A Clustering Algorithm for Bi-Criteria Stop Location Design with Elastic Demand

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Abstract: This paper proposes a bi-criteria formulation to find the optimal location of light rapid transit stations in a network where demand is elastic and budget is constrained. Our model is composed of two competing objective functions seeking to maximize the total ridership and minimize the total budget allocated. In this research, demand is formulated using the random utility maximization method with variables including access time and travel time. The transit station location problem of this study is formulated using mixed integer programming and we propose a heuristic solution algorithm to solve large scale instances which is inspired by the problem context. The elastic demand is integrated with the optimization problem in an innovative way which facilitates the solution process. The performance of our model is evaluated on two test problems and we carry out its implementation on a real-world instance. Due to the special shape of the Pareto front function, significant practical policy implications; in particular budget allocation, are discussed to emphasize the fact that the trade-off between cost and benefit may result in large investments with little outcomes and vice versa.

1 INTRODUCTION

The design of transit networks is a challenging task for transportation planners that span numerous often counter-intuitive considerations. Broadly, the Network Design Problem (NDP) can be approached through cost-benefit models that are capable of representing multiple aspects of system-wide behaviour, such as travel demand, transit route location, traffic congestion, or service frequency. In this paper, we address the problem of designing a Light Rapid Transit (LRT) network where budget expenditures modify system efficiency as well as the resulting transit demand in an optimization framework. A key feature of the integrated model is that transit demand and supply are endogenous variables, i.e. changing the design of the system affects the demand for the service provided and reciprocally different demand paradigms affect the best design of the system. This mutual interaction requires that a precise demand model be intertwined with a network design model in order to find efficient transit network designs.

Specifically, the proposed model seeks to minimize the total cost of opening stations at candidate locations while maximizing the total transit ridership. In this study, it is assumed that the general topology of the transit route is given; hence the model focuses exclusively on the location of stations and the impact on ridership in the network. As different designs are associated with different total budgets, access time and total travel times, we propose two objective functions that seek to maximize transit ridership and minimize the total construction budget required. The demand for LRT among origin destination pairs is estimated using an elastic demand model which is based on the access travel time and the in-vehicle travel time between the origin and the destination. A Mixed Integer Linear Programming (MILP) formulation is proposed to estimate the Pareto front associated with this bi-criteria optimization problem and introduce a heuristic solution algorithm to solve the problem on large instances. The behaviour of the model and the performance of the heuristic algorithm are demonstrated on two illustrative benchmark problems. Furthermore, we present a specific demonstration for the City of Sydney to examine the scalability of the model as well as practical insights for the approach. In this analysis, specific assumptions regarding the zoning system used for the travel demand model, the elasticity of the transit ridership against the walking distance and the travel time estimations are made using the actual mode choice models for the City of Sydney. As expected, we report that optimal solutions for the introduced bi-criteria NDP do not necessarily define a concave (nor convex) Pareto front, hence recognizing potentially significant trade-offs between ridership and budget in the cost-benefit function.

The contributions of this paper are four fold. First an elastic demand function is developed for the Light Rapid Transit-Network Design Problem (LRT-NDP) with a focus on stop location selection. Second, the elastic demand is discretized to a non-parametric model to facilitate the integration within the optimization method. Third, a bi-criteria formulation for the LRT-NDP is proposed which seeks to raise awareness, especially to practitioners, about the non-monotonic behaviour of the Pareto front function showing the financial consequences of superficial decision making in the context of the transit network design. Finally, a straightforward and efficient heuristic algorithm based on k-means clustering is proposed to approximate the Pareto front of the bi-criteria LRT-NDP on a city-wide instance. From a policy implication point of view, the proposed
model provides insights on budget allocation when more than one project is being considered.

The rest of the paper is structured as follows. First a review of the literature is provided to further elaborate the contribution of the present paper. Then the demand model, utilized data and assumptions made to estimate the origin-destination matrix are presented. Following that, the integer programming model used to solve the LRT station location problem with a budget objective function is elaborated. The results are presented afterwards and a discussion about project budget allocation strategies is provided. Finally in the last section the findings are summarized and future directions are stated.

