Qibla, and Related, Map Projections [1]

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Abstract

The Qibla problem - determination of the direction to Mecca - has given rise to retro-azimuthal map projections, an interesting, albeit unusual and little known, class of map projections. Principal contributors to this subject were Craig and Hammer, both writing in 1910. The projections thus introduced are examined. An unusual recent discovery from Iran suggests that Muslims might have been prior inventors of a similar projection, by at least several centuries. A property of retro-azimuthal projections is that the parallels are bent downwards towards the equator. The resulting maps, when extended to the entire world, thus must overlap themselves. A later corollary by Schoy leads to a new "cylindrical" azimuthal map projection with parallels bending away from the equator, here illustrated for the first time.

Keywords: Azimuthal directions, Map projections, Mecca, Qibla, retro-azimuthals

Background

In western literature the retro-azimuthal projection was first proposed in 1909 by Craig. An alternate version was proposed almost immediately thereafter by the German cartographer Hammer (1910). These projections have as their main property that the direction (azimuth) from every point on the map **to** the chosen origin are correct. In this respect the maps differ from the usual azimuthal projections that render azimuths **from** the center correctly. As is the case with the normal azimuthal projections, the retro-azimuthal condition is not sufficient to completely specify the projection, since two properties are required of most map projections. In Craig's (1910) version the meridians are equally spaced straight parallel lines perpendicular to a horizontal line through the origin. Hammer introduced the sensible alternate condition that all places should be at their proper distance from the center, a condition comparable to that defining the azimuthal equidistant projection. This condition changes both the spacing and the shape of the meridians and parallels; the meridians become curved lines and the curvature of the parallels is modified. Probably the radial distance from other azimuthal projections could be substituted in place of equidistance, but there is no obvious reason for doing this. I know of no projection which is equal area and retro-azimuthal; to date these conditions have not been demonstrated to be incompatible.

Jackson (1968) modified the spacing of the straight parallel meridians to obtain slight variants. When the meridians are spaced according to the sine of the longitude difference, he showed that the parallels of latitude are elliptical, and when they are spaced as the tangent of one half of the longitudinal difference then the parallels are parabolas. If the spacing is equal to the tangent of the difference then the parallels become hyperbolas. He also provided a variant with curved meridians, and a "retro-azimuthal stereographic" (but not conformal) projection.

The projection devised by Littrow in 1833 is not only conformal but also has the property that all places located on the central meridian have the property of being retro-azimuthal (Maurer 1919)[2]. It is also possible to have a projection that is retro-azimuthal with respect to two spherical locations (Maurer 1911, 1914). None of these projections can show the entire world without overlap. Craig's map, when expanded to the entire sphere, seems to look like a two dimensional orthographic view of a three dimensional hyperbolic paraboloid. The overlapping can be loosely envisioned as if coming from a projection of this onto the drawing plane. Of course the projection is not made in this way but rather is derived mathematically.

The initial incentive for these projections was to enable Muslims to determine the qibla, that is, the direction to the Ka'ba in Mecca[3]. Craig (1910) also suggested that navy ships could use such a map centered on their home port to determine the direction to a radio station. A comparable use was later made by Hinks (1929) with a map centered at Rugby in the United Kingdom (site of a long distance transmitter) for colonials in the British Empire. Hinks also produced a map focused on Malabar, apparently only to show how the appearance of the map projection changes with changes in the center's latitude. Additionally, de Henseler (1971) produced an equidistant retro-azimuthal map for the geophysical observatory in Addis Ababa. According to Snyder (1993: 228), Gilbert of Bell Telephone Laboratories satirically constructed a retro-azimuthal projection centered on Wall Street in New York City, the Mecca of the financial world. I am not aware of any other proposed uses. Others who have studied the problem of determining the azimuth of the qibla include Schoy (1917) and Berggren (1980). The qibla question was also addressed by Almakky & Snyder in 1996; they briefly mentioned the possibility of using ellipsoidal geodesics for the azimuth computation. All other discussions refer only to spherical geometry, as does the present text.

