CODERIVATIONS OF COCOMMUTATIVE COALGEBRAS

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We set our notation as follows: Let V be a graded vector space over some field and consider the graded vector spaces $\bigwedge^* V = \bigoplus \bigwedge^n V$ and $T^*V = \bigoplus T^n V$ with projection maps $p_n : \bigwedge^* V \longrightarrow \bigwedge^n V$ and $\pi_n : T^*V \longrightarrow T^n V$. Following [G - L], we recall that the comultiplication $\Psi : T^*V \longrightarrow T^*V \bigotimes T^*V$ is defined by the commutativity of the diagram

$$\begin{array}{ccc} T^*V & \xrightarrow{\Psi} & T^*V \bigotimes T^*V \\ {\scriptstyle \pi_{a+b}}\downarrow & & \downarrow{\scriptstyle \pi_a \otimes \pi_b} \\ T^{a+b}V & \xrightarrow[F(a,b)]{} & T^aV \bigotimes T^bV \end{array}$$

where $F(a,b) : T^{a+b} \longrightarrow T^a \bigotimes T^b$ and $E(a,b) : T^a \bigotimes T^b \longrightarrow T^{a+b}$ are the natural isomorphisms.

Let $\Delta : \bigwedge^* V \longrightarrow \bigwedge^* V \bigotimes \bigwedge^* V$ be the shuffle coproduct which gives $\bigwedge^* V$ the structure of the cofree cocommutative coalgebra generated by V. Let $\chi : \bigwedge^* V \longrightarrow T^*V$ be the injective coalgebra map given by

$$\chi(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in S_n} e(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

(cf. Tanre); here, $e(\sigma)$ denotes the Koszul sign of the commutations, which, recall, does not include the sign of the permutation; denote the inverse of χ (on the image of χ) by ρ.

Lemma 1. $\Delta : \bigwedge^* V \longrightarrow \bigwedge^* V \bigotimes \bigwedge^* V$ *is characterized by the commutative diagram*

$$\begin{array}{ccccc} \bigwedge^* V & \xrightarrow{\Delta} & \bigwedge^* V \bigotimes \bigwedge^* V & \xrightarrow{\chi \otimes \chi} & T^*V \bigotimes T^*V \\ {\scriptstyle p_{a+b}}\downarrow & & & & \downarrow{\scriptstyle \pi_a \otimes \pi_b} \\ \bigwedge^{a+b} V & \xrightarrow[\chi]{} & T^{a+b}V & \xrightarrow[F(a.b)]{} & T^aV \bigotimes T^bV \end{array}$$

Proof. Consider the diagram

$$
\begin{array}{ccccc}
\bigwedge^* V & \xrightarrow{\Delta} & \bigwedge^* V \bigotimes \bigwedge^* V & \xrightarrow{\chi\otimes\chi} & T^*V \bigotimes T^*V \\
\| & & & & \| \\
\bigwedge^* V & \xrightarrow{\chi} & T^*V & \xrightarrow{\Psi} & T^*V \bigotimes T^*V \\
{\scriptstyle p_{a+b}}\downarrow & & {\scriptstyle \pi_{a+b}}\downarrow & & {\scriptstyle \pi_a\otimes\pi_b}\downarrow \\
\bigwedge^{a+b} & \xrightarrow[\chi]{} & T^{a+b}V & \xrightarrow[F(a,b)]{} & T^aV \bigotimes T^bV
\end{array}
$$

where the top rectangle commutes since χ is a coalgebra map, the lower left square commutes since χ is injective, and the lower right square commutes by [G-L]. $\square$

Lemma 2. *Let $\Delta^{(n)} : \bigwedge^* V \longrightarrow T^n \bigwedge^* V$ be the iterated coproduct, and let $\chi : \bigwedge^n \bigwedge^* V \longrightarrow T^n \bigwedge^* V$ be as above. Then Image($\Delta^{(n)}$) $\subset$ Image(χ).*

Proof. Suppose that $I_1 \cup \dots I_n$ is a partition of I so that$v_{I_1} \otimes \dots \otimes v_{I_n} \in Im(\Delta^{(n)})$.