2 LITERATURE REVIEW

The NDP is generally formulated as a multi-objective nonlinear integer program (Current and Marsh, 1993), which is intractable for large instances (Hajabai and Ouyang, 2013). Due to the computational difficulty, most variants of the NDP are solved using tailored heuristics solution methods or meta-heuristics frameworks (Fan and Machemehl, 2004; Kepaptsoglou and Karlaftis, 2006; Szeto et al, 2014). Among the variants of the NDP, the Transit Route Network Design Problem (TRNDP) has been widely studied from three perspectives: the nature of the objective function, the complication of solving the problem under different assumptions, and the elasticity of demand (Ceder and Wilson 1986).

It is quite common to consider a multi-objective function in NDPs (Chow and Regan, 2012; Xie, 2014) and TRNDPs (see for example Farahani et al, 2013; Guihaire and Hao, 2008). Decision variables, definition of the objective function and solution methods have attracted most of the research, focusing on TRNDP while demand elasticity has been less studied (Kepaptsoglou and M. Karlaftis, 2006). In the context of TRNDP, demand models can be categorized into two groups: fixed demand models and elastic demand models. Due to the complexity of considering elastic demand, despite the inherent elastic nature of demand in all TRNDP problems, it is commonly considered to be fixed (Fan and Machemehl, 2006; Lee and Vuchic, 2005). Elastic demand also can be found in two forms; first, the demand between each pair of origin and destination is variable meaning that people may change their destination if the travel cost is high and second the demand between each OD pair is fixed but the share of each mode is variable (Farahani et al, 2013). Travel demand elasticity has other dimensions (Javanmardi et al, 2011; Rashidi and Mohammadian, 2011) which have been recently to some extent used in the general network design problem (Kang et al 2013).

The studies with elastic demand are dominantly occupied by research projects solving the traffic assignment problem for transit and car (Jiang and Xie, 2014) with decision variables including lane allocation (Wu et al, 2009), turning restrictions (Long et al, 2010), making some streets one-way (Miandoabchi and Farahani, 2010), new street construction (Poorzahedy and Turnquist, 1982), street capacity expansion (Leblanc, 1975) and pricing (Saha et al, 2014; Chen et al, 2012). TRNDP studies have used headway (Hu et al, 2005), frequency (Russo, 1998), route (Spasovic et al, 1994 and Szeco and Jiang, 2012, Schmid, 2014), stops (Yo and Yang, 2006) and links (Laporte et al, 2011) as the main decision variables to solve the problem. Among the decision variables being used for solving TRNDP, stop location has been less examined (Van Nes, 2001) while headway, frequency and route selection have been extensively studied (Gallo et al, 2011). Despite the importance of stop location design on other aspects of transit network design, only a handful of papers have discussed in a limited way of number of stops or stop spacing using approximate rule of thumb methods (Bagloee et al; 2011, Saka, 2001; Murray, 2003).

Rail transit network design with elastic demand has also been less studied because of the complication elastic demand imposes to the design problem which is already quite complicated. Martin and Garcia (2009) solved the problem with an elastic demand but they applied it on a small network where stations and centroids were at the edges of the network. They approximated the non-linear logit function by a polygonal curve (piecewise linear function) and the resulting model will be a linear integer programming model. The study of this paper focuses two of the least studied component of TRNDP which is rail network design with elastic demand and stop location finding problem.

