Simple Nonbinary Coding Strategy for Very Noisy Relay Channels

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SUMMARY From an information-theoretic point of view, it is well known that the capacity of relay channels comprised of three terminals is much greater than that of two terminal direct channels especially for low SNR region. Previously invented relay coding strategies have not been designed to achieve this relaying gain occurred in the low SNR region. In this paper, we propose a new simple coding strategy for relay channels of low SNR or equivalently very noisy relay channels. The multiplicative repetition is utilized to design this simple coding strategy. We claim that the proposed strategy is simple since the destination and the relay can decode with almost the same computational complexity by sharing the same structure of decoder. An appropriate static power allocation which yields the maximum throughput close to the optimal one in low SNRs is also suggested. Under practical constraints such as equal time-sharing etc., the asymptotic performance of this simple strategy is within 0.5 dB from the achievable rate for the relay channels. Furthermore, the performance at few thousand bits enjoy a relaying gain by approximately 1 dB.

key words: nonbinary LDPC code, multiplicative repetition, decode-and-forward, half duplex relaying, relay channel, very noisy relay channel

1. Introduction

Over recent years, cooperative communication has been studied extensively in order to increase the capacity of wireless networks [1]. The intermediate terminals between source and destination called relays in a cooperative network form a virtual multiple antenna system and hence allow a single-antenna user to transmit the information through the different and independent paths. By doing so, the terminals in cooperative networks work together to send their information and the spatial diversity still can be achieved without employing multiple antennas.

A three-terminal relay channel [2], which consists of a source (S), a relay (R) and a destination (D), is an elementary component of a cooperative network. By using the decode-and-forward protocol, R is capable to decode the received signal and forward the decoded information to D. This protocol yields the significant increase in the capacity compared to that of direct transmission (i.e., a system without R), when the quality of S-R link is good [3]. We shall refer to the increase in capacity with the cost of employing relay as relaying gain. At the present moment, full-duplex relaying is still impractical since R cannot receive and transmit simultaneously at the same time [4]. In this paper, we consider the more practical relay system known as time-division based half-duplex relaying.

It can be observed that the relaying gain is very large especially at low to medium signal to noise ratios (SNRs) [5]. We mention that the relay channel of low SNRs corresponds to the relay system operated at low rates, i.e. throughput below 0.4 bits. The recent progress on the development of relay coding strategies which offer the excellent performance in medium SNRs can be found in [4,6]–[9] while the efficient strategy to capture the relaying gain in low SNRs has not been proposed. In this study, we thus focus our attention on designing the coding strategy for this range of SNRs.

In order to function close to the theoretical limit of the relay channels, we have to deal with the coding strategy of very large code lengths (10^4 ~ 10^5 bits). For example, at medium SNRs, the optimized binary low density parity-check (BLDPC) codes of length 10^5 bits can perform within 1.0 ~ 1.5 dB from the theoretical limit [4,7,8]. Unfortunately, there is a limit on the size of code length in wireless communication with delay constraint. The large code lengths lead to the transmission latency and complex decoders which are not preferable in the real world communication. It is thus quite impractical to build such a coded relay system with the current wireless technologies in which short code lengths (10^2 ~ 10^3 bits) are feasible [10].

Inevitably, the low rate coding strategy is needed when operating in very noisy relay channels or relay channels of low SNR. If one would like to extend the optimization concept to design the low rate BLDPC codes for such case, the following problems, as occurred in direct transmission, could be arisen: 1) The Tanner graphs of optimized low-rate binary LDPC codes
tend to have check nodes of high degree, which result in the degraded decoding performance especially for small code lengths [11]; 2) It has been proved that the regularity of parity-check matrix facilitates hardware-friendly implementation e.g. low interconnect complexity between check and variable nodes [12] but the parity-check matrix of optimized LDPC code is highly irregular. Therefore, there is a trade-off in implementing the optimized LDPC decoder employing at both the R and D. Based on these potential problems, it is rather cumbersome to deal with BLDPC coding strategy for very noisy relay channels.

