Second-Order Resolvability, Intrinsic Randomness, and Fixed-Length Source Coding for Mixed Sources

Ryo Nomura, Member, IEEE, and Te Sun Han, Fellow, IEEE

Abstract

The second-order achievable rates in typical random number generation problems are considered. In these problems, several researchers have derived the first-order and the second-order achievability rates for general sources using the information-spectrum methods. Although these formulas are general, their computation are quite hard. Hence, an attempt to address explicit computation problems of achievable rates is meaningful. In particular, for i.i.d. sources, the second-order achievable rates have been determined simply by using the asymptotic normality. In this paper, we consider mixed sources of two i.i.d. sources. The mixed source is a typical case of nonergodic sources and whose self-information does not have the asymptotic normality. Nonetheless, we can explicitly compute the second-order achievable rates for these sources on the basis of two-peak asymptotic normality.

Index Terms

Second-Order Achievability, Random Number Generation, Source Coding, Mixed Source, Asymptotic Normality

R. Nomura is with the School of Network and Information, Senshu University, Kanagawa, Japan, e-mail: nomu@isc.senshu-u.ac.jp

T. S. Han is with the National Institute of Information and Communications Technology (NICT), Tokyo, Japan, email: han@aoni.waseda.jp

The first author with this work was supported in part by JSPS Grant-in-Aid for Young Scientists (B) No. 23760346.
I. INTRODUCTION

The problem of random number generation is one of the main topics in information theory [1]–[3]. There are several problem settings in random number generations. In particular, the resolvability problem and the intrinsic randomness problem are representative of them. The resolvability problem is formulated as follows [5]. We first use the term of “general source” to denote a sequence $X = \{X^n\}_{n=1}^{\infty}$ of random variables $X^n$ indexed by $n$ (taking values in countable sets), typically, $n$-dimensional random variables. Given an arbitrary general source $X = \{X^n\}_{n=1}^{\infty}$ (called the target random number), we generate or approximate it by using a discrete uniform random number whose size is requested to be as small as possible. One of the main objectives in this problem is to construct an efficient algorithm that transforms the discrete uniform random number to the specified target source. Recently, Han and Verdú [1], and Steinberg and Verdú [6] have determined the infimum of achievable uniform random number rates by using the information-spectrum methods. On the other hand, the intrinsic randomness problem is formulated as follows [2]. Given an arbitrary general source $X = \{X^n\}_{n=1}^{\infty}$ (called the coin source), we try to generate or approximate, by using $X = \{X^n\}_{n=1}^{\infty}$, a uniform random number with as large rates as possible. Vembu and Verdú [2], and Han [5] have determined the supremum of achievable uniform random number generation rates, again by invoking the information-spectrum methods. Since the class of general sources is quite large and it includes all nonstationary and/or nonergodic sources, their results are very basic and quite fundamental.

Furthermore, it turned out that these random number generation problems have close bearing with the fixed-length source coding problem (cf. Han and Verdú [1]). All the formulas established here may be said to be ones of the first-order.

On the other hand, the finer evaluation of the achievable rates, called the second-order achievable rates, have been investigated in several contexts. In the variable-length source coding problem, Kontoyiannis [7] has established the second-order source coding theorem. In the channel coding problem, Strassen (see, Csiszár and Körner [8]) and Hayashi [9] have determined the second-order capacity rate. Among others, Hayashi [10] has shown the second-order achievability theorems for the intrinsic randomness problem and the fixed-length source coding problem for general sources. In particular, for i.i.d. sources he has calculated the second-order optimal achievable rates by using the asymptotic normality in both problems.

In this paper, we address the computation problem concerning the second-order formulas for
resolvability problem, intrinsic randomness problem and fixed-length source coding problem, where the resolvability problem was first studied by Nomura and Matsushima [11]. In the resolvability problem or the intrinsic randomness problem the degree of approximation is measured by means of variational distance.

As we have mentioned in the above, the analysis based upon the asymptotic normality is effective in deriving the second-order achievable rates. However, it was only applied to the class which has a simple probabilistic structure, such as i.i.d. sources, Markovian sources or stationary DMC. On the other hand, we try to derive the second-order optimal achievable rates for mixed sources, which is a wider class of sources than the previous ones. Recall that mixed sources are typical cases of nonergodic sources. Nonetheless, we show that we can use still the two-peak asymptotic normality to exploit the second-order achievable rates for mixed sources.

This paper is organized as follows. In section II, we review the previous results on the first-order asymptotics for general sources. In section III, we derive the second-order asymptotics for general sources, analogously to section II. In Section IV, we define the mixed sources and state the lemmas which play the key role in the subsequent analysis. In Sections V, VI and VII, we establish the second-order achievability by using the two-peak asymptotic normality, for the resolvability problem, the intrinsic randomness problem and the fixed-length source coding problem, respectively. Finally, we conclude our results in Section VIII.

II. FIRST-ORDER ASYMPTOTICS

In this section we review the previous results on the first-order asymptotics of random number generation and fixed-length source coding.

To this end, we first give the necessary notations and definitions. In the sequel, let $Y = \{Y^n\}_{n=1}^{\infty}$ be a general source with values in countable sets $Y^n$. Let $Z$ be a countable set and let $Z, \overline{Z}$ be random variables with values in $Z$. Denote by $d(Z, \overline{Z})$ the variational distance

$$d(Z, \overline{Z}) \equiv \sum_{z \in Z} |P_Z(z) - P_{\overline{Z}}(z)|,$$

where $P_X(\cdot)$ denotes the probability distribution of random variable $X$. Moreover, let $U_M = \{1, 2, \cdots, M\}$ and let $U_M$ denote the random variable uniformly distributed on $U_M$. 

A. First-Order Resolvability

**Definition 2.1:** Rate $R$ is said to be $\delta$-achievable if there exists a mapping $\psi_n : U_M^n \to Y^n$ such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log M_n \leq R \quad \text{and} \quad \limsup_{n \to \infty} d(Y^n, \psi_n(U_M^n)) \leq \delta.
\]

**Definition 2.2 ($\delta$-resolvability):**
\[
S_r(\delta|Y) = \inf \{ R \mid R \text{ is } \delta\text{-achievable} \}.
\]

Then, we have

**Theorem 2.1 (Steinberg and Verdú [6]):**
\[
S_r(\delta|Y) = \inf \left\{ R \left| F(R) \leq \frac{\delta}{2} \right. \right\} \quad (0 \leq \forall \delta < 2),
\]
where
\[
F(R) = \limsup_{n \to \infty} \Pr \left\{ \frac{1}{n} \log \frac{1}{P_{Y^n}(Y^n)} \geq R \right\}.
\] (1)

B. First-Order Intrinsic Randomness

**Definition 2.3:** Rate $R$ is said to be $\delta$-achievable if there exists a mapping $\varphi_n : Y^n \to U_M^n$ such that
\[
\liminf_{n \to \infty} \frac{1}{n} \log M_n \geq R \quad \text{and} \quad \limsup_{n \to \infty} d(U_M^n, \varphi_n(Y^n)) \leq \delta.
\]