Table 1 presents a summary of several of the papers on rail transit network design problem. Guihaire and Hao (2008) prepared a review of main papers on transit network design until 2008. By looking at the second table of their paper, it can be discerned that there has been quite a few papers considered elastic (variable) demand in TRNDP (Lee and Vuchic, 2005). Farahani et al (2013) also provided a review of NDP papers. As it can be seen in Tables 5 and 6 in the paper by Guihaire and Hao, the transit NDPs have not considered a variable demand unless a multimodal network design is of interest in which a logit based modelling formulation has widely been used for only the competition between bus and car. Further, no stop location rail design problem with elastic demand has been reported by Farahani et al (2013). It is again obvious from all papers reviewed by Farahani et al (2013) that traffic assignment and frequency (fleet size) determination are the dominant topics in transit network design studies, mainly bus network design. In addition to the fact that no rail transit network design problem has considered stop location as the dependent variable, the elastic (variable) demand functions used in these models are quite simplistic and not as advanced as those used in the traffic assignment and route design
Due to its intrinsic complexity, the TRNDP has mostly been addressed through metaheuristics-based optimization models (Lin Xie, 2011). In particular, Genetic Algorithms (GA) have been extensively used to solve the Bus-TRNDP. This approach, first suggested by Pattnaik et al. (1998) and extended by Tom and Mohan (2003), was implemented to determine bus transit routes and schedules. Baaj and Mahmassani (1995) presented a hybrid route generation algorithm for the TRNDP based on artificial intelligence for solving vehicle routing problem. The solution method works from a given skeleton route connecting a few major demand centroids and progressively adds nodes to the routes. Israeli and Ceder (1995) formulated the TRNDP as a multi-objective nonlinear optimization problem which is designed to minimize the total passenger costs as well as the operator costs and proposed a mathematical programming approach to solve it. Their formulation includes route and schedule design, scheduling constraints and driver assignment constraint.

3 METHODOLOGY AND MATHEMATICAL FORMULATION

In this section, we present the methodology of our approach to solve the bi-criteria LRT-NDP and introduce the mathematical formulation of our model.

3.1 Demand model

We make some realistic assumptions regarding different factors that affect travel demand in order to maintain the problem computationally tractable. This supports the scalability of the approach for city-wide analysis while still being relevant for the analysis of budget allocation decisions.

The first assumption is related to the demand aggregation for city analysis. The specific Transport Analysis Zones (TAZ) used by planning agencies in Sydney are considered
in this analysis; if the demand for using LRT for a pair of origin and destination is greater than zero, this pair is kept in the origin-destination matrix. Both walking and transit modes are considered to access the LRT service and the travel times are estimated using the transport network for these two modes using ArcGIS software. Figure 1 shows the LRT line and the zones considered in the study. After dropping the zones with zero attraction and production, 102 (black dots in Figure 1) TAZ remain of interest for this analysis. 49 (orange circle with black dot inside) candidate sites are considered for the location of transit stations and 3 terminal stations (Circular Quay, Randwick and Kingsford Stations) are assumed fixed to ensure that the LRT line expands along the proposed plan.

![Figure 1](image)

**Figure 1** Centroids with greater than zero production/attraction are connected to the transport network and the LRT potential station sites, centroids are shown with black dots and station sites are in orange.

The total demand for LRT for each pair of centroids varies with the population at the origin and the employment size of the destination (Leigh, 1970). Thus, the demand for using LRT is simplified for each OD to be a function of the sum of the population of the origin and the employment of the destination and the probability of choosing LRT. Let \( Q_{ij}^{mn} \) be the demand function between \( i \) and \( j \) through stations \( m \) and \( n \). \( Q_{ij}^{mn} \) is determined using the following equations:

\[
Q_{ij}^{mn} = P_{ij}^{mn} D_{ij} \tag{1}
\]
\[
D_{ij} = \alpha (P_{ij} + Emp_i) \tag{2}
\]

where:

- \( P_{ij}^{mn} \): is the probability value of using LRT between \( i \) and \( j \) using station (entry) \( m \) and (exit) \( n \)
- \( D_{ij} \): is the total demand value for LRT from \( i \) to \( j \)
- \( \alpha \): is a factor for estimating trip distribution
- \( P_{ij} \): is the population of the origin zone \( i \)
- \( Emp_i \): is the employment of the destination zone \( i \)

A stochastic random utility model with a logit distribution function is used to estimate the probability of choosing LRT. The logit function is used to estimate \( P_{ij}^{mn} \) values for a specific set of origin, destination and stops (arrival and departure). This is an innovation of the modelling part of this paper that rather than directly using the parametric logit model within the network design problem, for all possible designs, non-parametric origin-destination-stop specific demand values \( Q_{ij}^{mn} \) are calculated and used within the network design problem.

