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On factorizations of directed graphs by cycles

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Abstract

In this paper we present a min-max theorem for a factorization problem in directed graphs. This extends the Berge-Tutte formula on matchings as well as formulas for the maximum even factor in weakly symmetric directed graphs and a factorization problem in undirected graphs. We also prove an extension to the structural theorem of Gallai and Edmonds about a canonical set attaining minimum in the formula. The matching matroid can be generalized to this context: we get a matroidal description of the coverable node sets.

1 Introduction

Let $G = (V, E)$ be a directed graph. A cycle (path) is the arc-set of a closed (unclosed) directed walk without repetition of arcs or nodes. A path/cycle-factor is the arc-set $M \subseteq E$ of a subgraph of G which is a node disjoint union of paths and cycles. We call an arc $e = uv \in E$ *symmetric*, if $vu \in E$, otherwise e is *asymmetric*. A cycle is *even*, if it consists of an even number of arcs; it is *asymmetric*, whenever it has at least one asymmetric arc. A loop is considered to be an odd cycle of length one, and is symmetric. Let $\mathcal{H}$ be a set of some cycles in G such that $\mathcal{H}$ contains all the even cycles and all the asymmetric cycles. (Note that we have the freedom to drop or include some symmetric odd cycles in $\mathcal{H}$.) If we have a pair $(G, \mathcal{H})$ as above, we say G is *$\mathcal{H}$-symmetric*. An *$\mathcal{H}$-factor* is a path/cycle-factor such that all cycles of it are cycles in $\mathcal{H}$. Let $\nu^{\mathcal{H}}(G)$ denote the maximum cardinality of an $\mathcal{H}$-factor. We consider the problem of determining $\nu^{\mathcal{H}}(G)$ in view of the formula below.

Some further definitions regarding formula (1): $N_G^+(X) := \{x \in V - X : \exists y \in X, yx \in E\}$. We say that some cycles of $\mathcal{H}$ *cover* a vertex-set $C \subseteq V$ if these cycles are node disjoint and the union of their node-sets is exactly C. The node-set of a directed graph can be partitioned into strongly connected components, the

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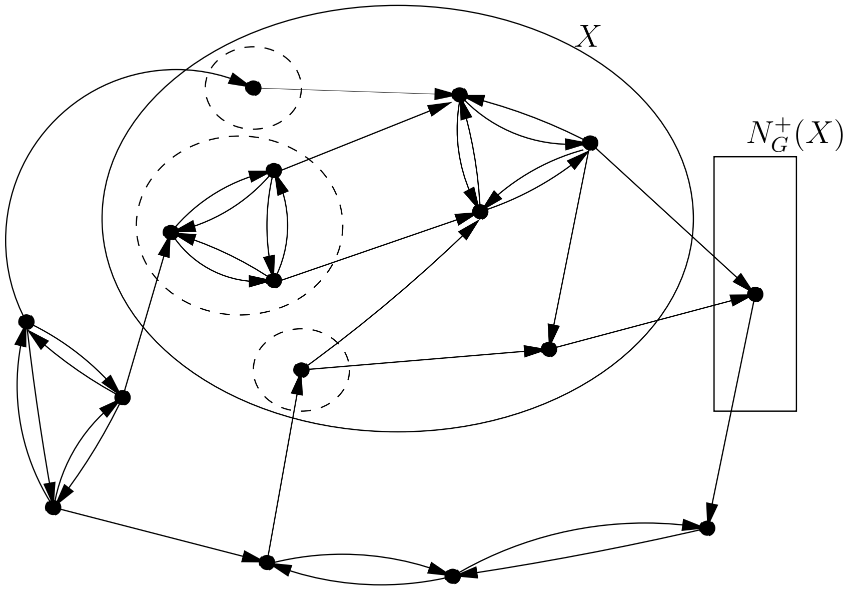
contraction of which leaves an acyclic graph. A strongly connected component will be called a *source-component* if it corresponds to a source-node in the contracted graph (a source-node is a node with no entering arc). For a node-set $X \subseteq V$, let $G[X]$ denote the subgraph of G spanned by X. $sc^{\mathcal{H}}G[X]$ denotes the number of those source components C in $G[X]$ that cannot be covered by some $\mathcal{H}$-cycles.

The main theorem of this paper is the following min-max formula for the maximum cardinality of an $\mathcal{H}$-factor.

Theorem 1.1. *If $G = (V, E)$ is an $\mathcal{H}$-symmetric directed graph, then*

$$\nu^{\mathcal{H}}(G) = \min_{X \subseteq V} |V| + |N_G^+(X)| - sc^{\mathcal{H}}G[X]. \tag{1}$$

We demonstrate the theorem with the following example:



Let $\mathcal{H}$ be the set of even cycles and asymmetric cycles in G, the graph G in the figure is $\mathcal{H}$-symmetric. The set X and $N_G^+(X)$ is indicated in the figure, the dashed parts are the source components of $G[X]$. Here $|V| + |N_G^+(X)| - sc^{\mathcal{H}}G[X] = 16 + 1 - 3 = 14$, and it is easy to find an $\mathcal{H}$-factor of size 14.