In this paper, we alleviate the aforementioned problems by considering nonbinary LDPC (NBLDPC) codes defined over GF(2^m) with ultra sparse regular parity-check matrix of column weight 2. This is motivated by the outstanding performance at short code lengths with the simplest form of regular parity-check matrix [13]–[15]. The concept of multiplicative repetition [11] is utilized to design a simple relay coding strategy. With the inherent rate compatibility property of this concept, R and D can decode their own received signals with almost the same computational complexity. We demonstrate via the simulation that the proposed strategy is a promising technique for capturing relaying gain at low SNRs.

The rest of this paper is organized as follows. The NBLDPC codes are introduced in Section 2. In Section 3, a system model is described. In Section 4, a simple NBLDPC coding strategy which is applicable to very noisy relay channels is presented. Good suboptimal parameters, including static power allocation, which can simplify the relay protocol are discussed in this section. In Section 5, we present the performance of the proposed relay coding strategy. The conclusions are finally given in Section 6.

2. Introduction to NBLDPC Codes

An NBLDPC code C over GF(2^m) is defined by the nullspace of a sparse \( M \times N \) parity-check matrix \( \mathbf{H} = (h_{ij}) \) defined over GF(2^m) [13]

\[
C = \{ \mathbf{x} \in \text{GF}(2^m)^N \mid \mathbf{Hx} = \mathbf{0} \in \text{GF}(2^m)^M \},
\]

(1)

where \( \mathbf{x} = (x_1, \ldots, x_N) \) is a codeword. The 4th parity-check equation for \( c = 1, \ldots, M \) is written as \( h_{i1}x_1 + h_{i2}x_2 + \cdots + h_{iN}x_N = 0 \) where \( h_{i1}, \ldots, h_{iN} \in \text{GF}(2^m) \) are the entries of \( i \)th row of \( \mathbf{H} \). The parameter \( N \) is the codeword length in symbol. An NBLDPC code is called \((d_v, d_c)\)-regular if \( \mathbf{H} \) has constant column weight \( d_c \) and row weight \( d_v \). In this paper, only \((d_v = 2, d_c)\)-regular NBLDPC codes defined over GF(2^m) are considered since they are empirically known as the best performing codes especially for short code lengths [13], [14]. The \( \mathbf{H} \) of an NBLDPC code can be represented by a Tanner graph with variable and check nodes [16]. Each variable node and check node represent a coded symbol and a parity-check equation, respectively. The code rate \( R \) of any NBLDPC code can be computed by \( R_{\text{LDPC}} = K/N = 1 - (d_v/d_c) \).

3. System Model

3.1 Half-Duplex Relay System

In time-division based half-duplex relaying [4], the transmission in relay channel takes place over two time slots of normalized duration. In the first time slot of duration \( t \), S sends information to R and D. In the second time slot of duration \( t' = 1 - t \), R and S send information to D. The first time slot is referred to as broadcast (BC) mode whereas the second time slot is referred to as multiple-access (MAC) mode. The parameter \( t \) is known as the time-sharing factor. Figure 1 illustrates the transmission in the time-division based half-duplex relay system. Parameters \( h_{SD}, h_{SR} \) and \( h_{RD} \) generally represent the channel effects such as fading, shadowing or path loss between two terminals.

By using decode-and-forward protocol, the transmission in half-duplex relay system can be described as follows. In BC mode, S firstly encodes the \( k \) information bits with a channel code of length \( tn \) and of rate \( R_{\text{BC}} = k/tn \). The \( tn \) coded bits are transmitted to both R and D. At the end of BC mode, R decodes its received signals and the estimation obtained from decoder will be further used in MAC mode. Meanwhile, D merely stores the received signals. In MAC mode, R transmits another set of \( t'n = n - tn \) bits to D by using the estimation decoded in BC mode. At the same time, S also simultaneously sends \( t'n \) bits to D. Finally, D can decode the received information from two modes at rate \( R = R_{\text{MAC}} = \frac{k}{tn + t'n} = \frac{k}{n} \).