**Definition 2.4 ($\delta$-intrinsic randomness):**
\[
S_\iota(\delta|Y) = \sup \{ R \mid R \text{ is } \delta\text{-achievable} \}.
\]

Then, we have

**Theorem 2.2 (Han [5]):**
\[
S_\iota(\delta|Y) = \sup \left\{ R \left| G(R) \leq \frac{\delta}{2} \right. \right\} \quad (0 \leq \forall \delta < 2),
\]
where
\[
G(R) = \limsup_{n \to \infty} \Pr \left\{ \frac{1}{n} \log \frac{1}{P_{Y^n}(Y^n)} \leq R \right\}.
\]
C. First-Order Fixed-length Source Coding

Let \( \varphi_n : \mathcal{Y}^n \to \mathcal{U}_{M_n} \), \( \psi_n : \mathcal{U}_{M_n} \to \mathcal{Y}^n \) be an encoder and a decoder, respectively, for source \( \mathcal{Y} = \{Y^n\}_{n=1}^{\infty} \). The decoding error probability \( \epsilon_n \) is given by \( \epsilon_n \equiv \Pr\{Y^n \neq \psi_n(\varphi_n(Y^n))\} \). Such a code is denoted by \((n, M_n, \epsilon_n)\).

**Definition 2.5:** Rate \( R \) is said to be \( \epsilon \)-achievable if there exists a code \((n, M_n, \epsilon_n)\) such that
\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{n} \log M_n \leq R.
\]

**Definition 2.6 (\( \epsilon \)-fixed-length source coding rate):**
\[
L_f(\epsilon|Y) = \inf \{R|R \text{ is } \epsilon \text{-achievable}\}.
\]

Then, we have

**Theorem 2.3 (Steinberg and Verdú [6]):**
\[
L_f(\epsilon|Y) = \inf \{R|F(R) \leq \epsilon\} \quad (0 \leq \forall \epsilon < 1),
\]
where \( F(R) \) is defined as in (1).

An immediate consequence of Theorem 2.1 and Theorem 2.3 is the following theorem, which reveals a deep relationship between resolvability and fixed-length source coding from the viewpoint of random number generation, that is,

**Theorem 2.4:**
\[
L_f(\epsilon|Y) = \mathcal{S}_r(2\epsilon|Y) \quad (0 \leq \forall \epsilon < 1).
\]

As for the operational meaning of this equivalence, see Han [5].

III. Second-Order Asymptotics

Having reviewed the results on the first-order asymptotics, we now focus on the second-order asymptotics, which will turn out to be in nice correspondence with the first-order asymptotics.

A. Second-Order Resolvability

**Definition 3.1:** Rate \( R \) is said to be \((a, \delta)\)-achievable if there exists a mapping \( \psi_n : \mathcal{U}_{M_n} \to \mathcal{Y}^n \) such that
\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{M_n}{e^{na}} \leq R \quad \text{and} \quad \limsup_{n \to \infty} d(Y^n, \psi_n(\mathcal{U}_{M_n})) \leq \delta.
\]

**Definition 3.2 ((a, \delta)-resolvability):**
\[
\mathcal{S}_r(a, \delta|Y) = \inf \{R|R \text{ is } (a, \delta) \text{-achievable}\}.
\]
Then, we have

**Theorem 3.1:**

$$S_r(a, \delta|Y) = \inf \left\{ R \middle| F_a(R) \leq \frac{\delta}{2} \right\} \quad (0 \leq \forall \delta < 2),$$

where

$$F_a(R) = \limsup_{n \to \infty} \Pr \left\{ \frac{1}{n} \log \frac{1}{P_{Y^n}(Y^n)} \geq a + \frac{R}{\sqrt{n}} \right\}.$$  \hspace{1cm} (2)

**Remark 3.1:** This theorem is an immediate consequence of Theorem 3.3 and Theorem 3.4 below.

**B. Second-Order Intrinsic Randomness**

**Definition 3.3:** Rate $R$ is said to be $(a, \delta)$-achievable if there exists a mapping $\varphi_n : Y^n \to U_{M_n}$ such that

$$\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log M_n e^{na} \geq R \quad \text{and} \quad \limsup_{n \to \infty} d(U_{M_n}, \varphi_n(Y^n)) \leq \delta.$$  

**Definition 3.4** ($(a, \delta)$-intrinsic randomness):

$$S_i(a, \delta|Y) = \sup \{ R | R \text{ is } (a, \delta)-\text{achievable} \}.$$  

Then, we have

**Theorem 3.2 (Hayashi [10]):**

$$S_i(a, \delta|Y) = \sup \left\{ R \middle| G_a(R) \leq \frac{\delta}{2} \right\} \quad (0 \leq \forall \delta < 2),$$

where

$$G_a(R) = \limsup_{n \to \infty} \Pr \left\{ \frac{1}{n} \log \frac{1}{P_{Y^n}(Y^n)} \leq a + \frac{R}{\sqrt{n}} \right\}.$$  

**C. Second-Order Fixed-length Source Coding**

**Definition 3.5:** Rate $R$ is said to be $(a, \varepsilon)$-achievable if there exists a code $(n, M_n, \varepsilon_n)$ such that

$$\limsup_{n \to \infty} \varepsilon_n \leq \varepsilon \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{M_n}{e^{na}} \leq R.$$  

**Definition 3.6** ($(a, \varepsilon)$-fixed-length source coding rate):

$$L_f(a, \varepsilon|Y) = \inf \{ R | R \text{ is } (a, \varepsilon)-\text{achievable} \}.$$  

Then, we have

**Theorem 3.3 (Hayashi [10]):**

$$L_f(a, \varepsilon|Y) = \inf \{ R | F_a(R) \leq \varepsilon \} \quad (0 \leq \forall \varepsilon < 1).$$
where $F_a(R)$ is defined as in [2].

On the other hand, we have the following theorem, which is a finer refinement of Theorem [2,4] to the second-order asymptotics:

**Theorem 3.4:**

$$L_f(a, \varepsilon|Y) = S_r(a, 2\varepsilon|Y) \quad (0 \leq \forall \varepsilon < 1).$$

**Proof:** It suffices to show that if $R > L_f(a, \varepsilon|Y)$ then $R + \gamma > S_r(a, 2\varepsilon|Y)$ for any small $\gamma > 0$, and vice versa with $R$ and $R + \gamma$ swapped. To this end, it is sufficient to literally follow the arguments described in Han [5, p.163] with $\gamma$ replaced by $\frac{\gamma}{\sqrt{n}}$.

**IV. Second-order asymptotics for mixed sources**

So far we have demonstrated the general formulas for typical first-order and second-order asymptotic problems (resolvability, intrinsic randomness and fixed-length source coding rate) of random number generation with any general source $Y = \{Y^n\}_{n=1}^{\infty}$.