Using the strategic public transport choice model of the City of Sydney, the parameters of in-vehicle travel time and access travel time are used. The probability of choosing LRT decreases as the travel time and the access time associated with LRT increase. A basic probability for choosing LRT is considered for each pair of origin and destination. This basic probability drops as the travel time and access time increase based on the first derivatives of the probability function shown later. If the access time between two zones is smaller than the access time to LRT, no demand is allocated to the pair of zones. More specifically, the value for probability of using LRT is estimated using the following equation:

\[
P_{ij}^{mn} = P \left( AT_{ij}^{mn}, TT_{ij}^{mn} \right)
\]
\[
= \left( P_0 - \frac{\partial P}{\partial AT} \Delta AT_{ij}^{mn} - \frac{\partial P}{\partial TT} \Delta TT_{ij}^{mn} \right)^+
\]

where: \( (X)^+ = \max \{X, 0\} \); \( \Delta AT_{ij}^{mn} = AT_{ij}^{mn} - AT_{ij}^0 \) and \( AT_{ij}^{mn} \) is the access travel time between \( i \) and \( j \) through \( m \) and \( n \); \( AT_{ij}^0 \) is the access travel time of the base case from \( i \) to \( j \); \( \Delta TT_{ij}^{mn} = TT_{ij}^{mn} - TT_{ij}^0 \) and \( TT_{ij}^{mn} \) is the travel time between \( i \) and \( j \) through \( m \) and \( n \); \( TT_{ij}^0 \) is the travel time of the base case from \( i \) to \( j \);
In the above quantities, the base case represents the pair of arrival and departure stations providing the fastest total travel time (access and in-vehicle) between \( i \) and \( j \):

\[ P_{ij}^{0} : \text{is the probability of using LRT from } i \text{ to } j \text{ with the shortest travel time and access time; } \]

\[ \Delta T: \text{is a variable representing the access travel time } \]

\[ TT: \text{is a variable representing the in-vehicle travel time } \]

\[ P: \text{is the probability cumulative function for using LRT which depends on } AT \text{ and } TT; \]

\[ \frac{\partial P}{\partial AT} \Delta AT_{ij}^{m} \frac{\partial P}{\partial TT} \Delta TT_{ij}^{m} \]

\[ + \]

\[ 015 \]

\[ \text{Otherwise. } \]

\[ m \] \[ \begin{align*} m_{\rightarrow} \sum_{(m, n) \in S} & \max \{ Q_{ij}^{mn} x_{m} x_{n} \} \quad [5] \\
\]

\[ z_{B} = \min \sum_{m \in S} x_{m} B_{m} \quad [6] \\
\]

\[ \text{Subject to: } \]

\[ x_{m} \in \{0, 1\} \forall m \in S \quad [7] \]

Model 1: Bi-criteria Integer Nonlinear Program for the LRT-NDP

\[ Q_{ij}^{mn} \text{ values provide non-parametric measures for the ridership from centroid } i \text{ to centroid } j \neq i \text{ entering through station } m \text{ and exiting through station } n \neq m. \text{ These values are determined } \text{a priori} \text{ by Equation [4], hence the ridership is a parameter in our optimization formulation. Nonetheless, it is important to note that the overall demand and even the demand for LRT between } i \text{ and } j \text{ are elastic and are functions of the design; or more specifically, functions of access time and travel time. In other words, } Q_{ij}^{nm} \text{ is the potential ridership between OD } i \rightarrow j \text{ only when stops } m \text{ and } n \text{ are available (i.e., } Q_{ij}^{mn} \text{ is a conditional potential ridership). The total realized demand between } i \text{ and } j \text{ will therefore vary greatly under different station designs resulting in demand being elastic as well as endogenously determined along with design. } \]