3.2 Relay Channel Model

Following [17], an equivalent physical model of the relay system displayed in Fig. 1 is shown in Fig. 2. In this model, R is placed on the direct line between S and D. The distance between S and D is normalized to 1 and the distance between S and R is denoted by \( d \).
Since, in this paper, the binary modulation is assumed and the relay system is coded by NBLDPC codes, we thus describe how to transmit the NBLDPC coded symbol via the binary modulation scheme. We can represent a coded symbols \( s_v \in \text{GF}(2^m) \) for \( v = 1, \ldots, N \) by the binary sequence of length \( m \) bits. For each \( m \), we fix a Galois field \( \text{GF}(2^m) \) with a primitive element \( \alpha \) and its primitive polynomial \( \pi \). Once a primitive element \( \alpha \) of \( \text{GF}(2^m) \) is fixed, each symbol is given an \( m \)-bit representation \[18, p. 110\]. For example, with a primitive element \( \alpha \) of \( \text{GF}(2^2) \) such that \( \pi(\alpha) = \alpha^3 + \alpha + 1 = 0 \), each symbol is represented as \( 0 = (0, 0, 0), 1 = (1, 0, 0), \alpha = (0, 1, 0), \alpha^2 = (0, 0, 1), \alpha^3 = (1, 1, 0), \alpha^4 = (0, 1, 1), \alpha^5 = (1, 1, 1) \) and \( \alpha^6 = (1, 0, 1) \).

At the \( th \) output of NBLDPC encoder, each coded symbol \( s_v \in \text{GF}(2^m) \) is mapped to the binary sequence of \( m \) bits \( (x_{v,1}, x_{v,2}, \ldots, x_{v,m}) \in \text{GF}(2^m) \) according to the primitive polynomial as described above. The obtained binary sequence \( (x_{v,1}, x_{v,2}, \ldots, x_{v,m}) \) is then mapped to \( m \) modulated signals through the mapper for the binary transmission.

For \( th \) transmission (consisting of two time slots), let \( s_{BC}^{(i)} \) be the modulated binary signals transmitted from \( S \) in \( BC \) mode. The variables \( s_{\text{MAC}}^{(i)} \) and \( s_{\text{MAC}}^{(i)} \) are interpreted in the same way but different terminals and modes. In similar way, \( y_{BC}^{(i)} \), \( y_{\text{MAC}}^{(i)} \) and \( y_{\text{BC}}^{(i)} \) are defined as the received signals at given terminals and modes. Throughout this paper, the alphabets in subscript are used to describe the terminal and mode of variables under consideration. From the notations above, the received signals at \( R \) and \( D \) in \( BC \) mode can be written as [4]

\[
y_{BC}^{(i)} = h_{\text{SD}} s_{BC}^{(i)} + n_{BC}^{(i)}, \quad (2)
y_{\text{MAC}}^{(i)} = h_{\text{SD}} s_{\text{MAC}}^{(i)} + n_{\text{MAC}}^{(i)}, \quad (3)
\]

In \( MAC \) mode, the received signal at \( D \) is given as

\[
y_{\text{BC}}^{(i)} = h_{\text{SD}} s_{\text{BC}}^{(i)} + h_{\text{BS}} n_{\text{BC}}^{(i)} + n_{\text{BC}}^{(i)}, \quad (4)
\]

where \( n_{\text{BC}}^{(i)}, n_{\text{MAC}}^{(i)} \) and \( n_{\text{BC}}^{(i)} \) are AWGN with zero mean and unit variance.

In this paper, we consider only the large scale path loss. Under this circumstance, \( |h_{\text{SD}}|^2 = 1 \), \( |h_{\text{BS}}|^2 = \frac{1}{\alpha} \) and \( |h_{\text{BS}}|^2 = \frac{1}{\alpha^2} \) where \( \alpha \) is the path loss exponent. For a fair comparison with direct transmission, the transmitted powers of \( S \) and \( R \) must satisfy an average global power constraint given as [4]

\[
\Theta : tP_{S,BC} + t'(P_{S,\text{MAC}} + P_{R,\text{MAC}}) \leq P, \quad (5)
\]

where \( P_{S,BC} \) is the average transmitted power of \( S \) in \( BC \) mode, \( P_{S,\text{MAC}} \) and \( P_{R,\text{MAC}} \) are similarly interpreted, and \( P \) is the total transmitted power of the relay system. The relationship between the total transmitted power and the signal to noise ratio (SNR) of the relay system is defined as follows [4].

\[
\text{SNR} = tP_{S,BC} + t'(P_{S,\text{MAC}} + P_{R,\text{MAC}}) = 2Rb/N_0, \quad (6)
\]

where \( R \) is the overall code rate of relay channel which is equal to \( k/n \) and \( E_b/N_0 \) is the the energy per bit to noise power spectral density ratio.