However, computation of these general formulas is quite hard in general or even formidable. Therefore, in this section we consider to introduce a class of tractable sources $Y$ for which the general formulas are computable but still of independent interest. One of such source classes would be the case where $Y$ is a mixed source of two i.i.d. sources $Y_1 = \{Y_1^n\}_{n=1}^{\infty}$ and $Y_2 = \{Y_2^n\}_{n=1}^{\infty}$. The computation problem of the first-order asymptotics for such mixed sources has already been solved, e.g., see Han [5], so in the sequel we now focus on the computation problem of the second-order asymptotics for mixed sources. As a result, it will turn out that we can explicitly compute the asymptotic formulas by virtue of *information-spectrum methods*.

Let begin the formal definition of mixed sources. Let $Y = \{0, 1, \cdots\}$ (countable) be a discrete source alphabet and $y = y_1 y_2 \cdots y_n \in Y^n$ denote a sequence emitted from the source of length $n$. Let $Y^n$ denote a random variable: a source sequence of length $n$.

We consider a mixed source consists of two stationary memoryless sources $Y_i = \{Y_i^n\}_{n=1}^{\infty}$, where $i = \{1, 2\}$. Then, the mixed source $Y = \{Y^n\}_{n=1}^{\infty}$ is defined by

$$P_{Y^n}(y) = w(1)P_{Y_1^n}(y) + w(2)P_{Y_2^n}(y),$$

where $w(i)$ are constants satisfying $w(1) + w(2) = 1$ and $w(i) > 0 \quad (i = 1, 2)$. Since two i.i.d. sources $Y_i \quad (i = 1, 2)$ are completely specified by giving just the first component $Y_i \quad (i = 1, 2)$, we may write simply as $Y_i = \{Y_i\} \quad (i = 1, 2)$ and define the variances:
**Definition 4.1 (variance):**

\[ \sigma_i^2 = E \left( \log \frac{1}{P_{Y_i}(Y_i)} - H(Y_i) \right)^2 \quad (i = 1, 2), \]

where we assume that these variances are finite, and

\[ H(Y_i) = \sum_{y \in Y} P_{Y_i}(y) \log \frac{1}{P_{Y_i}(y)}. \]

Since we consider the case where \( Y_i = \{Y_i\} \) \((i = 1, 2)\) is an i.i.d. source, the following asymptotic normality holds for each component i.i.d. source:

\[ \lim_{n \to \infty} \Pr \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \log \frac{1}{P_{Y_i}(Y_i)} - nH(Y_i) \right) \leq U \right\} = \int_{-\infty}^{U} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz, \]

where \( \sigma_i^2 \) denotes the variances defined in Definition 4.1 \((i = 1, 2)\).

The following lemma plays the key role in the sequel including the proof of Theorem 5.1, Theorem 6.1 and Theorem 7.1.

**Lemma 4.1 (Han [5]):** Let \( \{z_n\}_{n=1}^{\infty} \) be any real-valued sequence. Then for the mixed source \( Y \) defined in Section 2, it holds that, for \( i = 1, 2 \),

1) \[ \Pr \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \log \frac{1}{P_{Y_i}(Y_i)} - nH(Y_i) \right) \geq z_n \right\} \geq \Pr \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \log \frac{1}{P_{Y_i}(Y_i)} - nH(Y_i) \right) \geq z_n + \gamma_n \right\} - e^{-\sqrt{n\gamma_n}}, \]

2) \[ \Pr \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \log \frac{1}{P_{Y_i}(Y_i)} - nH(Y_i) \right) \geq z_n \right\} \leq \Pr \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \log \frac{1}{P_{Y_i}(Y_i)} - nH(Y_i) \right) \geq z_n - \gamma_n \right\}, \]

where \( \gamma_n > 0 \) satisfies \( \gamma_1 > \gamma_2 > \cdots > 0, \gamma_n \to 0, \sqrt{n\gamma_n} \to \infty. \)

**Proof:** See Appendix A. ■

**V. \((a, \delta)\)-Resolvability**

In this section we shall establish \( S_r(a, \delta | Y) \) for mixed sources. At first we introduce two fundamental lemmas as in Han [5]: Lemma 5.1 and Lemma 5.2.

Before describing lemmas, we need to define two sets. Let \( X = \{X^n\}_{n=1}^{\infty} \) and \( Y = \{Y^n\}_{n=1}^{\infty} \) be arbitrary general sources, then, given a sequence \( \{z_n\}_{n=1}^{\infty} \), define \( S_n(z_n) \) and \( T_n(z_n) \):

\[ S_n(z_n) = \left\{ x \in X^n \left| \frac{1}{\sqrt{n}} \log \frac{1}{P_{X^n}(x)} \geq z_n \right\} \right., \]

\[ T_n(z_n) = \left\{ y \in Y^n \left| \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(y)} \geq z_n \right\} \right.. \]
\[ T_n(z_n) = \left\{ y \in \mathcal{Y}^n \mid \frac{1}{\sqrt{n}} \log \frac{1}{P_{T_n}(y)} \leq z_n \right\}. \]

**Lemma 5.1:** Let \( X = \{X^n\}_{n=1}^{\infty} \) and \( Y = \{Y^n\}_{n=1}^{\infty} \) be arbitrary general sources, where \( X^n \) and \( Y^n \) are random variables taking values in \( \mathcal{X}^n \) and \( \mathcal{Y}^n \), respectively. Then, for an arbitrary sequence \( \{z_n\}_{n=1}^{\infty} \) and \( \gamma > 0 \), there exists a mapping \( \phi_n : \mathcal{X}^n \to \mathcal{Y}^n \) such that
\[
d(\phi_n(X^n), Y^n) \leq 2 \max \{ \Pr\{X^n \notin S_n(z_n + \gamma)\}, \Pr\{Y^n \notin T_n(z_n)\}\} + 2e^{-\sqrt{n}\gamma}.
\]

**Lemma 5.2:** Let \( X = \{X^n\}_{n=1}^{\infty} \) and \( Y = \{Y^n\}_{n=1}^{\infty} \) be arbitrary general sources, where \( X^n \) and \( Y^n \) are random variables taking values in \( \mathcal{X}^n \) and \( \mathcal{Y}^n \), respectively. Then, for an arbitrary sequence \( \{z_n\}_{n=1}^{\infty} \), \( \gamma > 0 \) and any mapping \( \phi_n : \mathcal{X}^n \to \mathcal{Y}^n \) it holds that
\[
d(\phi_n(X^n), Y^n) \geq 2 \Pr\{Y^n \notin T_n(z_n + \gamma)\} - 2 \Pr\{X^n \in S_n(z_n)\} - 2e^{-\sqrt{n}\gamma}, \tag{4}
\]

**Remark 5.1:** Also, (1) can be written as
\[
d(\phi_n(X^n), Y^n) \geq 2 \Pr\{X^n \notin S_n(z_n)\} - 2 \Pr\{Y^n \in T_n(z_n + \gamma)\} - 2e^{-\sqrt{n}\gamma}. \tag{5}
\]

The above lemmas are useful for the random number generation problem to approximate a probability distribution \( Y = \{Y^n\}_{n=1}^{\infty} \) by using an another probability distribution \( X = \{X^n\}_{n=1}^{\infty} \) [5]. Clearly, this problem includes the resolvability problem as a special case: the resolvability problem is the case of \( X^n = U_{M_n} \). So, in this case the condition in the above lemmas leads to
\[
\Pr\{X^n \notin S_n(z_n)\} = \begin{cases} 0 & z_n \leq \frac{1}{\sqrt{n}} \log M_n \\ 1 & z_n > \frac{1}{\sqrt{n}} \log M_n. \end{cases} \tag{6}
\]

Notice that the above lemmas are valid for general sources \( X \) and \( Y \).