To see the easy direction of inequality in (1), we prove that, for any $\mathcal{H}$-factor M and any set $X \subseteq V$, inequality $|M| \le |V| + |N_G^+(X)| - sc^{\mathcal{H}}G[X]$ holds. This implies that the left hand side is at most the right hand side in the formula (1). We get this inequality as the sum of the below inequalities:

$$|i_G(X) \cap M| \le |X| - sc^{\mathcal{H}}G[X], \tag{2}$$

$$|\delta_G(X) \cap M| \le |N_G^+(X)|, \tag{3}$$

$$|(i_G(V - X) \cup \delta_G(V - X)) \cap M| \le |V| - |X|, \tag{4}$$

where $i_G(X)$ denotes the set of the arcs of G with both ends in X and $\delta_G(X)$ denotes the set of the arcs of G with tail in X and head in $V - X$.

2 Preliminaries

For an introduction to matching theory see [10]; in this paper we will be supported by the following notions. An undirected graph is called *factor-critical* if the deletion of any node leaves a graph having a perfect matching (i.e. perfectly matchable). In case of directed graphs, *factor-critical* means that all arcs are symmetric, and the underlying undirected graph is factor-critical. Now, $fc^{\mathcal{H}}G[X]$ denotes the number of source components in $G[X]$ which cannot be covered by $\mathcal{H}$-cycles and are factor-critical. Let $Fc^{\mathcal{H}}G[X]$ denote the union of these components. A directed graph is said to be $\mathcal{H}$-critical, if it is factor-critical, and it cannot be covered by $\mathcal{H}$-cycles. Clearly, $fc^{\mathcal{H}}G[X] \le sc^{\mathcal{H}}G[X]$. The following strengthening of Theorem 1.1 will be easier to prove:

Theorem 2.1. *If $G = (V, E)$ is an $\mathcal{H}$-symmetric directed graph, then*

$$\nu^{\mathcal{H}}(G) = \min_{X \subseteq V} |V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X]. \tag{5}$$

The proofs in the paper are going to refer to the following well-known facts from matching theory (see [10]).

Lemma 2.2. *Suppose $s, t \in V$ are two (not necessarily distinct) nodes of the factor-critical graph $G = (V, E)$. Then there is a path $P(s, t)$ on an even number of edges such that $G[V - V(P)]$ is perfectly matchable. If $s = t$, then P is an empty path.*

Theorem 2.3 (Edmonds, Gallai). *For any graph $G' = (V', E')$ there is a set $A \subseteq V$ such that the following hold:*

1. *The components of $G' - A$ are factor-critical or perfectly matchable.*
2. *We construct a bipartite graph $G_0 = (A, D_0; E_0)$ as follows: delete the perfectly matchable components and contract the factor-critical components of $G' - A$, and delete the edges spanned by A. Then for any $v \in D_0$ there is a matching in G_0 covering A, and exposing v.*

Theorem 2.4. *Let $G = (U, V; E)$ be a bipartite graph. If there is a matching that covers a set $U' \subseteq U$ and there is a matching which covers a set $V' \subseteq V$, then there is a matching which covers $U' \cup V'$.*

3 Remarks

We mention two previous results which have motivated, and are special cases of Theorem 1.1 and 2.1.

A factorization problem was addressed in [1] by Cornuéjols and Hartvigsen, and in [2] by Cornuéjols and Pulleyblank. The so-called triangle-free 2-matching problem was discussed in detail: they gave a formula for the maximum number of nodes that can be covered by a node-disjoint collection of edges and odd cycles of length at least 5. The following related theorem due to M. Loebl and S. Poljak [13] characterizes a factorization problem in undirected graphs, see also J. Szabó and Z. Király [14]:

Theorem 3.1. *Let $G' = (V', E')$ be an undirected graph, and $\mathcal{H}'$ be a set of some (maybe none) odd cycles in G'. Let $\nu^{\mathcal{H}'}(G')$ denote the maximum number of nodes that can be covered by a node-disjoint collection of edges and $\mathcal{H}'$-cycles. Then*

$$\nu^{\mathcal{H}'}(G) = \min_{X \subseteq V} |V| - c^{\mathcal{H}'}(X) + |X| \tag{6}$$

*where $c^{\mathcal{H}'}(X)$ denotes the number of factor-critical components of $G - X$ that cannot be covered by edges and $\mathcal{H}'$-cycles (i.e. $\mathcal{H}'$-*critical *components).*

Each symmetric directed graph is $\mathcal{H}$-symmetric for any $\mathcal{H}$ containing the even cycles. For symmetric directed graphs Theorem 1.1 is equivalent to Theorem 3.1 applied to the underlying undirected graph. For completeness we include a proof of Theorem 3.1 first published by M. Loebl and Poljak [13].