For a Gaussian relay channel, a general time-division half-duplex relay channel with decode-and-forward protocol can achieve the following rate [4].

\[
\mathcal{R} = \sup_{\Theta, 0 \leq t, r \leq 1} \min \{ tC(w) + t'C(x), tC(y) + t'C(z) \}, \quad (7)
\]

where

\[
w = |h_{\text{SR}}|^2 P_{S,\text{BC}},
\]
\[
x = (1 - r^2) |h_{\text{SR}}|^2 P_{S,\text{MAC}},
\]
\[
y = |h_{\text{SR}}|^2 P_{R,\text{BC}},
\]
\[
z = |h_{\text{SR}}|^2 P_{S,\text{MAC}} + |h_{\text{SR}}|^2 P_{R,\text{MAC}} + 2r \sqrt{|h_{\text{SR}}|^2 |h_{\text{SR}}|^2 P_{S,\text{MAC}} P_{R,\text{MAC}}},
\]

\[
C(SNR) = \log_2(1 + SNR),
\]

and \( \Theta \) is defined in (5). The parameter \( r \) represents the correlation between \( s_{\text{MAC}}^{(i)} \) and \( n_{\text{MAC}}^{(i)} \) in MAC mode. The function \( C(.) \) is known as the Shannon formula used to calculate capacity of Gaussian direct link. In order to achieve the maximum rate given in (7), a joint optimization is needed over the correlation in MAC mode, time-sharing factor and power allocation.

4. Coding Strategy for Very Noisy Relay Channels

It is worth noting that maintaining the reliable communication in very noisy channels is to deal with low rate channel codes. We also note that the very noisy channel relates to the channel with low SNR, i.e. noise power is dominate. In this section, we first describe the multiplicative repetition which exactly is the method to construct high performance low rate NBLDPC codes. After that, good suboptimal parameters that can simplify relay protocol are discussed and we then suggest an appropriate power allocation for very noisy relay channel. Finally, the proposed coding strategy is developed based on the concept of multiplicative repetition.

4.1 Multiplicative Repetition

More recently, Kasai et al. have proposed an efficient
method called multiplicative repetition to construct low rate NBLDPC codes [11]. Over direct channels, the multiplicatively repeated NBLDPC codes constructed by this method outperform the previously found low rate codes. By using the NBLDPC code of rate $R_1$, we can easily construct code NBLDPC codes of lower rate $R < R_1$ described now as follows.

Let $C_1$ denotes an NBLDPC code of length $N$ and coding rate $R_1$. Since code $C_1$ is used to construct another codes, we refer to $C_1$ as a mother code. A code $C_2$ of length $2N$ and lower code rate $R_2 = \frac{1}{2}R_1$ can be constructed as follows. We select $N$ coefficients $r_{N+1}, \ldots, r_{2N}$ randomly from $\text{GF}(2^m) \setminus \{0\}$. Note that we define $r_v = 1 \in \text{GF}(2^m)$ for $v = 1, \ldots, N$ for simplicity of notation. We then multiplicatively repeat the coded symbols from $C_1$ with the coefficients to obtain the lower-rate code $C_2$ as follows

$$C_2 = \{(x_1, \ldots, x_{2N}) \in \text{GF}(2^m)^{2N} \mid x_{N+v} = r_{N+v}x_v, \\
\text{for } v = 1, \ldots, N, \{x_1, \ldots, x_N\} \in C_1\}.$$  

The codes $C_2, \ldots, C_T$ of lower code rates can also be constructed from code $C_1$ through the multiplicative repetition process. We refer to the parameter $T$ as repetition parameter. For $T \geq 3$, in a recursive fashion, $N$ coefficients $r_{(T-1)N+1}, \ldots, r_{TN}$ are chosen randomly from $\text{GF}(2^m) \setminus \{0\}$. The code $C_T$ of rate $R_T = \frac{1}{T}R_1$ can also be constructed as follows.

$$C_T = \{(x_1, \ldots, x_{TN}) \in \text{GF}(2^m)^{TN} \mid x_{(T-1)N+v} = r_{(T-1)N+v}x_v, \text{for } v = 1, \ldots, N, \\
\{x_1, \ldots, x_{(T-1)N}\} \in C_{T-1}\}.$$  