In the sequel, we consider the case that \( 0 \leq \delta < 2 \) and \( w(1) \neq \frac{\delta}{2} \) hold (cf. Remark 5.2 for the case of \( w(1) = \frac{\delta}{2} \)). Then, given \( 0 \leq \delta < 2 \) we divide the problem into three cases. Here, without loss of generality, we assume that \( H(Y_1) \geq H(Y_2) \) holds:

I \hspace{1cm} \( H(Y_1) = H(Y_2) \) holds.

II \hspace{1cm} \( H(Y_1) > H(Y_2) \) and \( w(1) > \frac{\delta}{2} \) hold.

III \hspace{1cm} \( H(Y_1) > H(Y_2) \) and \( w(1) < \frac{\delta}{2} \) hold.

In Case I, we shall establish \( S_r(H(Y_1), \delta|Y) \) (Obviously, this is equal to \( S_r(H(Y_2), \delta|Y) \)). In Case II and Case III we shall show \( S_r(H(Y_1), \delta|Y) \) and \( S_r(H(Y_2), \delta|Y) \), respectively. Now we have one of the main results:
**Theorem 5.1:** Given $0 \leq \delta < 2$, the following holds.

**Case I:**

$$S_r(H(Y_1), \delta | Y) = T_1,$$

where $T_1$ is specified by

$$\frac{\delta}{2} = \sum_{i=1}^{2} w(i) \int_{T_1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.$$  \hfill (8)

**Case II:**

$$S_r(H(Y_1), \delta | Y) = T_2,$$

where $T_2$ is specified by

$$\frac{\delta}{2} = w(1) \int_{T_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.$$  \hfill (9)

**Case III:**

$$S_r(H(Y_2), \delta | Y) = T_3,$$

where $T_3$ is specified by

$$\frac{\delta}{2} = w(1) + w(2) \int_{T_3}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.$$  \hfill (10)

**Remark 5.2:** It is easy to check that $T_2 = -\infty$, $T_3 = +\infty$ for $w(1) = \frac{\delta}{2}$. Also, it will turn out from the way of proving the above theorem that the second-order asymptotics gets trivial if $a \neq H(Y_1)$ and $a \neq H(Y_2)$, because this case necessarily implies that $\delta = 0$ or $\delta = 2w(1)$ or $\delta = 2$, depending on the value of $a$; then, we can formally set as $S_r(a, \delta | Y) = -\infty$.

**Proof of Case I:**

A simplest way to prove Theorem 5.1 is to first apply Theorem 3.1 to the present case of mixed sources and to proceed to the computation of necessary quantities. Here, however, more basically we start along with Lemma 5.1 and Lemma 5.2 in order to reveal the fundamental logic underlying the whole process of random number generation. Actually, the computations needed in the first way of proof are contained in those needed in the second way of proof; more exactly, both computations are substantially the same.

We show the proof of Case I by using Lemma 5.1 and Lemma 5.2. The proof consists of two parts.

1) **Direct Part:**
Set $M_n = e^{nH(Y_1)+T_1\sqrt{n}+\gamma}$, where $\gamma > 0$ is an arbitrarily small number. Then, trivially it holds that

$$\limsup_{n \to \infty} \frac{\log M_n - nH(Y_1)}{\sqrt{n}} \leq T_1.$$  

Thus, it is enough to show that there exists a mapping $\phi_n$ such that

$$\limsup_{n \to \infty} d(\phi_n(U_{M_n}), Y^n) \leq \delta.$$  

On the other hand, set $z_n = \frac{nH(Y_1)+T_1\sqrt{n}}{\sqrt{n}} - \gamma$, then $z_n + \gamma \leq \frac{\log M_n}{\sqrt{n}}$ holds. Thus, from Lemma 5.1 and (6) there exists a mapping $\phi_n$ such that

$$\frac{1}{2} d(\phi_n(U_{M_n}), Y^n) \leq \Pr \left\{ \frac{nH(Y_1)+T_1\sqrt{n}}{\sqrt{n}} - \gamma < \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \right\} + e^{-\sqrt{n}\gamma}.$$  

Moreover, from Lemma 4.1 there exists a mapping $\phi_n$ such that

$$\limsup_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n) \leq \limsup_{n \to \infty} \Pr \left\{ \frac{nH(Y_1)+T_1\sqrt{n}}{\sqrt{n}} - \gamma < \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \right\}$$

$$\leq \limsup_{n \to \infty} \sum_{i=1}^{2} \Pr \left\{ \frac{nH(Y_i)+T_1\sqrt{n}}{\sqrt{n}} - \gamma - \gamma_n < \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \right\} w(i)$$

$$\leq \sum_{i=1}^{2} \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n}(Y^n_i)-nH(Y_1)}{\sqrt{n}} > T_1 - 2\gamma \right\}$$

$$\leq \sum_{i=1}^{2} \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i)-nH(Y_1)}{\sqrt{n}\sigma_i} > \frac{T_1 - 2\gamma}{\sigma_i} \right\} w(i),$$

since $\gamma_n > 0$ is specified in Lemma 4.1 and so $\gamma_n < \gamma$ holds for sufficiently large $n$. Then, noting that $H(Y_1) = H(Y_2)$ holds, we have from the asymptotic normality (by virtue of the central limit theorem)

$$\limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i)-nH(Y_1)}{\sqrt{n}\sigma_i} > \frac{T_1 - 2\gamma}{\sigma_i} \right\}$$

$$= \int_{\frac{T_1 - 2\gamma}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz$$

$$= \int_{\frac{T_1 - 2\gamma}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz + \int_{\frac{T_1 - 2\gamma}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.$$
Here, by the continuity of the normal distribution,
\[ \int_{\frac{z_1}{\sigma_1}}^{\frac{z_2}{\sigma_1}} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] \, dz \to 0 \]
as \( \gamma \to 0 \). Thus, noting that \( \gamma > 0 \) is an arbitrarily small, we have
\[ \limsup_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}),Y^n) \leq \sum_{i=1}^{2} w(i) \int_{\frac{z_i}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] \, dz = \frac{\delta}{2}. \]

Therefore, Direct Part has been proved.

