Valuation Derived from Graded Ring and Module and Krull Dimension Properties

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ABSTRACT: In this paper we show if R is a graded ring then we can define a valuation on R induced by graded structure, and we prove some properties and relations for R. Later we show that if R is a graded ring and M a graded R-module then there exists a valuation on of M which is derived from graded structure and also we prove some properties and relations for R. In the following we give a new method for finding the Kurll dimension of a valuation ring.

Key Words: Graded Ring, Graded Moudule, Valuation Ring ,Valuation on a Module, Krull Dimension.

Contents

1. Introduction

In the algebra valuation ring and graded ring are two most important structures [10], [11], [12]. We know that the graded ring is also the most important structure since the graded ring is a base for dimension and [4], on the depth of the associated graded ring of a filtration [5], [6]. So, as these important structures, the relation between these structures are useful for finding some new structures, and if R is a discrete valuation ring then R has many properties that have many uses for example Decidability of the theory of modules over commutative valuation domains [12], Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices [11]. Graded ring is also most important structures especially for making some non-commutative structures with commutative rings [3].

In [1] M.H. Anjom SHoa and M.H. Hosseini show that there exists a filtered ring which derived from a and they find some new properties. Consequently M.H. Anjom SHoa and M.H. Hosseini in [2] proved that there exists a quasi valuation ring derived from filtered ring and also they proved if R is a strongly filtered ring then we can find a valuation ring which derived from R. In this article they find some new properties.

In this article we investigate the relation between graded ring and graded Module. We prove that if we have graded ring then we can find a valuation or quasi valuation ring. Continuously We prove that if M be a graded module then we can define a valuation on M. In following we prove a method for finding Krull dimension.

2. Preliminaries

Definition 2.1. *A filtered ring R is a ring together with a family $\{R_n\}_{n \geq 0}$ of subgroups of R satisfying in the following conditions:*

   i) $R_0 = R$;

   ii) $R_{n+1} \subseteq R_n$ for all $n > 0$;

   iii) $R_n R_m \subseteq R_{n+m}$ for all $n, m > 0$.

If R has conditions (i) , (ii) and instead of (iii) it has the following condition: $R_n R_m = R_{n+m}$ for all $n, m > 0$ then we say R has a strong filtration.

Definition 2.2. *Let R be a ring and I an ideal of R. Then $R_n = I^n$ is called I-adic filtration.*

Definition 2.3. *Let R be a filtered ring with $\{R_i\}$ and S a be subring of R. Then $\{\frac{R_i}{S}\}$ is a filtration on $\frac{R}{S}$.*

Definition 2.4. *A graded ring R is a ring R which can be expressed as a direct sum of subgroups $\{R_i\}$ i.e. $R = \sum_{n \geq 0} R_n$ such that $R_n R_m = R_{n+m}$ for all $n, m \geq 0$.*

Definition 2.5. *Let R be a filtered ring with filtration $\{R_n\}_{n \geq 0}$ and M be a $R\_$module with family $\{M_n\}_{n \geq 0}$ of subgroups of M satisfying the following conditions:*

   i) $M_0 = M$;

   ii) $M_{n+1} \subseteq M_n$ for all $n \geq 0$;

   iii) $R_n M_m \subseteq M_{n+m}$ for all $n, m \geq 0$,

Then M is called filtered $R\_$module.

Definition 2.6. *Let R be a filtered ring with filtration $\{R_n\}_{n \geq 0}$ and M be a $R\_$module together with a family $\{M_n\}_{n \geq 0}$ of subgroups of M satisfying the following conditions:*

i) $M_0 = R$;

ii) $M_{n+1} \subseteq M_n$ for all $n \geq 0$;

iii) $R_n M_m = M_{n+m}$ for all $n, m \geq 0$,

Then we say M has a strong filtration.