Let $X_v$ be the random variables with realization $x_v$ where $X_v$ represents the coded symbol and $v = 1, 2, \ldots, N$. Let $Y_v$ be the random variables with realization $y_v$, which is the received value from the channel $Pr(Y_v|X_v)$. The probability of transmitted symbol $Pr(X_v)$ is assumed to be uniform. We assume that the decoders at both $R$ and $D$ know the channel transition probability

$$Pr(X_v = x|Y_v = y_v), v = 1, 2, \ldots, N, \quad (8)$$

for $x \in \text{GF}(2^m)$. For memoryless binary-input output-symmetric (MBIOS) channels, the channel transition probability is given as

$$Pr(X_v = x|Y_v = y_v) = \prod_{i=1}^{m} Pr(X_{v,i} = x_i|Y_{v,i} = y_{v,i}), \quad (9)$$

where $X_{v,i}$ is the random variable of the transmitted bit, $(x_1, x_2, \ldots, x_m) \in \text{GF}(2^m)$ is the binary representation of $x \in \text{GF}(2^m)$ and the corresponding channel output $y_{v,i}$ and its random variable $Y_{v,i}$.

The decoding of NBLDPC codes is accomplished on the Tanner graph through the belief propagation (BP) algorithm [13]. The BP algorithm for multiplicatively repeated NBLDPC codes [11] is summarized into 4 steps as follows

**initialization**: We set the iteration round $\ell = 0$. Each variable node sends the initial message $p_{cv}^{(0)} = p_{cv}^{(0)}(0) = \xi_p$, to each adjacent check node $c$ where $v = 1, 2, \ldots, N$ and $c = 1, 2, \ldots, M$. For general NBLDPC codes ($T = 1$) and multiplicatively repeated NBLDPC codes ($T \geq 2$), the probability $p_{cv}^{(0)}(x)$ is initialized as follows

$$p_{cv}^{(0)}(x) = Pr(X_v = r_vx|Y_v = y_v)$$

$$\xi \prod_{i=1}^{T} Pr(X_{tN+v} = r_{N+v}x|Y_{tN+v} = y_{N+v}),$$

where $\xi$ represents the normalized constant so that

$$\sum_{x \in \text{GF}(2^m)} p_{cv}^{(0)}(x) = 1.$$  

**check to variable**: For each check node $c = 1, 2, \ldots, M$, let $\partial_c$ be the set of adjacent variable nodes of $c$. The check node $c$ sends the following message $p_{cv}^{(l)}(x) \in \mathbb{R}^{2^m}$ to each adjacent variable node $v \in \partial_c$

$$\bar{p}_{cv}^{(l)}(x) = p_{cv}^{(l)}(h_{cv}x) \quad \text{for } x \in \text{GF}(2^m),$$

$$\tilde{p}_{cv}^{(l+1)}(x) = \sum_{v' \in \partial_c \setminus \{v\}} \bar{p}_{cv}^{(l)}(x) \quad \text{for } x \in \text{GF}(2^m),$$

where $p_1 \otimes p_2 \in \mathbb{R}^{2^m}$ is a convolution of $p_1 \in \mathbb{R}^{2^m}$ which can be expressed as follows

$$\langle p_1 \otimes p_2 \rangle(x) = \sum_{y \in \text{GF}(2^m)} p_1(y)p_2(z) \quad \text{for } x \in \text{GF}(2^m).$$

The convolution appeared above can be efficiently calculated via FFT and IFFT [19]. Increment the iteration round as $\ell := \ell + 1$

**variable to check**: For each variable node $v = 1, 2, \ldots, N$, let $\partial_v$ be the set of adjacent check nodes of $v$. The message $p_{cv}^{(l)}$ sent from $v$ to $c$ is computed as follows

$$p_{cv}^{(l)}(x) = \xi p_{cv}^{(0)}(x) \prod_{v' \in \partial_v \setminus \{v\}} \bar{p}_{cv}^{(l)}(x) \quad \text{for } x \in \text{GF}(2^m),$$

where $\xi$ is again the normalized constant.