We claim that $v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} \in Im(\Delta^{(n)})$ for all $\sigma \in S_n$. To see this, consider $v_I = v_{I_1 \cup \dots \cup I_n} \in \bigwedge^* V$. Apply Δ to obtain

$$\Delta(v_{I_1 \cup \dots \cup I_n}) = \sum e(\sigma) v_{J_1} \otimes v_{J_2}$$

where the sum is taken over all unshuffles (J_1, J_2) of I. Among these summands are terms of the form

$$v_{I - I_{K_1}} \otimes v_{I_{K_1}}, K_1 = 1, \dots n.$$

Apply $\Delta \otimes 1$ to each such summand and obtain terms which include $n-1$ terms of the form

$$v_{I - I_{K_1} - I_{K_2}} \otimes v_{I_{K_2}} \otimes v_{I_{K_1}}, K_1 \neq K_2.$$

Now apply $\Delta \otimes 1 \otimes 1$ to each of the above terms and select the $n-2$ terms of the form

$$v_{I - I_{K_1} - I_{K_2} - I_{K_3}} \otimes v_{I_{K_3}} \otimes v_{I_{K_2}} \otimes v_{I_{K_1}}.$$

Continue this procedure which terminates with the application of $\Delta \otimes 1 \otimes \dots \otimes 1$ to obtain the $n!$ terms of the form

$$v_{I - I_{K_1} - \dots - I_{K_{n-1}}} \otimes v_{I_{K_{n-1}}} \otimes \dots \otimes v_{I_{K_1}}.$$

It is clear that these are all of the permutations of the original term. $\square$

Recall that a coderivation of a coalgebra A is a linear map $f : A \longrightarrow A$ such that $\pi_0 f = 0$ and

$$(f \otimes 1 + 1 \otimes f)\Delta_A = \Delta_A f$$

where Δ_A is the comultiplication on A.

In particular, a coderivation $f : T^*V \longrightarrow T^*V$ is characterized by

$$\pi_n f = \sum E(1, \dots, 1)(\pi_1 \otimes \dots \otimes \pi_1 f \otimes \dots \otimes \pi_1)\Psi^{(n)}$$

where

$$\Psi^{(n)} = (\Psi \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-2})) \circ \cdots \circ (\Psi \otimes 1) \circ \Psi$$

and

$$E(1, \ldots, 1) : T^1 V \otimes \cdots \otimes T^1 V \longrightarrow T^n V$$

is the isomorphism given by

$$E(1, \ldots, 1) = E(n-1, 1) \circ E(n-2, 1) \otimes 1 \circ \cdots \circ is(1, 1) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-2}.$$

See [G-L]for details.

We describe coderivations in the cocommutative setting in a similar fashion. First we have

Lemma 3. *The projection* $p_n : \bigwedge^* V \longrightarrow \bigwedge^n V$ *is given by*

$$p_n = \rho E(1, \ldots, 1)\chi^{\otimes n}(p_1 \otimes \cdots \otimes p_1)\Delta^{(n)}$$

Proof. By [G-L],

$$\pi_n = E(1, \ldots, 1)(\pi_1 \otimes \cdots \otimes \pi_1)\psi^{(n)}$$

and since $p_n = \rho\pi_n\chi$ and $\pi_1\chi = \chi p_1$ by Lemma 1,

$$\begin{aligned} p_n &= \rho E(1, \ldots, 1)(\pi_1 \otimes \cdots \otimes \pi_1)\psi^{(n)}\chi \\ &= \rho E(1, \ldots, 1)(\pi_1 \otimes \cdots \otimes \pi_1)\chi^{\otimes n}\Delta^{(n)} \\ &= \rho E(1, \ldots, 1)\chi^{\otimes n}(p_1 \otimes \cdots \otimes p_1)\Delta^{(n)}. \end{aligned}$$