Proof. It is easy to check one direction of inequality, thus we only show the existence of a set X and a factor that give equality in formula (6).

Take the set A from Theorem 2.3. Let $P \subseteq D_0$ be the set of nodes in G_0 which correspond to a factor-critical component of $G - A$ that cannot be covered by edges and $\mathcal{H}'$-cycles; let $Q := D_0 - P$. By Hall's theorem there is a set $Z \subseteq P$ and a matching M in G_0 that exposes $|Z| - |\Gamma_{G_0}(Z)|$ nodes in P. By part 2. of Theorem 2.3 there is a matching in G_0 that covers A, thus by Theorem 2.4 there is a matching M' that covers A, and exposes at most $|Z| - |\Gamma_{G_0}(Z)|$ nodes in P.

We can construct a factor M'' that exposes at most $|Z| - |\Gamma_{G_0}(Z)|$ nodes of G' as follows. We extend M' by using perfect matchings and near-perfect matchings in components of $G' - A$; except for nodes in Q exposed by M', use a cover by edges and $\mathcal{H}'$-cycles. Then M'' and $X = \Gamma_{G_0}(Z)$ gives equality in formula (6). □

Let G be a directed graph such that the odd cycles are symmetric, we call such a graph "hardly symmetric" (see [12, 5]). Let $\mathcal{H}_{even}$ be the set of even cycles of G, an *even factor* is by definition an $\mathcal{H}_{even}$-factor. Notice, that a directed graph is hardly symmetric if and only if it is $\mathcal{H}_{even}$-symmetric. These notions were introduced by W.H. Cunningham in [5]. In [12] Pap and Szegő gave a formula for $\nu^{\mathcal{H}_{even}}(G)$ for any hardly symmetric graph G, that formula is a special case of formula 1.1. We mention that M. Makai recently gave a TDI description of a polyhedron corresponding to even factors in a weakly symmetric graph [11]. We also mention that the following theorems can be deducted from the formula in [12] for $\nu^{\mathcal{H}_{even}}(G)$: Dilworth's theorem on the maximum number of independent elements in a partially ordered set, Menger's theorem on disjoint paths, a theorem of Gallai and Milgram on minimum number of directed path to cover all nodes in a directed graph, a theorem of S. Felsner in [6] on maximum number of arcs in a path/cycle-factor, a formula in [7] for path-matchings. For proofs, see [12].

Much of this research was motivated by the notions path-matching and even factors introduced by W.H. Cunningham and J.F. Geelen (see [5, 4, 7]). They gave good characterizations, as well as an algorithm based on the following algebraic method. For a directed graph $G = (V, E)$, we define a $V \times V$ matrix $M = M(G)$ of commuting, algebraically independent indeterminates:

$M_{u,v} := 0$ if $uv \notin E(G)$,
$M_{u,v} := x_{u,v}$ if $uv \in E(G)$ and $vu \notin E(G)$,
$M_{u,v} := x_{u,v}$ and $M_{v,u} := -x_{u,v}$ if $uv \in E(G)$ and $vu \in E(G)$.

Take an undirected graph G', let G'' be constructed from G' by replacing each edge uv in $E(G')$ by arcs uv and vu. The matching number of an undirected graph G' can be determined as half the rank of a matrix $M(G'')$. Thus, for symmetric directed graphs we have a combinatorial description for the rank of M. More generally, in case of G being hardly symmetric we have $rk(M) = \nu^{\mathcal{H}_{even}}(G)$.

Geelen discovered an algorithm to calculate the rank $rk(M)$ of matrix M for any directed graph G [8]. Since the rank is equal to the rank for some rational evaluation of the indeterminates, one has to find a nice evaluation. We get a randomized algorithm due to L. Lovász [9], if we put uniformly distributed independent values from $\{1, \dots, |V|\}$. Geelen's algorithm is a derandomization for this algorithm, which yields an algorithm to calculate the maximum cardinality of an even factor, and also to determine a maximum even factor, see [8].