2) Converse Part:

We consider a constant \( T'_1 < T_1 \). Assuming that \( T'_1 \) is \((H(Y_1),\delta)\)-achievable, we shall show a contradiction. Since we assume that \( T'_1 \) is \((H(Y_1),\delta)\)-achievable, there exists a mapping \( \phi_n \) such that
\[ \limsup_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}),Y^n) \leq \delta, \]
\[ \limsup_{n \to \infty} \frac{\log M_n - nH(Y_1)}{\sqrt{n}} \leq T'_1, \]
which means that there exists a constant \( \gamma > 0 \) satisfying
\[ \frac{\log M_n - nH(Y_1)}{\sqrt{n}} < T'_1 + \gamma < T_1, \]
for sufficiently large \( n \).

On the other hand, set \( z_n = \frac{nH(Y_1) + T'_1 \sqrt{n}}{\sqrt{n}} + \gamma \). Then \( z_n > \frac{\log M_n}{\sqrt{n}} \) holds. Thus, from Lemma \[5.2\] and \[6\], for any mapping \( \phi_n \) it holds that
\[ \frac{1}{2} d(\phi_n(U_{M_n}),Y^n) \geq \Pr \left\{ \frac{nH(Y_1) + T'_1 \sqrt{n}}{\sqrt{n}} + 2\gamma < \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \right\} - e^{-\sqrt{n}\gamma}. \]
Thus, from Lemma 4.1, for any mapping \( \phi_n \) we have

\[
\liminf_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n) \\
\geq \liminf_{n \to \infty} \Pr \left\{ \frac{nH(Y_1) + T'_1 \sqrt{n}}{\sqrt{n}} + 2\gamma < \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \right\} \\
= \liminf_{n \to \infty} \sum_{i=1}^{2} \Pr \left\{ \frac{nH(Y_1) + T'_1 \sqrt{n}}{\sqrt{n}} + 2\gamma < \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \right\} w(i) \\
\geq \sum_{i=1}^{2} \liminf_{n \to \infty} \Pr \left\{ \frac{-nH(Y^n_i) - nH(Y_1)}{\sqrt{n}} > T'_1 + 2\gamma + \gamma_n \right\} w(i) \\
= \sum_{i=1}^{2} \liminf_{n \to \infty} \Pr \left\{ \frac{-nH(Y^n_i) - nH(Y_1)}{\sqrt{n}} > T'_1 + 3\gamma \right\} w(i),
\]

(10)
since \( \gamma > \gamma_n \) holds for sufficiently large \( n \). Then, by virtue of the asymptotic normality, for \( i = 1, 2 \) it holds that

\[
\liminf_{n \to \infty} \Pr \left\{ \frac{-nH(Y^n_i) - nH(Y_1)}{\sqrt{n}\sigma_i} > \frac{T'_1 + 3\gamma}{\sigma_i} \right\} \\
= \int_{\frac{T'_1 + 3\gamma}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz \\
> \int_{\frac{T_1}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz,
\]
since we can let \( T'_1 + 3\gamma < T_1 \) hold by letting \( \gamma \to 0 \), and by substituting the above inequality into (10) for any mapping \( \phi_n \), we have

\[
\liminf_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n) \\
> \sum_{i=1}^{2} w(i) \int_{\frac{T_1}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz = \frac{\delta}{2}.
\]

This is a contradiction. Therefore, the converse part has been proved.

\[\blacksquare\]

**Proofs of Case II and Case III:**

The proofs of Case II and Case III are similar to the proof of Case I. Here, we show only differences in the proofs.

1) **Direct Part:**
a) Similarly to Case I, we have the proof of Case II as follows.

\[
\limsup_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n) \\
\leq \sum_{i=1}^{2} \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i) - n H(Y_1)}{\sqrt{n}} > T_2 - 2 \gamma \right\} w(i) \\
= \sum_{i=1}^{2} \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i) - n H(Y_1)}{\sqrt{n} \sigma_i} > \frac{T_2 - 2 \gamma}{\sigma_i} \right\} w(i) \\
= \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_1}(Y^n_1) - n H(Y_1)}{\sqrt{n} \sigma_1} > \frac{T_2 - 2 \gamma}{\sigma_1} \right\} w(1) \\
+ \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_2}(Y^n_2) - n H(Y_1)}{\sqrt{n} \sigma_2} > \frac{T_2 - 2 \gamma}{\sigma_2} \right\} w(2) \\
= \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_1}(Y^n_1) - n H(Y_1)}{\sqrt{n} \sigma_1} > \frac{T_2 - 2 \gamma}{\sigma_1} \right\} w(1) \\
+ \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_2}(Y^n_2) - n H(Y_1)}{\sqrt{n} \sigma_2} > \frac{n (H(Y_1) - H(Y_2)) + T_2 - 2 \gamma}{\sqrt{n} \sigma_2} \right\} w(2).
\]

Here, note that \( H(Y_1) > H(Y_2) \) holds. This means that for any constant \( W_1 > 0 \)

\[
\frac{n (H(Y_1) - H(Y_2)) + T_2 - 2 \gamma}{\sqrt{n} \sigma_2} > W_1,
\]

holds for sufficiently large \( n \). Thus, taking account of \( H(Y_1) > H(Y_2) \), we have

\[
\limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_2}(Y^n_2) - n H(Y_2)}{\sqrt{n} \sigma_2} > \frac{n (H(Y_1) - H(Y_2)) + T_2 - 2 \gamma}{\sqrt{n} \sigma_2} \right\} 
\leq \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_2}(Y^n_2) - n H(Y_2)}{\sqrt{n} \sigma_2} > W_1 \right\} 
= \int_{W_1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{y^2}{2} \right] dy.
\]

Since \( W_1 > 0 \) can be arbitrarily large, we have

\[
\lim_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_2}(Y^n_2) - n H(Y_2)}{\sqrt{n} \sigma_2} > \frac{n (H(Y_1) - H(Y_2)) + T_2 - 2 \gamma}{\sqrt{n} \sigma_2} \right\} = 0.
\]

Thus, from the asymptotic normality it follows that

\[
\sum_{i=1}^{2} \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_1}(Y^n_1) - n H(Y_1)}{\sqrt{n}} > T_2 - 2 \gamma \right\} w(i) \\
= \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n_1}(Y^n_1) - n H(Y_1)}{\sqrt{n} \sigma_1} > \frac{T_2 - 2 \gamma}{\sigma_1} \right\} w(1) \\
= w(1) \int_{\frac{T_2 - 2 \gamma}{\sigma_1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz \\
\rightarrow w(1) \int_{\frac{T_2 - 2 \gamma}{\sigma_1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz = \frac{\delta}{2},
\]

where \( \delta \) is a positive constant.
by letting $\gamma \to 0$. Thus, we have proved the direct part of Case II.

b) In Case III; we have

$$\limsup_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n)$$

$$\leq \sum_{i=1}^{2} \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_i^n}(Y_i^n) - nH(Y_2)}{\sqrt{n}} > T_3 - 2\gamma \right\} w(i)$$

$$\leq w(1) + \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_2^n}(Y_2^n) - nH(Y_2)}{\sqrt{n}\sigma_2} > \frac{T_3 - 2\gamma}{\sigma_2} \right\} w(2)$$

$$\to w(1) + w(2) \int_{\frac{T_3}{\sigma_2}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz = \frac{\delta}{2}$$

by letting $\gamma \to 0$. Consequently, similarly to the proof of Case I or Case II, we can prove the direct part of Case III.