Instead of using a retro-azimuthal projection, it is possible to indicate the magnitude of the retro-azimuthal directions (retro-azimuths) on any map by means of isolines. The preferred projection for this is a conformal one. A map like this, for Rugby on the Mercator projection, was reported by Reeves (1929), and one for Mecca, also on the Mercator projection, has been prepared by Kimerling (2001). The conformal property of the Mercator projection makes the direct reading of azimuth angles quite easy. De Henseler drew isolines on Miller's cylindrical projection and Hagiwara (1984: 843) portrayed lines of equal retro-azimuths on an azimuthal equidistant projection of the world centered on Mecca.

The Muslim Contribution

The scholars of the Muslim world recognized the qibla - direction to Mecca - problem much, much earlier, possibly as early as the ninth century (Ali, 1967, King 1975, King 1997, King and Lorch 1992). That is, they knew the equation $\tan q = \sin(\Delta 8)/\{\sin(N)\cos(\Delta 8)-\cos(N)\tan(N_o)\}$, where q is the retro-azimuthal direction, $\Delta 8 = 8 - 8_o$ is the longitudinal difference between Mecca and the location in question, N is the latitude of that location, and N_o is the latitude of Mecca ($N_o = 21^o27'$, $8_o = 39^o45'$). Schoy devoted sections 33-43, 83-86, and plate 12 of his 1923 monograph to this. Snyder (1993, 227-231) gave these same equations for the retro-azimuthal projections. The qibla function is rather complicated although the spherical trigonometry is relatively simple.

Physical maps are not extant but two amazing astrolable-like brass instruments, each containing locational positions, from 18th century Iran, apparently based on earlier models, have survived, and have been studied by King (1997, 1999). The quality of the engraving is comparable to that known from astrolabes. It is thus clear that Muslim scholars knew the solution to the qibla problem in both exact and approximate form for many centuries. Illustrations and analysis of the two recently (1989, 1995) discovered qibla instruments are given in King (1999, 199-201). The instrument discovered in 1989 is also shown by McKay (2000), and again by Mackenzie (2001). It was sold for $70,500 at auction by Christie's in London on June 5th, 2000. On this instrument the geographical graticule is quite similar to Craig's version of the projection, but predates it by more than two centuries. A two-degree graticule is indicated, but no coastlines are shown. Cities are indicated by small named circular marks. According to King (1997: 82, 1999: 242-254) the straight and parallel meridians are spaced according to the sine of the longitude difference, modulated by the cosine of the latitude of the map center. This decrease in spacing is barely perceptible in the illustrations. For the area covering the Middle East this spacing varies only slightly from that used by Craig. The parallels are perhaps drawn as circles (King, 1999: 242-254). According to Jackson's modification of Craig's projection, cited above, and my own computations, the parallels should be ellipses, but circles are easier to draw, and, for a small area around the map center hardly distinguishable from ellipses (Dekker, 2000). On the qibla instruments the cities are placed at distances proportional to the sine of the great circle distance to Mecca in order to maintain the correct direction to Mecca.

This unusual map projection can only be expanded to ninety degrees in longitude from the map center; beyond this distance the parallels converge and fold over on themselves (the sine function begins to come back on itself - to decline - beyond ninety degrees). The elliptical parallels also turn back causing the map to fold over. This is not critical because the instruments do not extend more than sixty degrees east and west of the center - they cover the principal Muslim countries of the time. The instruments also have a moveable diametrical rule attached at the center, at the latitude and longitude of Mecca. The rule is graduated according to the sine of the great circle distance, permitting distance measurement. Notations at the margin allow the direction of Mecca to be read off. Tables given by King (1999) demonstrate the precision with which the earlier Muslim astronomers gave answers to the qibla question, and to problems in spherical geometry. The exact relation of the brass maps to the extant tables is not yet established, nor is there as yet evidence of the existence of similar maps. Although possibly as much as three centuries old, the qibla intruments could still be used today. The main difficulty, as with all classical/medieval maps of the period, is the lack of scientific knowledge of the longitudes of geographical places[4].

Computational procedure

Stimulated by King's book, I have been able to construct a retro-azimuthal projection in a very simple manner, on the spherical assumption. In the computation of any azimuthal projection one needs a subroutine that calculates distance and direction from the map center. In such a routine, one fixes the center and varies the latitude and longitude of the points for the rest of the map. If the entries to the subroutine are switched then one gets the retro-azimuths and distances. In other words, vary what would normally be the center and fix the rest. The details are in the appendix. In this way I quickly obtained Hammer's equidistant retro-azimuthal projection, the graticule for which is shown in Figure 1.