Model 1 is nonlinear with regards to the decision variables of the formulation because of the term \[ \max_{m,n \in S} \{ Q_{ij}^{mn} x_{m} x_{n} \} \text{ in the objective function [5]. This maximum is used to select the pair of stations which maximizes the ridership from centroid } i \text{ to centroid } j \text{ among the possible combinations of entry and exit stations. In order to linearize this term, we introduce an auxiliary decision variable } y_{ij}^{mn} \text{ defined as: } \]

\[ y_{ij}^{mn} = \begin{cases} 1 \text{ if a station is open at candidate location } m \in L \\ 0 \text{ otherwise. } \end{cases} \]

Our objective is to maximize the total ridership and minimize the total budget of the project. The bi-criteria LRT-NDP can be represented using the mathematical formulation summarized in Model 1. Model 1 is a bi-criteria, integer, and nonlinear program. Objective function [5] seeks to maximize the total ridership while Objective function [6] seeks to minimize the total budget required for planning. The only decision in this program is represented by the binary variable \( x_{m} \) for each location \( m \in L \).
We can then substitute the nonlinear term using the sum: $\sum_{m \in S} \sum_{n \in S} y_{ij}^{mn} q_{ij}^{mn}$ and by adding three linear constraints to represent the fact that transit can only be routed through location where a station is built. The resulting linear formulation is summarized in Model 2.

$$z_R = \text{maximize} \sum_{i \in C} \sum_{j \in C} \sum_{m \in S} \sum_{n \in S} y_{ij}^{mn} q_{ij}^{mn}$$

Subject to

$$\sum_{m \in S} \sum_{n \in S} y_{ij}^{mn} = 1 \quad \forall i, j \in C: i \neq j$$

$$\frac{1}{|C|^2} \sum_{i \in C} \sum_{j \in C} y_{ij}^{mn} \leq x_m \quad \forall m, n \in S: m \neq n$$

$$0 \leq y_{ij}^{mn} \leq 1 \quad \forall i, j \in C: i \neq j, \forall m, n \in S: m \neq n$$

$$x_m \in \{0, 1\} \quad \forall m \in S$$

Model 2: Bi-criteria Mixed-Integer Linear Program for the LRT-NDP

Model 2 is a bi-criteria MILP. The objective function [8] together with the constraints [10], [11] and [12] is equivalent to the objective function [5]. Namely, constraint [10] (hereby referred to as the assignment constraint) enforces that all the transit between a pair of demand centroids is routed through a single pair of available stations. This is consistent with our assumption that all the demand between a pair of origin and destination centroids uses only the fastest route. Constraints [11] and [12] require that stations be opened at origin and destination locations for any transit to be routed through this pair of stations; combining these constraints with [10], it results that only a single pair of stations are selected to route the demand for each pair of origin and destination centroids. Since Objective function [8] seeks to maximize the total ridership, for each pair of demand centroids, the pair of stations which maximizes $Q_{ij}^{mn}$ is always selected. Hence decision variable $y_{ij}^{mn}$ needs not to be defined as integer, but only requires to be bounded between 0 and 1. This is a consequence of the assignment constraint which requires that only a single pair of stations may be used to route the transit between any two centroids. Further, since there is no capacity restrictions in our formulation, at the optimum, all the demand between $i$ and $j$ is routed through a single pair of stations $m$ and $n$ for which $x_m = x_n = 1$ and $y_{ij}^{mn} = 1$.

We propose to solve Model 2 using the ε-constraint method from multi-criteria optimization (Haimes et al., 1971). This approach consists of maintaining only one objective of a multi-objective program as an objective function and expressing the remaining objectives using a set of constraints. The Pareto front of the multi-objective program can then be determined by solving a range of single objective programs where the value of the ε-constraints (corresponding to the remaining objective functions) is gradually relaxed or tightened. We choose to pass the budget-related objective function as a ε-constraint; the single-objective MILP represented by Model 3 is obtained.

$$z_R = \text{maximize} \sum_{i \in C} \sum_{j \in C} \sum_{m \in S} \sum_{n \in S} y_{ij}^{mn} q_{ij}^{mn}$$