Definition 2.7. *Let R be a graded ring . An R-module is called a graded R-module if M can be expressed as a direct sum of subgroups $\{M_i\}$ i.e. $M = \sum_{n \geq 0} M_n$ such that $R_m M_n = M_{m+n}$ for all $n, m \geq 0$.*

Definition 2.8. *The elements of R_n or M_n in a graded ring or a module are called elements of the homogeneous degree n.*

Definition 2.9. *Let Δ be a totally ordered abelian group. A quasi valuation ν on R with values in Δ is a mapping $\nu : R^* \to \Delta$ satisfying :*

i) $\nu(ab) \geq v(a) + v(b)$;

ii) $v(a + b) \geq Min\{v(a), v(b)\}$.

Definition 2.10. *Let Δ be a totally ordered abelian group. A valuation ν on R with values in Δ is a mapping $\nu : R^* \to \Delta$ satisfying :*

i) $\nu(ab) = v(a) + v(b)$;

ii) $v(a + b) \geq Min\{v(a), v(b)\}$.

Definition 2.11. *Let M be an $R\_$module where R is a ring, and Δ an order set with maximum element ∞ and $\Delta \neq \{\infty\}$. A mapping v of M onto Δ is called a valuation on M, if the following conditions are satisfied:*

i) *For any $x, y \in M$, $v(x + y) \geq min\{v(x), v(y)\}$;*

ii) *If $v(x) \leq v(y)$, $x, y \in M$, then $v(ax) \leq v(ay)$ for all $a \in R$;*

iii) *Put $v^{-1}(\infty) := \{x \in M | v(x) = \infty\}$. If $v(az) \leq v(bz)$, where $a, b \in R$, and $z \in M \setminus v^{-1}(\infty)$, then $v(ax) \leq v(bx)$ for all $x \in M$*

iv) *For every $a \in R \setminus (v^{-1}(\infty) : M)$, there is an $a^{'} \in R$ such that $v((a^{'}a)x) = v(x)$ for all $x \in M$*

3. Valuation on Graded Ring and Module

Let R be a ring with unit and R a graded ring with subgroups $\{R_n\}_{n>0}$.

Remark 3.1. *Let R be a graded ring then for any nonzero $r \in R$ has a unique expression as a sum of homogeneous elements, $r = \sum_{n \in \mathbb{Z}} r_n$ where r_n is nonzero for a finite number of n in $\mathbb{Z}$.*

Proposition 3.2. *Let R be a graded ring R_0 is a subring of R and $1 \in R_0$.*

Proof: See ([8]). □

Remark 3.3. *Let R be a graded ring it is obvious that for all $i \in \mathbb{Z}$ we have $0 \in R_i$.*

Theorem 3.4. *Let R be a graded ring and Δ be ordered semigroup. If we define $\nu : R \to \Delta \cup \{\infty\}$ such that for every $x \in R$ we have $\nu(x) = \sum deg r_i$ where $r = \sum r_i$ and $deg r_i = i$. Then ν is a quasi valuation on R.*

Proof:

1) By proposition(3.2) we have $\nu(1) = 0$

2) By remake(3.3) we have $\nu(0) = \infty$

3) Let $r_i, r_j \in R$ such that $\nu(r_i) = i$ and $\nu(r_j) = j$ then $r_i + r_j \in R_i + R_j$ so $deg(r_i + r_j) \geq deg(r_i)$ and $deg(r_i + r_j) \geq deg(r_j)$. Now let $\alpha = \Sigma r_i$ and $\beta = \Sigma r_j$ then $deg(\alpha + \beta) \geq deg(\alpha)$ and $deg(\alpha + \beta) \geq deg(\beta)$.
So $\nu(\alpha + \beta) \geq min\{\nu(\alpha), \nu(\beta)\}$

4) Let $r_i, r_j \in R$ such that $\nu(r_i) = i$ and $\nu(r_j) = j$ then $r_i r_j \in R_i R_j \subseteq R_{i+j}$ there for $deg(r_i r_j) \geq i + j$.
Now let $\alpha, \beta \in R$ and $\alpha = \Sigma r_i$ and $\beta = \Sigma r_j$ then $\nu(\alpha\beta) \geq \nu(\alpha) + \nu(\beta)$

By above condition $\nu : R \to \Delta \cup \{\infty\}$ is a quasi valuation on R. □