Let G be an arbitrary graph, and let $\mathcal{H}_{even\ and\ asym.}$ be the collection of even cycles and asymmetric cycles. It is easy to see that

$$rk(M) = \nu^{\mathcal{H}_{even\ and\ asym.}}(G).$$

The above cited method gives a polynomial algorithm to compute this number. Theorem 1.1 gives a formula for a more general case: the only constraint to the set $\mathcal{H}$ is $\mathcal{H}_{even\ and\ asym.} \subseteq \mathcal{H}$. Of course one does not expected to have a polynomial algorithm in this generality. Theorem 1.1 is not a good characterization, since it would require to decide whether some $G[C]$ can be covered by some cycles of $\mathcal{H}$. Suppose, we have an oracle to decide for any factor-critical subgraph $G[C]$, if it can be covered by some cycles of $\mathcal{H}$, and it shows one covering, if any. Then Theorem 2.1 is a good characterization, and we may hope for a polynomial time algorithm.

Consider the following statement (for a proof see Cornuéjols, Hartvigsen and Pulleyblank [3]). If a factor-critical (undirected) graph can be factorized by edges and some cycles of $\mathcal{H}'$, then there is a factorization which uses exactly one odd cycle of $\mathcal{H}'$. Thus, Theorem 2.1 is a good characterization in the case when the odd cycles of $\mathcal{H}$ can be listed in polynomial time. This is the case, if the length of odd cycles in $\mathcal{H}$ is bounded.

We get another case when an oracle exists by the following lemma:

Lemma 3.2 (Cornuéjols, Pulleyblank, [2]). *Given an undirected graph G' and suppose $\mathcal{H}'$ is a set of odd cycles in G' such that the complement of $\mathcal{H}'$ has only triangles. Then G' is $\mathcal{H}'$-critical if and only if it is a triangle cluster of triangles not in $\mathcal{H}'$.*

A *triangle cluster* is the single node graph, and each graph we get by the following operation: choose an old node a, add new nodes b, c and arcs ab, bc, ca to the graph. A directed graph is called a triangle cluster, if it is symmetric, and the underlying undirected graph is a triangle cluster.

Theorem 3.3. *Suppose $G = (V, E)$ is a directed graph, such that each directed cycle of three arcs is symmetric. Then the maximum number of arcs in a path/cycle-factor without three-arc cycles is*

$$\min_{X \subseteq V} |V| + |N_G^+(X)| - tc\, G[X] \tag{7}$$

where $tc\, G[X]$ is the number of source components of $G[X]$ which are triangle clusters.

4 Proof

We extend the proof in [12] to prove Theorem 2.1. The proof of Theorem 2.1 will be presented in the following structure: A dividing procedure is presented in CASE 4 which gives two smaller graphs, and proves the formula by induction for most pairs $G, \mathcal{H}$. Cases where the dividing procedure does not lead to graphs with less edges will be discussed in the first three cases.

In a subgraph G' of G, if we use the letter $\mathcal{H}$, that means the truncation of $\mathcal{H}$ to those cycles of G that are also cycles in G'. This is legitimate since in this sense G' is $\mathcal{H}$-symmetric.

A set $X \subseteq V$ will be called a cut. A cut X is called *tight* if it minimizes $|V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X]$. A cut X is called *trivial* if one of the following holds:

(i) The source components of $G[X]$ are single nodes, $V = X \cup N_G^+(X)$ and there is no arc uv such that $u \in N_G^+(X)$.

(ii) X is a stable set in G, and there is no arc uv such that $u \in X$ and $v \in V - X$.

We have already proved in the introduction, that the left hand side is at most the right hand side in the formula (1). The proof that there is a cut X and an $\mathcal{H}$-factor K such that $|K| = |V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X]$ goes by induction on $|E| + |V|$. Take a counterexample with $|E| + |V|$ minimum. Without loss of generality we may assume that G is weakly connected, that is, its underlying undirected graph is connected.

Observation 4.1. $X = V$ is the only possibility for a tight cut of type (i).

Proof. If $X \neq V$ is a tight cut of type (i), then since G is weakly connected, $|N_G^+(X)| > 0$. Thus $|V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X] > |V| - fc^{\mathcal{H}}G[V]$, a contradiction. □

CASE 1. G is symmetric.

In this case formula (1.1) follows from Theorem 3.1. Thus from now on we may assume that G is not symmetric. For better reading in the forthcoming part, we use $\tau_G^{\mathcal{H}}(X) := |V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X]$ for the *value* of cut X in G. Let $\tau_G^{\mathcal{H}} := \min_{X \subseteq V} \tau_G^{\mathcal{H}}(X)$ be the value of a tight cut in G. Let $uv = e \in E$ be an arc such that $vu \notin E$. We observe that $G - e$ is an $\mathcal{H}$-symmetric digraph. For any cut X

$$\tau_{G-e}^{\mathcal{H}}(X) \leq \tau_G^{\mathcal{H}}(X) \leq \tau_{G-e}^{\mathcal{H}}(X) + 1 \tag{8}$$

with $\tau_G^{\mathcal{H}}(X) = \tau_{G-e}^{\mathcal{H}}(X) + 1$ if and only if for $e = uv$ either

A) $u \in X$ and $v \in V - X - N^+_{G-e}(X)$ or
B) $u \in X$ and $v \in Fc^{\mathcal{H}}(G-e)[X]$.