2) Converse Part:

a) In Case II; we consider a constant $T_2' < T_2$ and we assume that $T_2'$ is $(H(Y_1), \delta)$-achievable. Notice that there exists a constant $\gamma > 0$ satisfying $T_2' + \gamma < T_2$. Then, similarly to the proof of Case I, we have

$$\liminf_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n)$$

$$\geq \sum_{i=1}^{2} \liminf_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_i^n}(Y_i^n) - nH(Y_1)}{\sqrt{n}} > T_2' + 2\gamma + \gamma_n \right\} w(i)$$

$$\geq \liminf_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_1^n}(Y_1^n) - nH(Y_1)}{\sqrt{n}} > T_2' + 3\gamma \right\} w(1)$$

$$= w(1) \int_{\frac{T_2' + 3\gamma}{\sigma_1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz$$

$$> w(1) \int_{\frac{T_2}{\sigma_1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz = \frac{\delta}{2}$$

because we can let $T_2' + 3\gamma < T_2$ by letting $\gamma \to 0$. Hence, we have shown the converse part by using the similar argument to Case I.

b) Similarly, in Case III; we consider a constant $T_3' < T_3$ and we assume that $T_3'$ is $(H(Y_2), \delta)$-achievable. Notice that there exists a constant $\gamma > 0$ satisfying $T_3' + \gamma < T_3$. Then, we have

$$\liminf_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n)$$
Then, the first term of the right-hand side of the above inequality can be determined as follows. Notice that $H(Y_1) > H(Y_2)$ holds. This means that for any constant $W_1 > 0$

$$\frac{n(H(Y_2) - H(Y_1))}{\sqrt{n\sigma_1}} + \frac{T_3 + 3\gamma}{\sigma_1} < -W_1,$$

holds for sufficiently large $n$. Thus,

$$\lim_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_i^n}(Y_i^n) - nH(Y_i)}{\sqrt{n\sigma_1}} > \frac{n(H(Y_2) - H(Y_1))}{\sqrt{n\sigma_1}} + \frac{T_3' + 3\gamma}{\sigma_1} \right\}$$

$$\geq \lim_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_i^n}(Y_i^n) - nH(Y_i)}{\sqrt{n\sigma_1}} > -W_1 \right\}$$

$$= \int_{-W_1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{y^2}{2} \right] dy,$$

holds. Since $W_1 > 0$ can be arbitrarily large, we have

$$\lim_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_i^n}(Y_i^n) - nH(Y_i)}{\sqrt{n\sigma_1}} > \frac{n(H(Y_2) - H(Y_1))}{\sqrt{n\sigma_1}} + \frac{T_3' + 3\gamma}{\sigma_1} \right\} = 1.$$

Substituting the above equality into (11), by virtue of the asymptotic normality, it holds that

$$\lim_{n \to \infty} \frac{1}{2} d(\phi_n(U_{M_n}), Y^n)$$

$$\geq \sum_{i=1}^{2} \lim_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_i^n}(Y_i^n) - nH(Y_2)}{\sqrt{n}} > T_3' + 3\gamma \right\} w(i)$$

$$= w(1) + \lim_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y_2^n}(Y_2^n) - nH(Y_2)}{\sqrt{n\sigma_2}} > \frac{T_3' + 3\gamma}{\sigma_2} \right\} w(2)$$

$$= w(1) + w(2) \int_{\frac{T_3' + 3\gamma}{\sigma_2}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz$$

$$> w(1) + w(2) \int_{\frac{T_3' + 3\gamma}{\sigma_2}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz = \frac{\delta}{2},$$

because we can let $T_3' + 3\gamma < T_3$ by letting $\gamma \to 0$. Hence, we have shown the proofs similarly to Case I.
VI. \((a, \delta)\)-intrinsic randomness

Let us now turn to the computation problem of the \((a, \delta)\)-intrinsic randomness formula for mixed sources. To do so, without loss of generality, we consider the following three cases:

I \[ H(Y_1) = H(Y_2) \] holds.

II \[ H(Y_1) > H(Y_2) \] and \( w(2) > \frac{\delta}{2} \) hold.

III \[ H(Y_1) > H(Y_2) \] and \( w(2) < \frac{\delta}{2} \) hold.

Then, we have

**Theorem 6.1:** Given \( 0 \leq \forall \delta < 2 \), the following holds.

**Case I:**

\[ S_\iota(H(Y_2), \delta | Y) = T_4, \]

where \( T_4 \) is specified by

\[
\frac{\delta}{2} = \sum_{i=1}^{2} w(i) \int_{-\infty}^{T_4} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.
\]

**Case II:**

\[ S_\iota(H(Y_2), \delta | Y) = T_5, \]

where \( T_5 \) is specified by

\[
\frac{\delta}{2} = w(2) \int_{-\infty}^{T_5} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.
\]

**Case III:**

\[ S_\iota(H(Y_1), \delta | Y) = T_6, \]

where \( T_6 \) is specified by

\[
\frac{\delta}{2} = w(2) + w(1) \int_{-\infty}^{T_6} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.
\]

**Proof:** It suffices to proceed in parallel with the arguments as made in the proof of Theorem 5.1 while taking account of the duality between resolvability and intrinsic randomness. Note that (5) is used instead of (4) in the proof of Converse Part.

**Remark 6.1:** It is easy to see that \( T_5 = +\infty \) and \( T_6 = -\infty \) for \( w(2) = \frac{\delta}{2} \). The latter part of Remark 5.2 similarly applies here too with \( S_\iota(a, \delta | Y) = +\infty \) instead of \( S_r(a, \delta | Y) = -\infty \).
VII. \((a, \varepsilon)\)-fixed-length source coding

Let us now consider to compute the formula for \(L_f(a, \varepsilon|Y)\) for mixed sources. To do so, without loss of generality, we consider the following three cases:

I \(H(Y_1) = H(Y_2)\) holds.

II \(H(Y_1) > H(Y_2)\) and \(w(1) > \varepsilon\) hold.

III \(H(Y_1) > H(Y_2)\) and \(w(1) < \varepsilon\) hold.

Then, we have the following main result:

*Theorem 7.1:* Given \(0 \leq \varepsilon < 1\), the following holds.