Subject to

$$\sum_{m \in S} x_m B_m \leq \varepsilon_k$$

$$\sum_{m \in S} y_{ij}^{mn} = 1 \quad \forall i, j \in C: i \neq j$$

$$\frac{1}{|C|^2} \sum_{i \in C} \sum_{j \in C} y_{ij}^{mn} \leq x_m \quad \forall m, n \in S: m \neq n$$

$$0 \leq y_{ij}^{mn} \leq 1 \quad \forall i, j \in C: i \neq j, \forall m, n \in S: m \neq n$$

$$x_m \in \{0, 1\} \quad \forall m \in S$$

Model 3: Mixed-Integer Linear Program for the LRT-NDP

In Model 3, $\varepsilon_k$ is the available budget where $k$ is an index that represents an iteration in the estimation of the Pareto front of Model 2; specifically, we seek to solve Model 3 for the range of values: $\varepsilon_k \in [0, \sum_{m \in S} B_m]$. Without any loss of generality, we assume that the budget required to build the terminal stations is nil. Since ridership is always positive, this ensures that the terminal stations are always selected at any feasible point which image belongs to the Pareto front of Model 2.
3.3. Heuristic for solving large-scale problems

The heuristic algorithm uses k-means clustering (Hartigan and Wong, 1979) to select k locations and evaluate the efficiency of each solution using the concept of Pareto efficiency (Pareto, 1964). K-means clustering aims to partition a set of elements into a predefined number of clusters in order to minimize the average distance within clusters. While this problem is known to be NP-hard, Lloyd’s (1982) algorithm has been widely used as an efficient heuristic for k-means clustering. Recently, Rey et al. (2014) have successfully implemented a heuristic algorithm based on Lloyd’s algorithm to solve the transit route design problem on large transportation networks.

In our heuristic algorithm, we use k-means clustering to select a predefined number of locations which are evenly distributed across the LRT network according to the distance matrix $d_{mn}$ between any two locations $m, n \in L$. Every solution $s$, i.e. a set of locations, is evaluated by determining the total ridership, $R_s$ and budget $B_s$ associated with this combination of stations (terminals and selected locations). We distinguish two types of solutions:

Non-Dominated Solutions — A solution $s = \{l_1, ..., l_k\}$ with cost $(R_s, B_s)$ is said to be non-dominated if there does not exist any solution $s'$ with cost $(R'_s, B'_s)$ such that:
- $R_s < R'_s$ and $B_s \geq B'_s$, or
- $R_s \leq R'_s$ and $B_s > B'_s$.

Dominated Solutions — A solution $s = \{l_1, ..., l_k\}$ with cost $(R_s, B_s)$ is said to be dominated if there exist a solution $s'$ with cost $(R'_s, B'_s)$ such that:
- $R_s < R'_s$ and $B_s \geq B'_s$, or
- $R_s \leq R'_s$ and $B_s > B'_s$.

Determining the ridership and the budget of a solution is a quick procedure; hence every solution that is enumerated along the execution of the algorithm is evaluated and temporarily classified as non-dominated or dominated using the current set of solutions. In order to span the whole Pareto front, we run the k-means clustering step for all possible number of stations, that is, from 1 to $|L|$. Since the k-means clustering step relies on a random draw of $k$ locations, we repeat $T$ times (in practice, the process can be repeated until a sufficient number of different solutions have been obtained).

The pseudo-code of the heuristic algorithm is given in figure 2. The algorithm returns all dominated and non-dominated solutions that have been evaluated during its execution. The k-means clustering implementation relies on a random draw of $k$ locations and converges in $O(|L|^k)$ time where $w$ is the largest number of iterations required until two consecutives solutions are identical. Since the process is repeated $T$ times and for every possible number of locations, the time complexity of the heuristic is $O(|L|^2Tkw)$.

4 RESULTS

In this section, we demonstrate the performance and correctness of the model on two test problems and present the implementation of the solution algorithm on the real world instance of the City of Sydney.