**tentative decision**: For $v = 1, 2, \ldots, N$, the tentative decision of the $v$th symbol is given by

$$\hat{x}_v^{(l)} = \arg \max_{x \in \text{GF}(2^m)} p_{cv}^{(l)}(x) \prod_{v' \in \partial_v \setminus \{v\}} \bar{p}_{cv}^{(l)}(x),$$

Let $\mathbf{X} = (\hat{x}_1^{(l)}, \hat{x}_2^{(l)}, \ldots, \hat{x}_N^{(l)})$ be the estimated codeword of iteration round $\ell$. The decoder stops when the maximum iteration $\ell_{\text{max}}$ is reached or $H\mathbf{X} = \mathbf{0} \in \text{GF}(2^m)^M$. Otherwise repeat the latter 3 decoding steps.

4.2 Good Suboptimal Parameters for Relay Channel

Chakrabarti et al. have investigated the sensitivity of
achievable rate for relay channels and the major findings can be listed as follows [5]:
1) At low SNRs, the relaying gain is very large and $R$ under BPSK and Gaussian inputs are comparable.
2) The optimal correlation $r$ has a little effect on $R$ so we can restrict to $r = 1$ (completely correlated information).
3) Equal time-sharing $t = 1/2$ incurs a marginal loss compared with the optimal one.
4) Power allocation alone is sufficient to achieve most of the relaying gain.

Equal time-sharing, static power allocation, and completely correlated information in MAC mode are quite attractive from the practical point of view since the design of protocol, scheduling, and synchronization can be simplified. Based on the results listed above, we can restrict ourself to relay coding strategy with $r = 1$ because the loss in achievable rate is marginal. The remaining is to find the static power allocation (power allocation is static for all points of operating SNRs) that yields a good achievable rate.

We suggest that an appropriate static power allocation for low SNRs can be computed by averaging the optimal power allocation [5], [8]. This can be done as follows: Let $g = \{g_1, \ldots, g_b\}$ be the set of discrete values of SNR. The set of optimal power allocation at $S$ in BC mode which corresponds to $g$ is denoted by $P^*_S(BC) = \{P^*_S(BC)(g_1), \ldots, P^*_S(BC)(g_b)\}$. The sets $P^*_S(MAC)$ and $P^*_R(MAC)$ are defined in the same way. For any given SNR (in low SNR region), the fractions of power at each mode and terminal are given by

$$
\kappa_S(BC) = \frac{1}{b} \sum_{i=1}^{b} P^*_S(BC)(g_i),
\kappa_S(MAC) = \frac{t}{b} \sum_{i=1}^{b} P^*_S(MAC)(g_i),
\kappa_R(MAC) = 1 - \kappa_S(BC) - \kappa_S(MAC),
$$

where $\kappa_g = \frac{1}{b} \sum_{i=1}^{b} g_i$. Based on these fractions, the static power allocation for given SNR is expressed as

$$
tP^*_S(BC) = \kappa_S(BC)SNR,
tP^*_S(MAC) = \kappa_S(MAC)SNR,
tP^*_R(MAC) = \kappa_R(MAC)SNR.
$$

It is clearly seen that this static power allocation satisfies the power constraint in (5).

Various achievable rates obtained from the different set of parameters are displayed in Fig. 3. The direct capacity is also plotted to illustrate the relaying gain. The achievable rate obtained from the optimal parameters is the leftmost curve. Comparing to the optimal achievable rate, optimal power allocation, $t = 1/2$, and $r = 1$ incur a marginal loss in achievable rate (as we expected from [5]). By using $g = \{-16 \text{ dB}, -15 \text{ dB}, \ldots, -8 \text{ dB}\}$ which is the set of points in low SNR region, $\kappa_S(BC) = 0.6456$, $\kappa_S(MAC) = 0.0634$, and $\kappa_R(MAC) = 0.2910$ are obtained. It can be seen from the figure that this suggested static power allocation with $t = 1/2$ and $r = 1$ exhibits very good achievable rate at low SNRs (or equivalently $R < 0.4$). For BPSK input, the suggested static power allocation theoretically achieves the relaying gain in low SNRs by more than 3 dB. We also show that the coarse power allocation i.e. equal power allocation suffers from the significant loss more than 2 dB in achievable rate.

**Fig. 3** Various achievable rates obtained from the different set of parameters. In this figure, the attenuation is $\alpha = 2$ and the $S$-$R$ distance is $d = 2$.

### 4.3 Proposed Coding Strategy

In this subsection, we present a simple low rate coding strategy for $t = 1/2$ and $r = 1$. For $x_v \in GF(2^m)$, $v = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, m$, the proposed low rate relay coding strategy can be described as follows.