CASE 2. There exits a trivial tight cut of type (ii).

Claim 4.2. *If there is a trivial tight cut of type (ii), then the formula holds.*

Proof. Take an arc $e = uv$ such that $vu \notin E$. If $\tau^{\mathcal{H}}_{G-e} = \tau^{\mathcal{H}}_{G}$, then we are done by induction. Otherwise (8) implies that for any tight cut X in G

$$\tau^{\mathcal{H}}_{G} = \tau^{\mathcal{H}}_{G}(X) = \tau^{\mathcal{H}}_{G-e}(X) + 1.$$

Take a tight cut X in G. By assumption, X is a trivial cut in G. Arc e accords to A) or B), so X cannot be of type (ii), a contradiction. □

CASE 3. Every tight cut is trivial.

Take a tight cut X in G. By assumption, X is a trivial cut in G, and by Claim 4.2 it must be of type (i). By Claim 4.1 $X = V$.

Now $X = V$ is a tight cut of type (i), so there must be at least one source-node in G. Take arc $e' = u'v'$ such that $\{u'\}$ is a source-node in G. $e' = u'v'$ must be of type B), and then $V - u'$ is a tight cut in G. By assumption, $V - u'$ is tight and by Observation 4.1 it can only be of type (ii), a contradiction.

CASE 4. In any other case, let us consider a minimal nontrivial tight cut X.

Claim 4.3. *Each source component of $G[X]$ is $\mathcal{H}$-critical.*

Proof. If a source component C of $G[X]$ can be covered with cycles in $\mathcal{H}$, then $X - C$ is also a tight cut. If $X - C$ is nontrivial, then this contradicts the minimality of X. Thus $X - C$ is trivial, by Observation 4.1 $X - C$ is of type (ii), and by Claim 4.2 we are done.

Suppose a source component $G[C]$ of $G[X]$ cannot be $\mathcal{H}$-factorized, but is not factor-critical. $C \neq V$, thus $G[C]$ has less arcs, than G has. Then by induction there is a subset $Y \subseteq C$ with value $\tau^{\mathcal{H}}_{G[C]}(Y) \leq |C| - 1$. Since $G[C]$ is not factor-critical, $\tau^{\mathcal{H}}_{G[C]}(C) = |C|$ and $\tau^{\mathcal{H}}_{G[C]}(\emptyset) = |C|$, thus Y is a proper nonempty subset of C. It is easy to see that $X - C \cup Y$ is a tight cut in G. C is strongly connected, therefore $X - C \cup Y$ must be entered as well as left by arcs of G. Then $X - C \cup Y$ is a nontrivial tight cut, a contradiction. □

Delete the arc-set $F := \{uv \in E : u \in V - X,\ v \in N^+_G(X)\}$ and contract each component of $Fc_G(X)$ to a node. Let $G_Q = (V_Q, E_Q)$ denote the graph obtained this way. Q denotes the set of new nodes, define $X_Q := X - Fc^{\mathcal{H}}G[X] \cup Q$.

Let $G_1 = (V_1, E_1)$ denote the graph having node set $V_1 := X_Q \cup N^+_G(X)$ and arc set $E_1 := \{uv \in E_Q : u \in X_Q\}$.

Let $G_2 = (V_2, E_2)$ denote the graph having node set $V_2 := Q \cup (V_Q - X_Q)$ and arc set $E_2 := \{uv \in E_Q : v \in V_2 - N^+_G(X)\}$.

The cycles in G_1 and G_2 are also cycles in G. When using $\mathcal{H}$ for G_i, it stands for the truncation of $\mathcal{H}$ to E_i. Clearly, G_1 and G_2 are $\mathcal{H}$-symmetric. Since X is nontrivial, $|E_1| < |E|$ and $|E_2| < |E|$.

Claim 4.4. *Suppose K_1, K_2 are $\mathcal{H}$-factors in G_1, G_2, respectively. Then G has an $\mathcal{H}$-factor K with cardinality $|K| = |K_1| + |K_2| + (|Fc^{\mathcal{H}}G[X]| - fc^{\mathcal{H}}G[X])$.*

Proof. Let K' denote the set of arcs of G corresponding to $K_1 \cup K_2$. We claim that K' can be completed in G so that it has the desired cardinality. To this end let C denote a component of $Fc^{\mathcal{H}}G[X]$, and let c denote its corresponding node in G_Q. By Claim 4.3, C is $\mathcal{H}$-critical.