Figure 1 goes about here.

Hammer's projection is normally presented in polar coordinate form, and the map of the whole world contains a hole and overlaps itself. This is not a problem if the projection is only used for an area near the center. But the appearance of the map changes quite dramatically when the area to be shown is increased, as in Figure 1, to depict the graticule for the entire earth. Depending on the latitude of the center of attention, the size of the interior hole and the area of overlap also changes. This can be demonstrated quickly by use of the computer program "Cartographic Map Projections of the World" from Axion Spatial Imaging Ltd. (http://www.axionspatial.com). Snyder and Voxland (1989: 152-3; Snyder 1993: 230-231) showed the overlapping parts separately.

In the subroutine, as used for the example of Hammer's projection given in Figure 1, the distance and direction to Mecca are calculated in spherical coordinates as one does for the normal azimuthal projections. The numerical answers are then taken to be polar coordinates in the plane. The usual practice is to convert these polar coordinates to rectangular coordinates for automatic plotting. Instead of using polar coordinates in this manner, the map can also be displayed by setting the distance proportional to one rectangular coordinate and the direction proportional to the other (orthogonal) rectangular coordinate. The center of interest (Mecca) is then the top edge of the map (Figures 2 & 3).

Figures 2 & 3 go about here

Measuring down from the top edge yields the angular spherical distance, from zero degrees to 180 degrees, of places on the map from Mecca. This new map still overlaps itself and direction to Mecca is measured horizontally, starting in the center and ranging from zero to -180 degrees to the left, and zero to +180 degrees to the right of the center. This puts the sausage-shaped hole from Hammer's version in the middle of the map. One can think of the construction of these maps (Figures 2 & 3) as taking place by a winding around the surface of a cylinder with the height on the cylinder representing distance and the position along the circumference representing direction. Then the maps represent an unrolling of this cylinder. The overlap is still seen and the magnitude of the open space depends on the latitude of the center of attention. The position of the open space depends on where the cylinder is unrolled - it can be centered as shown in figures 2 & 3 or can be set to the edges of the map. Hagiwara (1984) gives a diagram similar to this in his presentation.

Distance - Direction Diagrams

To better understand the qibla map in its rectangular form, consider the equivalent to the azimuthal equidistant projection in its rectangular form. Take Y as spherical distance (top to bottom; zero to pi, in radians) and X as direction (left to right; zero to two pi). Such a map, with origin at Santa Barbara is given in Figure 4. Santa Barbara itself is the line across the top of the map. Figure 5

<u>Figures 4 & 5 go about here</u>

presents the graticule for this map – the poles are easily visible. Do you recognize that this so-called distance – direction diagram is an oblique version of the Plate Carrée projection (also known as the rectangular projection, the simple cylindrical projection, or, in at least one GIS as "unprojected")? It could also be called a Cassini projection in the oblique form in which the edge, rather than the central meridian, is the important location. A version based on Lambert's cylindrical projection is also possible but, in that case, while directions are still correct, distances from the top are not, though the map would be equal area. Portions of the world can also be represented in this manner with distances as on the gnomonic, stereographic, or perspective azimuthal projections. These would of course not have the properties of the azimuthal projections with similar names. This is because the azimuthal projections are conventionally thought of as having a circular form, and these cylindrical versions do not have this form.

I have thus taken the Hammer (1910) equidistant retro-azimuthal projection and plotted it in rectangular coordinates, instead of the usual polar coordinates. Arden-Close (1938), in a very terse note, has done the same but in the opposite direction. He has taken the normal cylindrical coordinate version of Craig (1910) with straight equidistant and parallel meridians, and has plotted it in polar coordinates with straight meridians radiating at equal angles from a center. He refers to this as "a polar azimuthal retro-azimuthal projection", rather unfortunate terminology. A better, and simpler, name would be "a polar retro-azimuthal projection", which is not to be confused with Hammer's equidistant retro-azimuthal projection (Figure 1), also given in polar coordinates. From the preceding it is clear that one could also produce a "conical" version by multiplication of the azimuths by the equivalent of a cone constant. Jackson (1968: 322) has in fact done this.