Proposition 3.5. *Let R be a strongly graded ring then $\nu : R \to \Delta \cup \{\infty\}$ is a valuation on R.*

Proof: By theorem(3.4) R is a quasi valuation ring. Since R is a strongly graded ring then we have $R_i R_j = R_{i+j}$ then $deg(r_i r_j) = i + j$ there for if $\alpha, \beta \in R$ and $\alpha = \Sigma r_i$ and $\beta = \Sigma r_j$ then $\nu(\alpha\beta) = \nu(\alpha) + \nu(\beta)$. □

Theorem 3.6. *Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $\omega : R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $\omega(\alpha) = \min\{i \,|\, \alpha \in R_i \backslash R_{i+1}\}$. Then ω is a quasi valuation on R.*

Proof: See ([2]) □

Theorem 3.7. *Let $G(R)$ be an associated graded ring where R is a filtered ring. Then $\nu : R \to \Delta \cup \{\infty\}$ is a quasi valuation on $G(R)$ which derived from quasi valuation ω on R as defined in theorem(3.6).*

Proof: Since $\omega : R \to \mathbb{Z}$ such that for every $\alpha \in R$ and

$$\omega(\alpha) = \min\{i \,|\, \alpha \in R_i \backslash R_{i+1}\}$$

for every $r_i \in G(R_i)$ we have $deg(r_i) = \omega(\alpha)$ where $r_i = \alpha + R_{i+1}$. □

Remark 3.8. *If R is a ring, we will denote by $Z(R)$ the set of **zero-divisors** of R and by $T(R)$ the **total ring of fractions** of R.*

Definition 3.9. *A ring R is said to be a **Manis valuation ring** (or simply a **Manis ring**) if there exists a valuation ν on its total fractions $T(R)$, such that $R = R_\nu$.*

Definition 3.10. *A ring R is said to be a **Prüfer ring** if each overring of R is integrally closed in $T(R)$.*

Definition 3.11. *A Manis ring R_ν is said to be ν-**closed** if $R_\nu/\nu^{-1}(\infty)$ is a valuation domain (see Theorem 2 of [11]).*

Corollary 3.12. *Let R be a strongly graded ring. Then R is Manis ring.*

Proposition 3.13. *Let R_ν be a Manis ring. If R_ν is ν-closed, then R_ν is Prüfer.*

Proof: See proposition 1 of [12]   □

Proposition 3.14. *Let R be a strongly graded ring. Then R is ν-closed.*

Proof: By proposition (3.5) and definition (3.9) we have R is a Manis ring and $R = R_\nu$.
Now let $\alpha, \beta \in R$ and
$$\nu(\alpha) = i \ and \ \nu(\beta) = j$$
Consequently if
$$(\alpha + \nu^{-1}(\infty))(\beta + \nu^{-1}(\infty)) \in \nu^{-1}(\infty)$$
Then $i + j \geq \infty$ so $\alpha \in \nu^{-1}(\infty)$ or $\beta \in \nu^{-1}(\infty)$. Hence by definition (3.11) R is ν-closed.   □

Corollary 3.15. *Let R be a strongly graded ring. Then R is Prüfer.*

Proof: By proposition (3.14) R is ν-closed so by corollary (3.15) R is Prüfer.   □

Corollary 3.16. *Let R be a noetherian and strongly graded domain. Then R is a Dedekind domain.*

Proof: By corollary (3.15) R is Prüfer domain so R is Dedekind domain.   □

Theorem 3.17. *Let R be a domain. Then the following conditions are equivalent.*

   i) R is a Prüfer domain;

  ii) Every tow-generated ideal of R is invertible;

 iii) R_P is a valuation for every prime ideal P of R;

  iv) R_m is a valuation for every maximal ideal m of R;

  v) Each finitely-generated non-zero ideal I of R is a cancelation ideal, that is
     $IJ = IJ$ for ideals J, K , implies $J = K$;

  vi) R is integrally closed and there exists integer $n > 0$ such that for every two
      elements $a, b \in R$, $(a, b)^n = (a^n, b^n)$;