K' has at most one arc in $\delta_G(C)$: choose $t \in C$ as the tail of this arc if present, otherwise choose t arbitrarily. K' has at most one arc in $\varrho_G(C)$: choose $s \in C$ as the head of this arc if present, otherwise choose s arbitrarily. By Lemma (2.2), there is an path/cycle-factor K_C in $G[C]$ of size $|C| - 1$, consisting of two-arc cycles and an $s - t$ path on an even number of arcs.

$K := K' \cup \bigcup_{c \in Q} K_C$ is a path/cycle-factor with cardinality $|K| = |K_1 \cup K_2| + (|Fc^{\mathcal{H}}G[X]| - |Q|)$. We only have to check, if the cycles traversing $Fc^{\mathcal{H}}G[X]$ are in $\mathcal{H}$:

Suppose that a cycle $W \subseteq K$ is not in $\mathcal{H}$. Let W_Q be the cycle in G_Q corresponding to W. All arcs in W are symmetric in G, hence W_Q has no arc from Q to $X - Q$, and from $V - X - N^+_G(X)$ to X. By the definition of G_2, W_Q has no arc from $V - X$ to $N^+_G(X) \cup (X - Q)$. Then W_Q can only be a cycle alternating between Q and $N^+_G(X)$, thus W_Q, W are even cycles, W is in $\mathcal{H}$. □

Claim 4.5. *G_1 has an $\mathcal{H}$-factor K_1 with cardinality $|V_1| - fc^{\mathcal{H}}G[X]$.*

Proof. By induction, it is enough to prove that $\tau_{G_1}(Y) \geq |V_1| - fc^{\mathcal{H}}G[X]$ holds for all $Y \subseteq V_1$.

$\tau_{G_1}(Y) \geq \tau_{G_1}(Y \cup N^+_G(X))$, hence we suppose that $N^+_G(X) \subseteq Y \subseteq V_1$. Let $S := \{v \in N^+_G(X) :$ there is no arc uv with $u \in Y - N^+_G(X)\}$.

We have $N^+_{G_1}(X_Q \cap Y) = N^+_{G_1}(Y) \cup (N^+_G(X) - S)$, thus

$$|N^+_{G_1}(X_Q \cap Y)| \leq |N^+_{G_1}(Y)| + |N^+_G(X)| - |S|, \tag{9}$$

$$fc^{\mathcal{H}}G_1[Y] - |S| = fc^{\mathcal{H}}G_1[X_Q \cap Y]. \tag{10}$$

Let Y_G denote the set we get from Y after replacing the nodes of $Y \cap Q$ by the corresponding nodes in G. Since X is a tight cut in G,

$$|V| + |N^+_G(X)| - fc^{\mathcal{H}}G[X] \leq |V| + |N^+_G(X \cap Y_G)| - fc^{\mathcal{H}}G[X \cap Y_G]. \tag{11}$$

It is easy to see, that $fc^{\mathcal{H}}G[X] = |Q| = fc^{\mathcal{H}}G_1[X_Q]$, $N^+_G(X \cap Y_G) = N^+_{G_1}(X_Q \cap Y)$, and $fc^{\mathcal{H}}G[X \cap Y_G] = fc^{\mathcal{H}}G_1[X_Q \cap Y]$. Then by inequality (11) we get

$$|N^+_G(X)| - fc^{\mathcal{H}}G_1[X_Q] \leq |N^+_{G_1}(X_Q \cap Y)| - fc^{\mathcal{H}}G_1[X_Q \cap Y]. \tag{12}$$

By adding up (9), (10) and (12)

$$fc^{\mathcal{H}}G_1[Y] - fc^{\mathcal{H}}G_1[X_Q] \leq |N^+_{G_1}(Y)|. \tag{13}$$

Thus,

$$|V_1| - fc^{\mathcal{H}}G[X] = |V_1| - fc^{\mathcal{H}}G[X_Q] \leq |V_1| + |N^+_{G_1}(Y)| - fc^{\mathcal{H}}G_1[Y] = \tau_{G_1}(Y). \tag{14}$$

□

Claim 4.6. *G_2 has an $\mathcal{H}$-factor K_2 with cardinality $|V_2| - |Q|$.*

Proof. By induction, it is enough to prove that $\tau_{G_2}(Z) \geq |V_Q| - |X_Q|$ holds for all $Z \subseteq V_2$.

$\tau_{G_2}(Z) \geq \tau_{G_2}(Z \cup Q)$, hence we suppose that $Q \subseteq Z \subseteq V_2$. Let Z_G denote the set we get from Z after replacing the nodes of Q by the corresponding nodes in G.