1) **Encoding in BC mode:** By using a multiplicatively repeated code $C_T$ of rate $R_{S,BC} = \frac{K}{TN}$ constructed from mother code $C_1$ of rate $R_1$ with $T \geq 1$, $S$ encodes $K$ information symbols ($mk$ bits) into a codeword $x = (x_1, x_2, \ldots, x_{TN}) \in GF(2^m)^{TN}$. This codeword $x \in C_T$ is mapped to a binary sequence of length $ln = mTN$ bits. Then the BPSK modulated signals of the binary sequence are sent to $R$ and $D$. $S$ also stores the codeword $x$ for using in MAC mode. We note that there is no multiplicative repetition if $T = 1$.

2) **Decoding in BC mode:** $R$ decodes its received signals by using the BP algorithm as described in section 4.1. $R$ first calculates the input $p^{(0)}_i(x)$ for BP algorithm which is given by

$$
p^{(0)}_i(x) = \prod_{r=1}^{m} \exp(\frac{1}{2}(g^{(t-1)m+i} - h_{SR} s_i)^2)
$$

where $g^{(t-1)m+i}$ is the $i$-th element of the set of $t-1$ correlated inputs to $R$ from $D$ and $S$. $h_{SR}$ is the channel gain.
We claim that the proposed coding strategy is very simple due to the following two reasons. 1) The encoder at \( R \) uses only the multiplicative repetition which requires only \( TN \) multiplications over \( GF(2^m) \). 2) The decoding step at both \( S \) and \( R \) is accomplished on the same Tanner graph of \( N \) variable nodes, i.e., the Tanner graph of mother code \( C_1 \). The difference of decoding at \( R \) and \( D \) is only the initialization step. Thus, the computational complexity of the decoders is almost the same even the code rate at \( D \) is lower than that of \( R \).

![Fig. 4 Block diagram of the proposed coding strategy.](image_url)

5. Results and Discussions

In this section, we examine the decoding performance of the proposed coding strategy under \( t = 1/2 \). For all results, the regular \( (d_v = 2, d_e = 4) \) NBLDPC codes defined over \( GF(2^8) \) are employed. The FFT-based BP algorithm [19] with the maximum iteration \( \ell_{\max} = 100 \) is used. The BPSK modulation is deployed for all the links. We set the attenuation exponent to \( \alpha = 2 \) and the \( S-R \) distance is \( d = 0.5 \). The bit error rate and frame error rate are denoted by BER and FER, respectively.

We first assess the proposed coding strategy by using the Monte Carlo density evolution [20, pp.22-23]. By assuming very large code length and cycle free Tanner graph, the Monte Carlo density evolution is a tool
for calculating the minimum SNR (known as decoding threshold) of which NBLDPC codes can reliably decode the noisy received signal. The Monte Carlo density evolution for direct transmission is extended to calculate the decoding threshold of the proposed NBLDPC relay coding strategy. The decoding thresholds obtained from this method are reported in Table 1. It is obviously seen that the proposed coding strategy can asymptotically perform within 0.5 dB from \( R \).

<table>
<thead>
<tr>
<th>( R ) (bit)</th>
<th>( R_{5,DC} ) (bit)</th>
<th>Min. ( E_b/N_0 ) (dB)</th>
<th>Threshold (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/6</td>
<td>1/3</td>
<td>-4.66</td>
<td>-4.24</td>
</tr>
<tr>
<td>1/4</td>
<td>1/2</td>
<td>-4.08</td>
<td>-3.62</td>
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<td>3/5</td>
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</tbody>
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After looking at the performance in the limit of very large code length, the performances at small and medium code lengths are presented. Figure 5 shows the performances of the proposed coding strategy at \( R = 1/4 \) and \( R = 3/10 \). Different information sizes \( k \) are used to show that the BER curve is improved by increasing \( k \) (also \( n \)). At both rates, the proposed strategy of medium size can perform within 1.8 dB from their corresponding \( R \). For \( R = 0.25 \), it is seen that the performance within 1.2 dB from \( R \) can be achieved with \( k = 6000 \) bits. The direct capacities for \( R = 1/4 \) and \( R = 3/10 \) are also plotted to show that the proposed strategy of small \( k \) can surpass these limit.