Proof: See [3], [4], [5].   □

Proposition 3.18. *Let R be a strongly graded domain. Then we have following
condition for R:*

  i) *Every tow-generated ideal of R is invertible;*

  ii) R_P *is a valuation for every prime ideal P of R;*

  iii) R_m *is a valuation for every maximal ideal m of R;*

  iv) *Each finitely-generated non-zero ideal I of R is a cancelation ideal, that is*
      $IJ = IJ$ *for ideals J, K , implies $J = K$;*

  v) *R is integrally closed and there exists integer $n > 0$ such that for every two*
     elements $a, b \in R$, $(a, b)^n = (a^n, b^n)$;

Proof: By Corollary(3.15) R is a Prüfer domain so by Theorem(3.17) we have the
conditions   □

Remark 3.19. *Let R be a graded ring and M a graded R-module, then for any
nonzero $a \in M$ has a unique expression as a sum of homogeneous elements, $a =
\sum_{n \in \mathbb{Z}} m_n$ where a_n is nonzero for a finite number of n in $\mathbb{Z}$.*

Remark 3.20. *Let M be a graded ring it is obvious that for all $i \in \mathbb{Z}$ we have
$0 \in M_i$.*

Theorem 3.21. *Let R be a graded ring and M a graded R-module. Let Δ be an
order semigroup. Then $\nu : M \to \Delta \cup \{\infty\}$ where $\nu(a_i) = deg(a_i)$ is a valuation on
M.*

Proof:

  i) Let $a_i, a_j \in M$ such that $\nu(a_i) = i$ and $\nu(a_j) = j$ then $a_i + a_j \in M_i + M_j$
     so $deg(a_i + a_j) \geq deg(a_i)$ and $deg(a_i + a_j) \geq deg(a_j)$. Now let $x = \Sigma a_i$ and
     $y = \Sigma a_j$ then $deg(x + y) \geq deg(x)$ and $deg(x + y) \geq deg(y)$.
     So $\nu(x + y) \geq min\{\nu(x), \nu(y)\}$

  ii) Let for $x, y \in M$ we have $\nu(x) = i$, $\nu(y) = j$ and $i \leq j$. For every $r \in R$
      where $r \in R_k$ we have $ax \in R_k M_i \subseteq M_{k+i}$ and $ay \in R_k M_j \subseteq M_{k+j}$. There
      for $deg(ax) \geq k + i$ and $deg(ay) \geq k + j$ since $i \leq j$ then $k + i \leq k + j$ so for
      every $a \in R$ we have $\nu(ax) \leq \nu(ay)$

iii) Let $\nu(az) \leq \nu(bz)$, where $a, b \in R$ such that $a \in R_i$, $b \in R_j$ and $z \in M \setminus \nu^{-1}(\infty)$. Since $z \in M \setminus \nu^{-1}(\infty)$ then there exists $k \neq \infty$ such that $z \in M_k$ and $\nu(z) = k$. So we have $i \leq j$. Now for every $z \in M$ we have $\nu(az) \leq \nu(bz)$.

iv) Let $a \in R \setminus (\upsilon^{-1}(\infty) : M)$. If $x \in \nu^{-1}(\infty)$ then for all $a', a \in R$ $\upsilon((a'a)x) = \upsilon(x) = \infty$.
Now let $x \notin \nu^{-1}(\infty)$ and for all $a' \in R$ we have $\nu((a'a)x) \neq \nu(x)$. Since $1 \in R_0$ and for all $x \in M$ we have $1x = x$ so we should have $deg(1) \neq 0$ and it is contradiction.

