write explicitly

$$S^{(\pm)}(\pm x) = \pm i \int \frac{d^2p}{(2\pi)^2} 2\pi \delta(p^2 - m^2) \Theta(p^0) (\pm \not{p} + m) \exp[-ip \cdot x] \quad (6)$$

$\tilde{\rho}(-\not{D}_x^2)$  cannot be pulled out of the integral since  $\tilde{\psi}(p)$  is a distribution.

With these expressions for  $S^{(\pm)}(\pm x)$  and after tracing over the  $\gamma$  matrices  $\widehat{P}_{\mu\nu}(k)$  becomes

$$\begin{aligned} \widehat{P}_{\mu\nu}(k) &= -2 \int d^2p \delta(p^2 - m^2) \delta(k^2 - 2k \cdot p) \Theta(p_0) \Theta(k_0 - p_0) \\ &\quad [p_\mu k_\nu + p_\nu k_\mu - 2p_\mu p_\nu - \frac{k^2}{2} g_{\mu\nu}]. \end{aligned} \quad (7)$$

The integral over  $p$  is finite and  $k^\mu \widehat{P}_{\mu\nu}(k) = 0$ . It is important to note the role of the  $\delta$  functions in getting this result. They are specific to the form of the propagators  $S^{(\pm)}(\pm x)$ . Hence  $\Pi_{\mu\nu}(k) = e^2 (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) k^2 \widehat{d}(k)$  with

$$\widehat{d}(k) = \lim_{m^2 \rightarrow 0} \frac{4m^2}{k^4} \frac{1}{\sqrt{1 - \frac{4m^2}{k^2}}} \Theta(k^2 - 4m^2) \text{sign}(k_0) = 2\delta(k^2) \text{sign}(k_0). \quad (8)$$

Causality imposes redefining  $k^2 \widehat{d}(k)$  through a one-time subtracted dispersion relation

$$\begin{aligned} k^2 \widehat{d}(k) \rightarrow \widehat{r}(k) &= \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\widehat{d}(tk) (tk)^2}{(t - i\epsilon)(1 - t + i\epsilon)} \\ &= \frac{im^2}{\pi} \left[ \frac{1}{m^2} + \frac{2}{k^2} \frac{1}{\sqrt{1 - \frac{4m^2}{k^2}}} \log \left[ \frac{\sqrt{1 - \frac{4m^2}{k^2}} + 1}{\sqrt{1 - \frac{4m^2}{k^2}} - 1} \right] \right]. \end{aligned} \quad (9)$$

Here  $k^2 > 4m^2, k_0 > 0$ . The first term in  $\frac{1}{m^2}$  comes from the subtraction itself. Hence  $\lim_{m^2 \rightarrow 0} \widehat{r}(k) = \frac{i}{\pi}$ , giving  $\Pi_{\mu\nu}(k) = i \frac{e^2}{\pi} (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})$ , ie the right boson mass. In the process it is essential to identify properly the singular order of the propagator to determine the subtraction needed in the dispersion relation since it determines completely the limit  $m^2 \rightarrow 0$ . If not, then UV divergences and violation of gauge invariance will show up.

## 5 Conclusion

The necessity of formulating Quantum Field Theory in the continuum requires treating fields as OPVD with specific test functions. Thereby a non-standard regularization scheme is obtained which is in the line of the Epstein and Glaser extension of singular distributions. For abelian gauge theories the usual translation operation on distributions has been modified following Dirac's early analysis of the possible phase factor leading to the proper gauge transformation of the initial Fermi field itself. The procedure meets the necessary requirements of leaving the solution of the Schwinger model unaltered and yet still permits a path integral formulation in terms of the smeared field. It provides naturally an interpretation of Fujikawa's analysis of the abelian anomaly. Finally recognizing the filiation of our approach with Epstein and Glaser's treatment indicates the essential role of causality in restoring gauge invariance which is otherwise violated by any regularisation with UV damping test functions only. This opens up the very interesting perspective of building up a gauge invariant LC-quantization framework free of

divergences by construction from the outset. Only finite renormalization will occur in connection with the intrinsic scale present in the decomposition of unity in Fourier space or in the parametrization of Epstein and Glaser's weight function  $w(x), w(0) = 1$  in configuration space.

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