□

Now if $f : \bigwedge^* V \longrightarrow \bigwedge^* V$ is a coderivation, the equation

$$(f \otimes 1 + 1 \otimes f)\Delta = \Delta f$$

is equivalent to

$$\begin{aligned} (\pi_a \otimes \pi_b)(\chi \otimes \chi)(f \otimes 1 + 1 \otimes f) &= (\pi_a \otimes \pi_b)(\chi \otimes \chi)\Delta f \\ &= (\pi_a \otimes \pi_b)\Psi\chi f \qquad \text{since } \chi \text{ is a coalgebra map} \\ &= F(a, b)\pi_{a+b}\chi f \qquad \text{by [G-L]} \\ &= F(a, b)\chi p_{a+b} f. \end{aligned}$$

As a result,

$$\chi p_{a+b} f = E(a, b)(\pi_a \otimes \pi_b)(\chi \otimes \chi)(f \otimes 1 + 1 \otimes f)\Delta$$

or

$$p_{a+b} f = \rho E(a, b)(\pi_a \otimes \pi_b)(\chi \otimes \chi)(f \otimes 1 + 1 \otimes f)\Delta.$$

Note that since the right hand side of the penultimate equation is in the image of Δ, it makes sense to apply ρ to it.

Lemma 4. *A coderivation* $f : \bigwedge^* V \longrightarrow \bigwedge^* V$ *is characterized by*

$$p_n f = \rho E(1,\ldots,1)\chi^{\otimes n}(\sum p_1 \otimes \cdots \otimes p_1 f \otimes \cdots \otimes p_1)\Delta^{(n)}.$$

Proof. We can see that $p_2 f$ satisfies the claim by letting $a = b = 1$ in the previous remark. By induction on n, we may write

$$p_{n+1} f = \rho E(n,1)(\pi_n \otimes \pi_1)(\chi \otimes \chi)(f \otimes 1 + 1 \otimes f)\Delta$$

$$= \rho E(n,1)(\chi \otimes \chi)(p_n \otimes p_1)(f \otimes 1 + 1 \otimes f)\Delta$$

$$= \rho E(n,1)(\chi \otimes \chi)(p_n f \otimes p_1 + p_n \otimes p_1 f)\Delta$$

$$\rho E(n,1)(\chi \otimes \chi)[\rho E(1,\ldots,1)\chi^{\otimes n}(\sum p_1 \otimes \cdots \otimes p_1 f \otimes \cdots \otimes p_1)\Delta^{(n)} \otimes p_1$$

$$+\rho E(1,\ldots,1)\chi^{\otimes n}(p_1 \otimes \cdots \otimes p_1)\Delta^{(n)} \otimes p_1 f]\Delta$$

$$= \rho E(n,1)(\chi \otimes \chi)(\rho \otimes 1)(E(1,\ldots,1) \otimes 1)(\chi^{\otimes n} \otimes 1)[(\sum p_1 \otimes \cdots \otimes p_1 f \otimes \cdots \otimes p_1) \otimes p_1$$

$$+(p_1 \otimes \cdots \otimes p_1) \otimes p_1 f](\Delta^{(n)} \otimes 1)\Delta$$

$$= \rho E(n+1,1)\chi^{\otimes n+1}[\sum p_1 \otimes \cdots \otimes p_1 f \otimes \cdots \otimes p_1]\Delta^{(n+1)}$$

after some minor simplifications. □

Remark. If $g : \bigwedge^* V \longrightarrow V$ is a linear map, then the extension of g to a coderivation $\hat{g} : \bigwedge^* V \longrightarrow \bigwedge^* V$ with $p_1 \hat{g} = g$ is given by the equation in Lemma 4.

Example. A familiar example of this construction may be seen in the following situation. Suppose that $g : V \bigwedge V \longrightarrow V$ is a linear map (e.g. a Lie bracket). Regard g as a map $g : \bigwedge^* V \longrightarrow V$ by defining $g | \bigwedge^n V = 0$ when $n \neq 2$. To define $\hat{g} : \bigwedge^3 V \longrightarrow \bigwedge^2 V$, we calculate

$$p_2 \hat{g} = \rho E(1,1)(\chi \otimes \chi)(g \otimes p_1 + p_1 \otimes g)\Delta.$$

(Signs and degrees are suppressed here for clarity.)