$\square$

Proposition 3.22. *Let M be an R_module where R is a ring, and let $\upsilon : M \to \Delta$ be a valuation on M. Then the following statements are true:*

i) If $\nu(x) = \nu(y)$ for $x, y \in M$, then $\nu(ax) = \nu(ay)$ for all $a \in R$;

ii) $\nu(-x) = \nu(x)$ for all $x \in M$;

iii) If $\nu(x) \neq \nu(y)$, then $\nu(x + y) = min\{\nu(x), \nu(y)\}$;

iv) If $\nu(az) = \nu(bz)$ for some $a, b \in R$ and $z \in M \setminus \nu^{-1}(\infty)$, then $\nu(ax) = \nu(bx)$ for all $x \in M$;

v) If $\nu(az) < \nu(bz)$ for some $a, b \in R$ and $z \in M$, then $\nu(ax) < \nu(bx)$ for all $x \in M \setminus \nu^{-1}(\infty)$;

vi) The core ν^{-1} of ν is prime submodule of M;

vii) The following subsets constitute a valuation pair of R with core $(M : \nu^{-1}(\infty))$:

$$A_\nu = \{a \in A | \nu(ax) \geq \nu(x) \ for \ all \ x \in M\},$$

$$P_\nu = \{a \in A | \nu(ax) \geq \nu(x) \ for \ all \ x \in M \setminus \nu^{-1}(\infty)\}$$

Proof: see proposition 1.1 [6] $\square$

Corollary 3.23. *If M is a graded R_module, then $\nu : M \to \mathbb{Z}$ has all of the properties that explained in the Proposition(3.22).*

Proposition 3.24. *Let M be an R_module, where R is a ring. Then there is a valuation on M, if and only if there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P.*

Proof: see (Proposition 1.3 [6]) $\square$

Corollary 3.25. *Let M be a graded $R\_$module, where R is a graded ring. Then there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P.*

Proof: By theorem(3.21) there is a valuation on M, then by proposition(3.24) there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P. □

Definition 3.26. *Let M be an $R\_$module where R is a ring, and let ν be a valuation on M. A representation system of the equivalence relation $\sim_\nu$ is called a skeleton of ν.*

Definition 3.27. *A subset S of M is said to be ν-independent if $S \cap \nu^{-1}(\infty) = \phi$, and $\nu(x) \notin \nu(Ry)$ for any pair of distinct elements $x, y \in S$. Here, we adopt the convention that the empty subset ϕ is ν-independent.*

Corollary 3.28. *Let M be a graded $R\_$module, where R is a graded ring. Then there is a skeleton on M.*

Proof: By theorem(3.21) there is a valuation on M,then by definition(3.26) we have there is a skeleton on M. □

Proposition 3.29. *Let M be a graded $R\_$module where R is a graded ring, and ν a valuation on M. If Λ is a skeleton of ν, then the following conditions are satisfied:*

  i) Λ is a ν-independent subset of M;

  ii) For every $x \in M \setminus \nu^{-1}(\infty)$, there exists a unique $\lambda \in \Lambda$ such that $\nu(x) = \nu(R\lambda)$.

Proof: By corollary(3.28) Λ is a skeleton of ν and by proposition(1.4, [6]) we have the above conditions. □

Proposition 3.30. *Let M be a graded $R\_$module where R is a graded ring, and ν a valuation on M. If Λ is a skeleton of ν. If $a_1\lambda_1 + \cdots + a_n\lambda_n = 0$ where $a_1, \cdots a_n \in R$ and $\lambda_1 \cdots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu^{-1}(\infty) : M), i = 1, \cdots, n$.*

Proof: By corollary(3.28) Λ is a skeleton of ν and by proposition(1.5, [6]) we have If $a_1\lambda_1 + \cdots + a_n\lambda_n = 0$ where $a_1, \cdots a_n \in R$ and $\lambda_1 \cdots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu^{-1}(\infty) : M), i = 1, \cdots, n$. □

4. Krull Dimension Derived from Valuation on Graded Ring and Module

Definition 4.1. *Let $(E, \leq)$ be an order set. For $a, b \in E$ we write $[a, b]$ for the set $\{x \in E, a \leq x \leq b\}$ and we put $\Gamma(E) = \{(a, b), a \leq b\}$.*

Remark 4.2. *By transfinite recurrence we define on $\Gamma(E)$ the following filtration:*

    i) $\Gamma_{-1}(E) = \{(a, b), a = b\}$;

    ii) $\Gamma_0 = \{(a, b) \in \Gamma(E), [a, b] is Artinian\}$;

    iii) Supposing Γ_β has been defined for all $\beta \leq \alpha$, then
        $\Gamma_\alpha = \{(a, b) \in \Gamma(E), \forall b \geq b_1 \geq \ldots \geq b_n \geq \ldots \geq a,$ **there is an** $n \in$ $\mathbb{N}$ **such that** $[b_{i+1}, b_i] \in \Gamma_\beta(E)$ **for all** $i \geq n\}$.