4.1. Model behaviour – Trade-off between objectives and economic implications

In order to illustrate the behaviour of the proposed model and highlight the economic trade-offs that may arise from different network designs, we consider two instances and examine their associated Pareto fronts. Consider the LRT project depicted in Figure 3.

For $k = 1..|L|$
for $t = 1..T$
randomly select $k$ locations in $L$: $s \leftarrow \{l_1, ..., l_k\}$
evaluate dominance of $(R_s, B_s)$, store $s$
while convergence $= \text{false}$
do
for $m \in \{l_1, ..., l_k\}$ do
$C_m \leftarrow \{n \in L | s : n = \arg\min d_{mn}\}$
$s' \leftarrow \emptyset$
for $C_m \in \{C_1, ..., C_k\}$ do
$m \leftarrow \arg\min \{\sum_{n \in C_m \setminus \{m\}} d_{mn}\}$
$s' \leftarrow s' \cup \{m\}$
evaluate dominance of $(R'_s, B'_s)$, store $s'$
if $s' = s$ then convergence $= \text{true}$
else $s \leftarrow s'$
end do
end do
end while
return all dominated and non-dominated solutions

Figure 2 Heuristic Algorithm for the bi-criteria LRT design problem

In this project, an LRT between two terminal stations is to be constructed and several candidate sites for transit stations are available. In instance A, we assume an elastic demand between every pair of centroids that is proportional to the distance from the centroids to the LRT and random budgets for the candidate locations. In instance B, we assume a uniform budget across candidate locations and random demand between the different pairs of centroids.
Figure 3 Instance study of a LRT design problem, each candidate location as well as the terminal stations are surrounded by two demand centroids.

4.1.1. Instance A: Elastic Distance-based Ridership with Random Budget

In this instance, we assume an elastic demand between every pair of centroids that is proportional to the distance from the centroids to the LRT. In other words, if a station is opened at every candidate location, then the ridership between any two demand centroids is maximal and equal to $R_{\text{max}}$. Furthermore, we assume that centroids are uniformly located around the LRT, hence the shortest (resp. longest) distance from a centroid to a station is $d_{\text{min}}$ (resp. $d_{\text{max}}$) and the ridership $Q_{ij}^{mn}$ is determined as:

$$Q_{ij}^{mn} = R_{\text{max}} \left(1 - \frac{a_i^m + d_{ij}^n - 2d_{\text{min}}}{2(d_{\text{max}} - d_{\text{min}})}\right)$$

where $d_{ij}^m$ is the distance from centroid $i$ to station $m$. The budget required to open a station $m \in L$ at a candidate location is uniformly and randomly chosen in a range $[B_{\text{min}}, B_{\text{max}}]$. We define the step budget, $\bar{B}$ as the average budget required to open a station at a candidate location:

$$\bar{B} \equiv \frac{1}{10|L|} \sum_{m \in L} B_m$$

We propose to solve $10|L|$ programs of the form of Model 3, where the right hand side of the $\epsilon$-constraint is equal to:

$$\epsilon_k = k\bar{B}, \ \forall \ 1 \leq k \leq 10|L|$$

Hence $z_1$ represents the maximal ridership obtained when a unit of step budget is available and $z_{10|L|}$ represents the maximal ridership obtained when the entire budget is available, i.e. when a station can be opened at every candidate location.

4.1.2. Instance B: Uniform Budget with Random Ridership

In contrast to instance A, in this instance study, we assume a uniform budget across candidate locations and random demand between the different pairs of centroids. Hence, every candidate location is treated identically from a budget perspective, and in this instance the step budget can be defined as a normalized budget, that is: $\bar{B} = 1$, therefore $\epsilon_k = k$ is the number of stations that can be opened at each iteration $k$. In this instance, we seek to evaluate the performance of the model when the potential ridership across demand centroids may take arbitrary values. To do so, we randomly generate the ridership for every combination of centroids and locations, that is $Q_{ij}^{mn}$ is chosen randomly and uniformly within a range $[R_{\text{min}}, R_{\text{max}}]$ for all $i \neq j \in C$ and for all $m \neq n \in L$.