*Case I:*

\[
L_f(H(Y_1), \varepsilon|Y) = T_7, \\
\text{where } T_7 \text{ is specified by}
\]

\[
\varepsilon = \sum_{i=1}^{2} w(i) \int_{\frac{T_7}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.  \\
\text{(13)}
\]

*Case II:*

\[
L_f(H(Y_1), \varepsilon|Y) = T_8, \\
\text{where } T_8 \text{ is specified by}
\]

\[
\varepsilon = w(1) \int_{\frac{T_8}{\sigma_1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.  \\
\text{(14)}
\]

*Case III:*

\[
L_f(H(Y_2), \varepsilon|Y) = T_9, \\
\text{where } T_9 \text{ is specified by}
\]

\[
\varepsilon = w(1) + w(2) \int_{\frac{T_9}{\sigma_2}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz.  \\
\text{(15)}
\]

*Remark 7.1:* It is easy to check that \(T_8 = -\infty, T_9 = +\infty\) for \(w(1) = \varepsilon\). The latter part of Remark 5.2 similarly applies here too with \(L_f(a, \varepsilon|Y) = -\infty\) instead of \(S_r(a, \delta|Y) = -\infty\).

*Proof:* Although the proof is immediate from Theorem 3.4 and Theorem 5.1 with \(\epsilon = \frac{\delta}{2}\), we give in Appendix B another information-spectrum approach that is of independent interest, where
Lemma 7.1 and Lemma 7.2 (due to Han [5]) as described below are invoked. This is to see the operational duality between resolvability and fixed-length source coding.

**Lemma 7.1:** Let $M_n$ be an arbitrary given positive integer. Then, for all $n = 1, 2, \ldots$, there exists an $(n, M_n, \epsilon_n)$ code such that

$$
\epsilon_n \leq \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \geq \frac{1}{\sqrt{n}} \log M_n \right\}.
$$

**Lemma 7.2:** For all $n = 1, 2, \ldots$, any $(n, M_n, \epsilon_n)$ code satisfies

$$
\epsilon_n \geq \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \geq \frac{1}{\sqrt{n}} \log M_n + \gamma \right\} - e^{-\sqrt{n}\gamma},
$$

where $\gamma > 0$ is an arbitrary constant.

**VIII. Concluding Remarks**

We have so far considered the second-order achievability to evaluate the finer structure of random number generation for mixed sources. The class of mixed sources is very important, since all of stationary sources can be regarded as forming mixed sources obtained by mixing stationary ergodic sources with respect to probability measures. However, in general, mixed sources do not have an asymptotic normality. So, our result is also meaningful since we have demonstrated that the analysis based on the two-peak asymptotic normality is still effective also for sources whose self-information spectrum does not have a single asymptotic normality such as mixed sources.

As shown in the proofs of the present paper, the information-spectrum approach is substantial in the analysis of the second-order achievable rates. In particular, Lemma 4.1 shown by Han [5] is the simple but key lemma, which enables us to work with the two-peak asymptotic normality for mixed sources.

Finally, we state some comments about the mixed source treated in this paper. We only consider the case that the mixed source consists of two i.i.d. sources. From the viewpoint of computation, it can be considered as the simplest case. We can consider more general mixed sources such as

$$
P_{Y^n}(y) = \sum_{i=1}^{\infty} P_{Y^n_i}(y)w(i),
$$
or

$$
P_{Y^n}(y) = \int_{\theta \in \Theta} P_{Y^n}(y)dw(\theta),
$$

where $\theta \in \Theta$ indicates the parameter of i.i.d. source and $\Theta$ is a probability sample space. We can easily extend our results to the former case on the basis of Lemma 4.1. However, in the latter
case, we cannot use the similar method as described in this paper, although we can obtain a due extension of Lemma 4.1. In that case, we need to use the continuous asymptotic normality instead of the two-peak asymptotic normality. Such a class of function is studied in the field of probability theory [12].

APPENDIX A
PROOF OF LEMMA 4.1

Although the proof to be shown below literally mimics the proof given in Han [5], we will repeat it here for the reader’s convenience.

At first we show the first inequality. Set a sequence \{γ_n\}_n=1^\infty satisfying \γ_1 > \γ_2 > \cdots > 0, γ_n → 0, \sqrt{n}γ_n → \infty. Then it holds that
\[
\Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}} - \frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}} \leq -γ_n \right\} = \sum_{y \in D_n(i)} P_{Y^n_i}(y) \leq \sum_{y \in D_n(i)} P_{Y^n_i}(y)e^{-\sqrt{n}γ_n} \leq e^{-\sqrt{n}γ_n},
\]
for \(i = 1, 2\), where
\[
D_n(i) = \left\{ y \in Y^n \mid \left| -\log P_{Y^n_i}(y) - \log P_{Y^n_i}(y) \right| \leq -γ_n \right\}.
\]
This means that
\[
\Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}} - γ_n < \frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}} \right\} \geq 1 - e^{-\sqrt{n}γ_n},
\]
holds for \(i = 1, 2\). So we have
\[
\frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}} - γ_n < \frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}}
\]
with probability \(1 - e^{-\sqrt{n}γ_n}\). So we have for \(i = 1, 2\)
\[
\Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}} \geq z_n \right\} \geq \Pr \left\{ \frac{-\log P_{Y^n_i}(Y^n_i)}{\sqrt{n}} - γ_n \geq z_n \right\} - e^{-\sqrt{n}γ_n},
\]
which is the first inequality of the lemma.

Secondly, we show the second inequality of the lemma. Set
\[
S_n(η_n) = \left\{ y \in Y^n \mid P_{Y^n_i}(y) \leq e^{-\sqrt{n}z_n} \right\}
\]
\[
S_n^{(i)}(η_n) = \left\{ y \in Y^n \mid P_{Y^n_i}(y) \leq \frac{e^{-\sqrt{n}z_n}}{w(i)} \right\}
\]
From the property of mixed sources, \( y \in S_n(i)(\eta) (i = 1, 2) \) holds for \( y \in S_n(\eta) \). This means that
\[
S_n(\eta) \subset S_n(i)(\eta)
\]
\((i = 1, 2)\). Moreover, since \( \gamma_n \geq -\frac{\log w(i)}{\sqrt{n}} \) \((i = 1, 2)\) hold for sufficiently large \( n \), we have
\[
\Pr \left\{ -\frac{\log P_{Y^n}(Y^n_i)}{\sqrt{n}} \geq z_n \right\} = \Pr \left\{ P_{Y^n}(Y^n_i) \leq e^{-\sqrt{\pi}z_n} \right\} = P_{Y^n}(S_n(\eta)) \\
\leq P_{Y^n}(S_n(i)(\eta)) = \Pr \left\{ -\frac{\log P_{Y^n}(Y^n_i)}{\sqrt{n}} \geq z_n - \frac{\log w(i)}{\sqrt{n}} \right\} \\
\leq \Pr \left\{ -\frac{\log P_{Y^n}(Y^n_i)}{\sqrt{n}} \geq z_n - \gamma_n \right\},
\]
for sufficiently large \( n \), which is the second inequality. ■

**APPENDIX B**

**PROOF OF THEOREM 7.1**

1) **Direct Part:**

We prove that \( L_1 = T_7 + \gamma \) is an \((H(Y_1), \varepsilon)\)-achievable, where \( \gamma > 0 \) is an arbitrary small constant.