Remark 4.3. *We obtain an ascending chain $\Gamma_{-1}(E) \subset \Gamma_0(E) \subset \ldots \subset \Gamma_\alpha(E) \subset \ldots$.*
There exists an ordinal ξ such that $\Gamma_\xi(E) = \Gamma_{\xi+1}(E) = \ldots$.

Definition 4.4. *If there exists an ordinal α such that $\Gamma(E) = \Gamma_\alpha(E)$ then E is said to have a* Krull Dimension*. The smallest ordinal with the property that $\Gamma(E) = \Gamma_\alpha(E)$ will be called the* Krull Dimension *of E and we denoted by $KdimE$*

Definition 4.5. *A ring R is said to be a ring with an involution if there exists a mapping $* : R \longrightarrow R$ such that forr every $a, b \in R$:*

    i) $a^{**} = a$

    ii) $(a + b)^* = a^* + b^*$

    iii) $(ab)^* = b^* a^*$

The symmetric element of R respect to (*) is the set $Sym = \{x \in R | x^* = x\}$

Remark 4.6. *If R is associative with 1.* $SymR$ *will be the set of its symmetric elements. A subset $P \subseteq R$ is called a $* - Ordering$ if*

    i) $P + P \subseteq P$

    ii) $rPr^{-1} \subseteq P$ for all $r \in R$

    iii) $P \bigcup -P = \mathrm{Sym}R$

    iv) $P \bigcap -P = \{0\}$

    v) P is closed under the Jordan multiplication $\{a, b\} = ab + ba$

Definition 4.7. *For every $0 \leq \gamma \in \Delta$ we define a relation $\sim_\gamma$ on $R^\times$ by $x \sim_\gamma y :\Leftrightarrow \nu(x) + \gamma < \nu(x - y)$. We write $\sim_\nu := \sim_0$*

Lemma 4.8. *These relations are semigroup congruences and it is easy to see that they are cancellative and $*$-invariant.*

Proof: See Lemma 1.5,1.6 in Da   □

Definition 4.9. *Let R be a ring and Δ be order semigroup. If $\nu : R \to \Delta \cup \{\infty\}$ be a valuation on R and $R^\times = R \setminus \{0\}$.*

   i) ν *is compatible with a *-ordering P of R iff $x \sim_\nu y \in P$ for all $x, y \in SymR^\times$*

   ii) ν *is called quasi-commutative iff for all $a, b \in R^*$ we have $ab \sim_\nu ba$*

   iii) ν *is weakly quasi-Ore iff for all $a, b \in R^*$ there exist $r, s \in R^\times$ such that $ra \sim_\nu sb$ Note that this condition is left-right symmetric since ν is a $*-valuation$. Clearly, ν is weakly quasi-Ore iff the semigroup $R^\times / \sim_\nu$ satisfies the Ore condition*

Proposition 4.10. *Let R be a ring and Δ be order semigroup. If we define $\nu : R \to \Delta \cup \{\infty\}$ such that ν is a quasi valuation on R, and for all $x, y \in R^\times$ we define $x \sim_\nu y$ if and only if $\nu(x) = \nu(y)$ then:*

   i) ν *is compatible with a *-ordering P of R;*

   ii) ν *is called quasi-commutative;*

   iii) ν *is weakly quasi-Ore;*

Proposition 4.11. *Let R be a ring and Δ be order semigroup. If we define $\nu : R \to \Delta \cup \{\infty\}$ such that ν is a quasi valuation on R and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $SymR$ is an order set.*

Proof: Since R is a quasi valuation, then by proposition 4.10 then $SymR$ is an order set.   □

Corollary 4.12. *Let R be a ring and Δ be order semigroup. If we define $\nu : R \to \Delta \cup \{\infty\}$ such that ν is a quasi valuation on R and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $SymR$ has Krull Dimension.*