An unusual azimuthal projection

An obvious thought is now to take a normal azimuthal projection and redraw it with meridians as equidistant straight parallel lines as Craig did for his retro-azimuthal projection. It is then necessary to calculate the location of the points for each of the parallels, at the correct azimuth from the center of the map. Schoy (1913a, b) described this and demonstrated mathematically, for the equatorial case, that the parallels are everywhere concave with respect to the equator and that they finally end as tangents to the meridians at ±180 degrees at either side of the map, and that they reach this at infinity. This is not a particularly useful map, and Schoy did not draw it. It is given here for the first time, centered on the equator at the Greenwich meridian, as Figure 6. The equations, very

<u>Figure 6 goes about here</u>

complicated in Schoy's first 1913 paper, are quite simple if one has a spherical direction subroutine, namely

$$X =)8, \quad Y = X^*\tan(\pi/2 - 2),$$

Where 2is the spherical direction to places on the sphere and on the map (the azimuth from the map center). Delta lambda is of course the difference in longitude between the map center and the places on the map. This unusual azimuthal projection is nowhere mentioned in the cartographic literature. We are accustomed to thinking of azimuthal projections as circular in outline and thus this map contradicts our preconception. Other spacings of the meridians are possible. Obvious candidates for the spacing of the straight and parallel meridians is to take the radial values from the usual azimuthal projections and apply them to the spacing of the meridians. Thus one's choices for the meridian spacing could include an 'orthographic' (sin()8), |)8| < π/2), 'Lambert equal area' (2 sin()8/2)), 'stereographic' (2 tan()8/2)), 'gnomonic' (tan()8), |)8| < π/2), or general 'perspective' version. Calculation of the location of the parallels to preserve the azimuthal property are then required. These maps would of course not have the properties - conformality, etc. - of the normal azimuthal projections with similar names.

Concluding remarks

The discovery of two Muslim qibla instruments in the last years of the millennium has stimulated research on the little-known retro-azimuthal map projections. There have been recent publications announcing this discovery in *Imago Mundi*, *Mercator's World* and *The American Scientist*, and some of its implications have been examined by King, Dekker, and Kimerling (2001). In the present instance this has led to the presentation of a variant of this class of projections, and to a new depiction of a curious azimuthal projection. In today's computer environment the computation of the direction of an azimuth to a particular location is no longer tedious, which makes the early Muslim achievements even more noteworthy.

Acknowledgements:

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Notes:

[1] *Cartography and Geographic Information Science*, 29 (2002), 1: 17-23

An extensively illustrated power point presentation of this paper was delivered to a University of California, Santa Barbara, Geography Department colloquium in November 2000, and to the California Map Society meeting at the University of Southern California in January 2001. This presentation may be seen at http://www.ncgia.ucsb.edu/projects/tobler/Qibla.

[2] According to Maurer (1911), Littrow (1833) was not aware of the retro-azimuthal property of his projection. Maurer claims that he himself first proposed this projection in 1905. The projection is also known as Weir's azimuth diagram, of use in navigation, and apparently has been re-invented several times.

[3] The literature on the qibla is quite large. A detailed overview is given by Schoy (1927). Reference should also be made to Wensinck's (1927) articles on the Ka'ba and the Hadjdj.

[4] For example, see J.K. Wright (1923)

Appendix:

The changes to a spherical direction subroutine necessary to obtain a retro-azimuthal projection follow. The basic notion is to switch the first entries with the next ones. The inputs and results are of course in radians.

The usual routine required to compute spherical distances and directions takes as input the following variables within a loop going through all of the latitudes and longitudes for the map - the order is specific to my subroutine.

1) The origin longitude 8_o

2) The sine of the origin latitude, sin(N_o)

3) The cosine of the origin latitude, cos(N_o)
4) The latitude of the point whose distance and direction are desired, N
5) The longitude of the point whose distance and direction are desired, 8
Repeating, in complete form the subroutine entries are {8_o, sin(N_o), cos(N_o), N, 8, r, 2}.
The result is given as spherical distance, r, and spherical direction, 2, both in radians on a sphere.