Set \( M_n = e^{nH(Y_1) + \sqrt{n}L_1} \). Then, obviously we have
\[
\lim_{n \to \infty} \sup \log \frac{M_n - nH(Y_1)}{\sqrt{n}} \leq L_1.
\]

Thus, it is enough to show that there exists an \((n, M_n, \varepsilon_n)\) code satisfying
\[
\lim_{n \to \infty} \sup \varepsilon_n \leq \varepsilon.
\]

From Lemma 7.1, there exists an \((n, M_n, \varepsilon_n)\) code that satisfies
\[
\varepsilon_n \leq \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \geq \frac{1}{\sqrt{n}} \log M_n \right\} = \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \geq \frac{nH(Y_1) + \sqrt{n}L_1}{\sqrt{n}} \right\} = \Pr \left\{ -\frac{\log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}} \geq L_1 \right\} = \sum_{i=1}^{2} \Pr \left\{ -\frac{\log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}} \geq L_1 \right\} w(i).
\]
The last equality is derived from the definition of the mixed source. Then, from Lemma 4.1, we have
\[
\limsup_{n \to \infty} \sum_{i=1,2} \Pr \left\{ \frac{- \log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}} \geq L_1 \right\} w(i) \\
\leq \sum_{i=1}^2 \limsup_{n \to \infty} \Pr \left\{ \frac{- \log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}} \geq L_1 - \gamma_n \right\} w(i) \\
\leq \sum_{i=1}^2 \limsup_{n \to \infty} \Pr \left\{ \frac{- \log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}\sigma_i} \geq \frac{L_1 - \gamma}{\sigma_i} \right\} w(i) \\
= \sum_{i=1}^2 \limsup_{n \to \infty} \Pr \left\{ \frac{- \log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}\sigma_i} \geq \frac{T_7}{\sigma_i} \right\} w(i),
\]
since \(\gamma_n\) is specified in Lemma 4.1, and so \(\gamma > \gamma_n\) holds for sufficiently large \(n\).

Then, noting that \(H(Y_1) = H(Y_2)\) holds, from the asymptotic normality, we have
\[
\limsup_{n \to \infty} \Pr \left\{ \frac{- \log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}\sigma_i} \geq \frac{T_7}{\sigma_i} \right\} = \int_{\frac{T_7}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{z^2}{2} \right] dz,
\]
for \(i = 1, 2\). Thus, we have
\[
\limsup_{n \to \infty} \sum_{i=1}^2 \Pr \left\{ \frac{- \log P_{Y^n}(Y^n_i) - nH(Y_1)}{\sqrt{n}} \geq L_1 \right\} w(i) \leq \sum_{i=1}^2 w(i) \int_{\frac{T_7}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{z^2}{2} \right] dz.
\]
Substituting the above inequality into (15), we have
\[
\limsup_{n \to \infty} \epsilon_n \leq \sum_{i=1}^2 w(i) \int_{\frac{T_7}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{z^2}{2} \right] dz = \epsilon,
\]
where the last equality is derived from the definition of \(T_7\) given by (13). Therefore, Direct Part has been proved.

2) Converse Part:
Let us assume that \(L_2\) satisfying \(L_2 < T_7\) is \((H(Y_1), \varepsilon)\)-achievable. Then we shall show a contradiction. Notice that there exists a constant \(\gamma > 0\) such that \(L_2 + 3\gamma < T_7\) holds.

Then from the assumption there must exist an \((n, M_n, \epsilon_n)\) code such that
\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon,
\]
\[
\limsup_{n \to \infty} \log M_n - nH(Y_1) \leq L_2,
\]
This means that for sufficiently large \(n\), there must exist an \((n, M_n, \epsilon_n)\) code such that
\[
\log M_n \leq \frac{nH(Y_1)}{\sqrt{n}} + L_2 + \gamma.
\]
for any \( \gamma > 0 \). On the other hand, from Lemma 7.2, any \((n, M_n, \epsilon_n)\) code satisfies
\[
\epsilon_n \geq \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \geq \frac{1}{\sqrt{n}} \log M_n + \gamma \right\} - e^{-\sqrt{n} \gamma}.
\]
Thus, from (16) it follows that
\[
\epsilon_n \geq \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{P_{Y^n}(Y^n)} \geq \frac{1}{\sqrt{n}} \log M_n + \gamma \right\} - e^{-\sqrt{n} \gamma}.
\]
for sufficiently large \( n \), where the last equality is derived from the definition of the mixed source.

From Lemma 4.1, we have
\[
\lim inf_{n \to \infty} \epsilon_n \geq \sum_{i=1}^{2} \Pr \left\{ \frac{-\log P_{Y^n}(Y^n_i) - nH(Y^1_i)}{\sqrt{n}} \geq L_2 + 2\gamma \right\} w(i) - e^{-\sqrt{n} \gamma},
\]
in (17) since \( \gamma_n \) is specified in Lemma 4.1 and so \( \gamma_n < \gamma \) holds for sufficiently large \( n \). Then, by virtue of the asymptotic normality we have
\[
\sum_{i=1}^{2} \lim inf_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n}(Y^n_i) - nH(Y^1_i)}{\sqrt{n}} \geq L_2 + 3\gamma \right\} w(i) = \sum_{i=1}^{2} \lim inf_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n}(Y^n_i) - nH(Y^1_i)}{\sqrt{n}} \geq \frac{L_2 + 3\gamma}{\sigma_i} \right\} w(i),
\]
and
\[
\sum_{i=1}^{2} \lim inf_{n \to \infty} \Pr \left\{ \frac{-\log P_{Y^n}(Y^n_i) - nH(Y^1_i)}{\sqrt{n}} \geq \frac{L_2 + 3\gamma}{\sigma_i} \right\} w(i) = \sum_{i=1}^{2} w(i) \int_{\frac{L_2 + 3\gamma}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] \, dz
\]
\[
> \sum_{i=1}^{2} w(i) \int_{\frac{L_2 + 3\gamma}{\sigma_i}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] \, dz = \epsilon,
\]
since we can let \( L_2 + 3\gamma < T_7 \) holds by letting \( \gamma \to 0 \) and the last equality is from (13). Thus, substituting the above into (17), it must hold that
\[
\lim inf_{n \to \infty} \epsilon_n \geq \epsilon.
\]
This is a contradiction and we conclude that $L_2$ satisfying $L_2 < T_7$ is not an $(H(Y_1), \varepsilon)$-achievable.

Proofs of Case II and Case III:
The proofs of Case II and Case III can be shown by the same arguments as the proof of Case I of Theorem 7.1.

Acknowledgment
The first author is very grateful to Prof. Toshiyasu Matsushima of Waseda University for his valuable discussions and comments.

References