Proof: By proposition(4.11) R is order set so R have Krull Dimension.   □

Corollary 4.13. *Let R be a ring and Δ be order semigroup. If we define $\nu : R \to \Delta \cup \{\infty\}$ such that ν is a quasi valuation on R and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $\Gamma_{-1}\Delta\Delta(SymR) = 0$.*

Remark 4.14. *Let R be a ring and Δ be order semigroup. If we define $\nu : R \to \Delta \cup \{\infty\}$ such that ν is a quasi valuation on R and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $KdimR \leq KdimSymR$.*

Example 4.15. *Let R be a DVR (Discreet valuation ring), We know $KdimR = 1$. Now we prove it by above structure. Since R is DVR so R is a quasi valuation ring.*

   i) $\Gamma_{-1}(SymR) = 0$;

ii) $\Gamma_0 = \{(a,b) \in \Gamma(R), [a,b] isArtinian\}$;

iii) $\Gamma_1(SymR)$ *has been defined for all* $\beta \leq 1$ *, then*
$\Gamma_1(R) = \{(a,b) \in \Gamma(SymR), \forall b \geq b_1 \geq \ldots \geq b_n \geq \ldots \geq a,$ *there is an* $n \in$
$\mathbb{N}$ *such that* $[b_{i+1}, b_i] \in \Gamma_0(E)$ *for all* $i \geq n\}$.
There for as for every $0 \neq r_i \in R$ *and* $i \geq n$ *we have* $[r_{i+1}, ri] \in \Gamma_{-1}$ *or*
$[r_{i+1}, ri] \in \Gamma_0$ *and so* $\Gamma_1(SymR) = \Gamma(SymR)$. *Now by definition(4.4) we*
have $KdimSymR = 1$

Now we have $KdimR \leq 1$ *since* $KdimR \neq 0$ *so* $KdimR = 1$

References

1. M.H. Anjom SHoa, M.H. Hosseini, Filtered Ring Derived from Discrete Valuation Ring and Its Properties, Advances in Pure Mathematics, 2014, 4, 71-75

2. M.H. Anjom SHoa, M.H. Hosseini, Quasi Valuation and Valuation Derived From Filtered Ring And Their Properties, Asian Journal of Algebra,2014

3. W.C. Brown, Matrix Over Commutative Ring, Marcel Dekker, Inc.

4. F.R. Cohen , Aaron Heap , Alexandra Pettet On the Andreadakis–Johnson filtration of the automorphism group of a free group (Journal of Algebra 329 (2012) 72–91).

5. N. S. Gopalakrishnan, Commutative algebra,oxonian press,1983.

6. M. Hazewinkel, V.V. KirichenkoKiev, Algebras, Rings and Modules, CWI,Amsterdam, The Netherlands Nadiya Gubareni Technical University of CzĀłstochowa,Poland and Taras Shevchenko University,Kiev, Ukraine KLUWER.

7. L. Klep , Embedding .-ordered domains into skew fields,Journal of Algebra 303 (2006) 416ï£¡429.

8. T.Y. Lam , A First Course in Noncommutative Rings,Springer-Verlag ,1991.

9. H. Marubayashi, H. Miyatmoto, A. Ueda , Noncommutative Valuation Rings and semi-Hereditary Orders ,KLUWER ACADEMIC PUBLISHERS ,1991.

10. Koji Nishida On the depth of the associated graded ring of a filtration (Journal of Algebra 285 (2005) 182–195).

11. G. Puninskia, V. Puninskayab, C. Toffalorib Decidability of the theory of modules over commutative valuation domains, Annals of Pure and Applied Logic, 145 (2007), (258 - 275).

12. David E. Rush, Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices, Journal of Algebra, 308 (2007), (295 - 320).

13. Paolo Zanardo, On ν-closed Manis vluation rins,Communication in algebra, 18(3), 775-788(1990).

14. Paolo Zanardo, Construction of manis vluation rins,Communication in algebra, 21(12), 4183-4194(1993).

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