The comparison between the performance of proposed coding strategy with the direct performance is also presented. We select the multiplicatively repeated NBLDPC codes of \( R = 1/4 \) for comparison since these codes are known as the best performing channel code at low code rates and small code lengths [11]. Under the same information size (here are 144 and 576 bits), it is shown in Fig. 5 that the proposed coding strategy of \( R = 1/4 \) provides coding gain of more than 3 dB over their corresponding direct performances. Therefore, at the cost of employing relay, the relay system governed by the proposed coding strategy is attractive since we can save more than two time of the transmitted power. With this evidence, it is worth to implement the relay system instead of the conventional direct transmission.

To further illustrate the performance of the proposed strategy, the performances at lower rates \( R = 1/6 \) and \( R = 1/12 \) are shown in Fig. 6. The medium information size \( k = 576 \) bit is employed. The gap from \( R \) is about 2.2 dB for both rates. The relaying gain of about 1.5 dB can also be obtained. As clearly seen from the figure, FER performance is also excellent. We note that the optimized LDPC codes do not possess this advantage and a high rate outer code i.e. BCH or RS codes is needed [4]. This implies small overall rate loss.

![Fig. 5 The performance of proposed coding strategy at \( R = 1/4 \) and \( R = 3/10 \). The mother code \( C_1 \) of the proposed strategy of \( R = 1/4 \) and \( T = 1 \) is \((2,4)\)-regular NBLDPC code while \((2,5)\)-regular NBLDPC code is used as the mother code \( C_2 \) for the proposed strategy of \( R = 3/10 \) and \( T = 1 \). The vertical line labelled with “Relay” corresponds to achievable rate of relay channel. The dashed vertical line labelled with “Direct” corresponds to capacity of direct transmission.](image)

Although the transmission model in MAC mode described in (4) is widely used (see e.g. [4], [8]) but the signals from \( S \) and \( R \) must be coherently combined. This may cause a high complexity of receiver. This problem can be easily alleviated by keeping \( S \) silent in MAC mode (this definitely incurs a loss in achievable rate). In this circumstance, \( R \) receives the additional information of length \( t'n \) only from \( R \). It is seen from Fig. 6 that the performance with silent \( S \) in MAC mode is away from its original performance by about 0.5 dB. The relaying gain of 1 dB is still can be achieved even \( S \) is forced to be silent in MAC mode. Therefore, the application of relay system governed by the proposed strategy is still attractive even we consider very practical situation. We note that the power allocation for \( S \) in BC mode and \( R \) in MAC mode is static. This static power allocation is calculated from the method provided in Section 4.2 but the difference is only \( P_{B,MAC} = 0 \).

We mention that the mother code for the proposed strategy used in this section is regular \((2,d_c)\)-NBLDPC codes defined over \( GF(2^8) \). Hence, two problems addressed in the introduction can be easily alleviated. The mother code used in this study is regular and the degree of check node is low, i.e \( d_c = 5 \) for proposed strategy of \( R = 3/10 \). Before ending this section, it is important to note that the proposed coding strategy for very noisy relay channels is based on the multiplicative repetition not a simple repetition. If one try to use a simple repetition instead of multiplicative repetition for encoding in MAC mode, the coded relay performance will be degraded as shown in the case of binary LDPC
6. Conclusion

In this paper, the new simple NBLDPC coding strategy for very noisy relay channel is presented. We have demonstrated that this NBLDPC coding strategy is very promising to capture the relaying gain in half duplex relay channel at low SNRs under very practical constraints. The advantages of the proposed strategy can be listed as follows: 1) At BER of $10^{-5}$, the performance within 2 dB from the theoretical limit of relay channel can be obtained by using the NBLDPC codes of medium sizes. 2) We do not need the optimization process for optimizing the degree profile of LDPC codes. We just employ the $(2, d_c)$-regular NBLDPC defined over GF($2^m$). 3) The FER performances obtained from the $(2, d_c)$-regular NBLDPC codes are excellent even the outer BCH or RS codes are not employed to lower the error floors. 4) The relay and the destination can decode with almost the same computational complexity since both the relay and the destination use the same Tanner graph for decoding. 5) The encoding processes for both the source and the relay in MAC mode are very simple since we use the multiplicative repetition which requires only $TN$ multiplications over GF($2^m$).

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References