To obtain Hammer's retro-azimuthal projection use the following order within the loop that computes distances and directions for items to be placed on the map.
1) The longitude of the point whose distance and direction are desired, 8
2) The sine of the latitude of the point whose distance and direction are desired, sin(N)
3) The cosine of the latitude of the point whose distance and direction are desired, cos(N)
4) The latitude of the map center (Mecca), N_o
5) The longitude of the map center (Mecca), 8_o
The subroutine entries are thus changed to {8, sin(N), cos(N), N_o, 8_o, r, 2}. The subroutine itself is not modified in any way. The result gives Hammer's projection as polar distance, r, and angular direction, 2. These can be converted to rectangular coordinates in the usual manner for plotting. Alternatively, set X= 2, and Y= r for the distance - direction retro-azimuthal diagram. It may be necessary to add pi (3.14159265) to X to put the hole in the middle of the map instead of at the edges.

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Figure Captions:

Figure 1. The graticule for Hammer’s equidistant retro-azimuthal projection, with center near the latitude and longitude of Mecca. A version with landmasses and in color can be seen using the computer program “Cartographic Map Projections of the World” by Axion Spatial Limited.

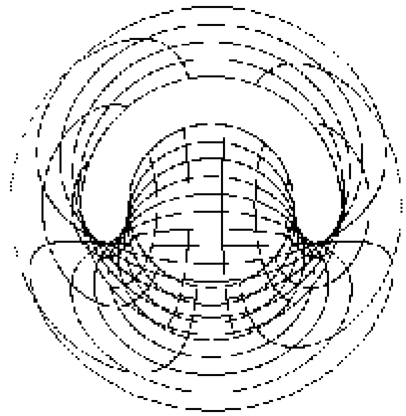
Figure 2. On this diagram angular distances to Mecca are to be measured along the vertical axis from the top down, zero degrees at the top to 180 degrees at the bottom. Directions to Mecca are measured along the horizontal axis as azimuth values from –180 degrees on the left to +180 degrees on the right.

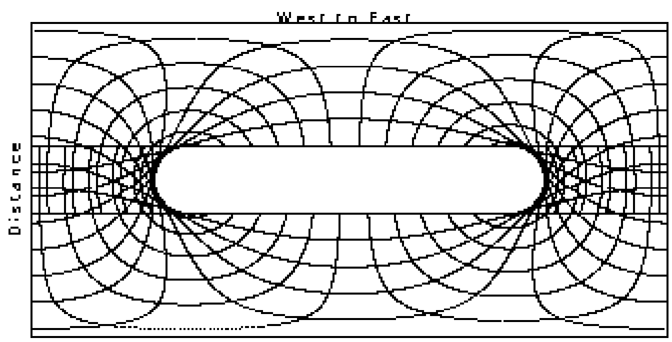
Figure 3. Here is the world coastline corresponding to the preceding diagram. Mecca is along the top. Observe how the map overlaps itself, making it difficult to read.

Figure 4. This is the equidistant azimuthal distance - direction diagram (oblique Plate Carée). Santa Barbara is represented by the line across the top. Distances from Santa Barbara are to be measured down, and are correct (to scale). Directions from Santa Barbara are measured along the horizontal axis, as azimuth values from –180 degrees on the left to +180 degrees on the right. There are no overlapping areas and no holes.

Figure 5. Here is the latitude – longitude graticule for the previous map. The location of the poles is easily identified.

Figure 6. An azimuthal projection with parallel meridians in the manner of Craig, centered on the equator at the Greenwich meridian. Spherical directions from the center are correctly preserved. When not centered on the equator the parallels have more complicated shapes.

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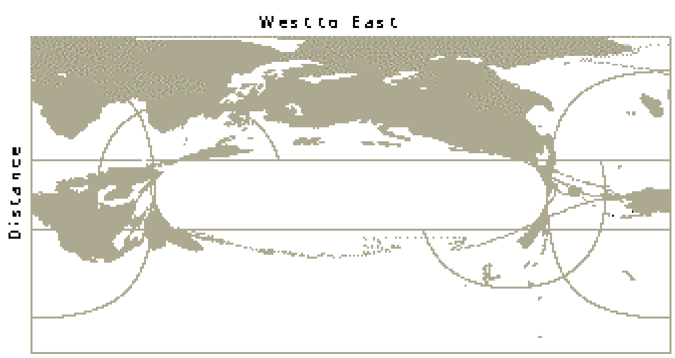
West to East
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West to East
Distance