$$x \wedge y \wedge z \xrightarrow{\Delta} (x \wedge y) \otimes z + (x \wedge z) \otimes y + (y \wedge z) \otimes x + z \otimes (x \wedge y) + y \otimes (x \wedge z) + x \otimes (y \wedge z)$$

$$\xrightarrow{g \otimes p_1 + p_1 \otimes g} g(x \wedge y) \otimes z + g(x \wedge z) \otimes y + g(y \wedge z) \otimes x + z \otimes g(x \wedge y) + y \otimes g(x \wedge z) + x \otimes g(y \wedge z)$$

$$\xrightarrow{\rho(\chi \otimes \chi)} g(x \wedge y) \wedge z + g(x \wedge z) \wedge y + g(y \wedge z) \wedge x.$$

Here χ is the identity map on $\bigwedge^1 V = V$.

Coderivations on cofree cocommutative coalgebras are related to coderivations on cofree coalgebras by the following proposition.

Proposition. *Suppose that $f : T^*V \longrightarrow V$ is a linear map which extends to a coderivation $\hat{f} : T^*V \longrightarrow T^*V$. Then the diagram*

$$\begin{array}{ccccc} \bigwedge^* V & \xrightarrow{\chi} & T^*V & = & T^*V \\ \widehat{f\circ\chi}\uparrow & & \hat{f}\uparrow & & \downarrow\pi_1 \\ \bigwedge^* V & \xrightarrow{\chi} & T^*V & \xrightarrow{f} & V \end{array}$$

commutes. Here, $\widehat{f\circ\chi}$ is the extension of the map $f\circ\chi : \bigwedge^* V \longrightarrow V$ to a coderivation $\widehat{f\circ\chi} : \bigwedge^* V \longrightarrow \bigwedge^* V$.

Proof. To compute $\chi\widehat{f\circ\chi}$, we compute

$$\pi_n\chi\widehat{f\circ\chi} = \chi p_n\widehat{f\circ\chi}$$

$$= \chi\rho E(1,\ldots,1)\chi^{\otimes n}[\sum p_1\otimes\cdots\otimes p_1\widehat{f\circ\chi}\otimes p_1\otimes\cdots\otimes p_1]\Delta^{(n)}$$

$$= E(1,\ldots,1)[\sum \chi p_1\otimes\cdots\otimes \chi f\chi\otimes\cdots\otimes\chi p_1]\Delta^{(n)}$$

$$= E(1,\ldots,1)\sum[\pi_1\chi\otimes\cdots\otimes\chi f\chi\otimes\cdots\otimes\pi_1\chi]\Delta^{(n)}$$

$$= E(1,\ldots,1)\sum[\pi_1\otimes\cdots\otimes\pi_1\hat{f}\otimes\cdots\otimes\pi_1]\chi^{\otimes n}\Delta^{(n)}$$

(since $\chi f = f$ and $\pi_1\hat{f} = f$)

$$= E(1,\ldots,1)\sum[\pi_1\otimes\cdots\otimes\pi_1\hat{f}\otimes\cdots\otimes\pi_1]\Psi^{(n)}\chi$$

$= \pi_n\hat{f}\chi$ □.

The relationship between sha structures and sh Lie structures now follows:

Theorem. *Suppose that $\{m_n\} : T^*V \longrightarrow V$ is an sha structure on V. Then $\{m_n\circ\chi\}$ is an sh Lie structure on V.*

Proof. Recall that the sha structure induces linear maps $T^nsV \longrightarrow sV$ which we also denote by m_n . These maps extend to coderivations with the property that $D = \sum\widehat{m_n} : T^*sV \longrightarrow T^*sV$ is a differential. We show that $D' : \bigwedge^* sV \longrightarrow \bigwedge^* sV$ defined by $D' = \sum\widehat{m_n\circ\chi}$ is also a differential. The commutative diagram

$$\begin{array}{ccc} \bigwedge^* sV & \xrightarrow{\chi} & T^*sV \\ D'\uparrow & & \uparrow D \\ \bigwedge^* sV & \xrightarrow{\chi} & T^*sV \\ D'\uparrow & & \uparrow D \\ \bigwedge^* sV & \xrightarrow{\chi} & T^*sV \end{array}$$

yields $D'^2 = 0$ since $D^2 = 0$ and χ is injective.

□