We have $N_G^+(X \cup Z_G) = (N_G^+(X) - (Z \cap N_G^+(X))) \cup N_{G_2}^+(Z)$, thus

$$|N_G^+(X \cup Z_G)| = |N_G^+(X)| - |Z \cap N_G^+(X)| + |N_{G_2}^+(Z)|. \tag{15}$$

Since X is tight in G,

$$|V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X] \leq |V| + |N_G^+(X \cup Z_G)| - fc^{\mathcal{H}}G[X \cup Z_G]. \tag{16}$$

Now we prove inequality (17). Consider the $\mathcal{H}$-critical source components of $G_2[Z]$. These are all the nodes in $Z \cap N_G^+(X)$ as single node components and some other components disjoint from $N_G^+(X)$. The latter type components give $\mathcal{H}$-critical source components of $G[X \cup Z_G]$, too. This proves

$$fc^{\mathcal{H}}G_2[Z] - |Z \cap N_G^+(X)| \leq fc^{\mathcal{H}}G[X \cup Z_G]. \tag{17}$$

By adding up (15), (16) and (17)

$$fc^{\mathcal{H}}G_2[Z] - |Q| = fc^{\mathcal{H}}G_2[Z] - fc^{\mathcal{H}}G[X] \leq |N_{G_2}^+(Z)|. \tag{18}$$

Thus,

$$|V_2| - |Q| \leq |V_2| + |N_{G_2}^+(Z)| - fc^{\mathcal{H}}G_2[Z].$$

□

By Claims 4.4, 4.5 and 4.6, G has an $\mathcal{H}$-factor K of cardinality $|V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X]$. This completes the proof in CASE 4.

5 Structural description

Theorem 5.1 (Structure Theorem). *Let $G = (V, E)$ be an $\mathcal{H}$-symmetric graph. Let $D := \{v \in V :$ there exists a maximum $\mathcal{H}$-factor K such that $\delta_K(v) = 0\}$.*

1. *$\nu(G) = |V| + (|N_G^+(D)| - fc^{\mathcal{H}}G[D])$, and*
2. *the source components of $G[D]$ are $\mathcal{H}$-critical.*

Proof. Let X be a tight cut such that $|X|$ is minimum. We are going to prove that $X = D$. It follows from Claim 4.3 that each source component of $G[X]$ is $\mathcal{H}$-critical. First we prove that $D \subseteq X$. Take any node $v \in D$. Let K_v be an even factor of size $|K_v| = \tau_G = \tau_G(X)$, with $\delta_{K_v}(v) = 0$. For $K = K_v$, we must have equality in (2)-(4). From equality in (4) we get that $v \notin V - X$.

Now we prove $X \subseteq D$. Consider G_Q, G_1 and G_2 which were defined for any tight cut in the proof of Theorem 2.1. By Claims 4.4 and 4.6, the following claim finishes the proof of Theorem 5.1.

Claim 5.2. *For any $v \in X_Q$, there is an $\mathcal{H}$-factor K_1 with cardinality $|V_1| - fc^{\mathcal{H}}G[X]$ such that $\delta_{K_1}(v) = 0$.*

Proof. Let G_1' denote the $\mathcal{H}$-symmetric graph obtained from G_1 by deleting the arcs coming out of v. We have to prove that there is a $\mathcal{H}$-factor in G_1' of cardinality $|V_1| - fc^{\mathcal{H}}G[X]$.

We are going to prove, that $\tau_{G_1}(Y) \geq |V_1| - fc^{\mathcal{H}}G[X] + 1$ for any $Y \subseteq V_1 - v$. Then by Theorem 2.1 we will be done, since $\tau_{G_1'}(Y+v) \geq \tau_{G_1}(Y) - 1$ for any set $Y \subseteq V_1 - v$.

If $Y \subseteq V_1 - v$, then $\tau_{G_1}(Y) \geq \tau_{G_1}(Y \cup N_G^+(X))$, hence we suppose, that $N_G^+(X) \subseteq Y \subseteq V_1 - v$. Let $S := \{w \in N_G^+(X) :$ there is no arc uw with $u \in Y - N_G^+(X)\}$. We have $N_{G_1}^+(X_Q \cap Y) = N_{G_1}^+(Y) \cup (N_G^+(X) - S)$, thus

$$|N_{G_1}^+(X_Q \cap Y)| \leq |N_{G_1}^+(Y)| + |N_G^+(X)| - |S|, \tag{19}$$

$$fc^{\mathcal{H}}G_1[Y] - |S| = fc^{\mathcal{H}}G_1[X_Q \cap Y]. \tag{20}$$

Let Y_G denote the resulting set after replacing the nodes of $Y \cap Q$ by the corresponding source components of $G[X]$ in Y. Since X is a minimum tight cut in G,

$$|V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X] + 1 \leq |V| + |N_G^+(X \cap Y_G)| - fc^{\mathcal{H}}G[X \cap Y_G]. \tag{21}$$

It is easy to see, that $fc^{\mathcal{H}}G[X \cap Y_G] = fc^{\mathcal{H}}G_1[X_Q \cap Y]$, then by inequality (21):

$$|N_G^+(X)| - fc^{\mathcal{H}}G[X] + 1 \leq |N_{G_1}^+(X_Q \cap Y)| - fc^{\mathcal{H}}G_1[X_Q \cap Y]. \tag{22}$$

By adding up (19), (20) and (22) we get

$$fc^{\mathcal{H}}G_1[Y] - fc^{\mathcal{H}}G[X] + 1 \leq |N_{G_1}^+(Y)|,$$

thus,

$$|V_1| - fc^{\mathcal{H}}G[X] + 1 \leq |V_1| + |N_{G_1}^+(Y)| - fc^{\mathcal{H}}G_1[Y] = \tau_{G_1}(Y).$$

□

6 Matroidal description

In an undirected graph, the system of node-sets which can be covered by a matching gives the independent sets of the so-called matching matroid. The Tutte-matrix gives a linear representation of the matching matroid. As a generalization, we give a matroid corresponding to an $\mathcal{H}$-symmetric graph G, however we could not give a linear representation so far.

For an $\mathcal{H}$-factor M define $V_+(M) := \{v \in V : \delta_M(v) = 1\}$.

Theorem 6.1. *Let $G = (V, E)$ be an $\mathcal{H}$-symmetric graph. The following family is the family of independent sets of a matroid:*

$$\mathcal{I}(G, \mathcal{H}) := \{I \subseteq V : \text{ there is a maximum } \mathcal{H}\text{-factor } M \text{ such that } I \subseteq V_+(M)\}. \tag{23}$$

To show a version of the matroid exchange axiom, it suffices to prove the following lemma:

Lemma 6.2. *Suppose M_1 and M_2 are $\mathcal{H}$-factors with $|M_1| < |M_2|$. Then there is an $\mathcal{H}$-factor M_1' such that $V_+(M_1) \subset V_+(M_1')$ and $V_+(M_1') - V_+(M_1) \subseteq V_+(M_2)$.*

Proof. Consider the $\mathcal{H}$-symmetric graph G' we get by deleting all arcs $uv \in E(G)$ for $u \in V - (V_+(M_1) \cup V_+(M_2))$. It is clear that M_1 and M_2 are $\mathcal{H}$-factors in G'.

Let $k := |V_+(M_2) - V_+(M_1)| - 1$. We construct the $\mathcal{H}$-symmetric graph G'' from G' as follows. We add a set U of k new nodes, that is $V(G'') := V(G') \cup U$. We add $k \cdot (k+1)$ new arcs, each possible arc uv for $u \in V_+(M_2) - V_+(M_1)$ and $v \in U$. We are going to prove that there is an $\mathcal{H}$-factor M in G'' with $|M_1|+k+1 = |V_+(M_2) \cup V_+(M_1)|$ arcs. If there is such an $M \subseteq E(G'')$, then $M_1' := M \cap E(G)$ will do.

Suppose for a contradiction that $\nu^{\mathcal{H}}(G'') \leq |M_1| + k$. Then $\nu^{\mathcal{H}}(G'') = |M_1| + k$, since we can add to M_1 k disjoint arcs from $V_+(M_2) - V_+(M_1)$ to U. By Theorem 5.1 we get for $D'' = D(G'')$

$$|M_1| + k = |V(G'')| - fc^{\mathcal{H}}G''[D''] + |N^+_{G''}(D'')|. \tag{24}$$

Since there is no arc leaving any node in U we get $U \subseteq D''$, thus $N^+_{G''}(D'') = N^+_{G'}(D'' - U)$. For each node v in $V_+(M_2) - V_+(M_1)$ one can construct an $\mathcal{H}$-factor in G'' of $|M_1| + k$ arcs with no arc leaving v, thus $V_+(M_2) - V_+(M_1) \subseteq D''$. Then the source-components in $G''[D'']$ are disjoint from U, thus $fc^{\mathcal{H}}G''[D''] = fc^{\mathcal{H}}G'[D'' - U]$.

$$\begin{aligned}\tau^{\mathcal{H}}_{G'}(D'' - U) &= |V(G')| - fc^{\mathcal{H}}G'[D'' - U] + |N^+_{G'}(D'' - U)| = \\ &= |V(G'')| - k - fc^{\mathcal{H}}G''[D''] + |N^+_{G''}(D'')| = \nu^{\mathcal{H}}(G'') - k = |M_1| \end{aligned} \tag{25}$$

$\tau^{\mathcal{H}}_{G'}(X)$ is an upper bound for the cardinality of any $\mathcal{H}$-factor in G', then there cannot be any greater than $|M_1|$. This is in contradiction with the existence of M_2. ☐

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