We consider an instance which contains $|S| = 12$ station locations (including 2 terminals) and therefore 24 demand centroids. The budget required per station is chosen in the range $[1,10]$ for instance A and is constant in instance B. We use the following parameters values: $R_{\text{min}} = 1, R_{\text{max}} = 100, B_{\text{min}} = 1, B_{\text{max}} = 10, d_{\text{min}} = 0$ and $d_{\text{max}} = |S| - 1$.

This configuration ensures that a potential ridership of 100 is achieved for any combination of centroids and stations such that both centroids are assigned to the closest candidate location and that a unit ridership is obtained when centroids are assigned to the furthest candidate location. We generate an instance of the bi-criteria LRT design problem using a C++ routine and coordinate the resolution of the $10|L| = 100$ single-objective programs with the AMPL/CPLEX framework, all the programs are solved to optimality in less than 1 minute with default parameters. For both instances, we also measure the performance of the heuristic algorithm for the bi-criteria LRT design problem described in Figure 2. Namely, we report every non-dominated solution when the algorithm is executed with $T = 10$. The results obtained are for instance A are presented in Figure 4 and the results obtained for Instance B are presented in Figure 5.
Figure 4 Results for an instance on a LRT design problem with a uniformly distributed demand centroids and a randomly chosen required budget per station.

The left-hand side figure in Figure 4 shows the evolution of the ridership according to the step budget and represents the Pareto front (left-hand side axis, red points) and the evolution of the number of stations (right-hand side vertical axis, blue points). The right-hand side figure of figure 4 shows the remaining slack in the available budget for every iteration $k$. While the ridership increases monotonically with the step budget, the Pareto front exhibits some inflexion points and even flat segments. This shows that in some configurations, the design of such an LRT service may not always benefit from an increase in the budget. Indeed, the flat segments observed in the right-most part of the Pareto front show that an increase in the available resources does not always improve the total ridership provided by the LRT service. Similarly, this figure also highlights that the number of stations to be opened globally grows linearly with the step budget unless an increase in the step budget is not enough to open an additional station or if an alternative station improving the total ridership may be opened. The budget constraint is quite tight, as depicted by the right-hand side figure in Figure 4, apart from some iterations, for $k \geq 50$, where the incremental budget does not provide any improvement of the total ridership. This study illustrates the trade-offs involved in the design of the LRT service. The additional ridership obtained by adding an additional station to the transit service varies significantly depending on the current configuration. This also confirms that an increase in the total ridership may be obtained without opening an additional station but only increasing the available budget. From an optimization perspective, the heuristic algorithm is shown to fairly approximate the Pareto front although some non-dominated solutions appear to non-optimal. It should also be stressed that step budget used in the MILP model to span the Pareto front only reflects a discredited picture of the Pareto front.

Figure 5 presents the results obtained in instance B, i.e., with a uniform budget for candidate locations. In contrast to instance A, on the left-hand side figure, we observe that the ridership evolves logarithmically with the step budget (which directly corresponds to the number of stations that may be opened). Furthermore, the heuristic algorithm is able to accurately find non-dominated solutions of the bi-criteria LRT design problem. The right-hand side figure indicates that there is a remaining slack in the budget constraint for every iteration $k$ which is not a multiple of 10. This is to be expected since in this particular instance, every term $\epsilon_k$ for which $k$ is a multiple of 10 corresponds to the exact budget required to open $k/10$ stations.
4.2. Realistic Instance – LRT Design within the City of Sydney

In this section, we present the results obtained for a realistic instance within Sydney Metropolitan Area. The network used for this study is the one presented in Figure 1; this network encompasses 101 demand centroids and a total of 52 candidate locations for transit stations. Among these 52 locations, 3 of them represent terminals stations and are thus fixed in the model.

Due to the large size of the instance, the single-objective LRT design problem represented by Model 2 cannot be solved to optimality using conventional MILP algorithms, i.e. Branch-and-Bound or Branch-and-Cut, in reasonable time. Therefore to estimate the Pareto front of Model 2 for the City of Sydney instance, we use a tailored heuristic algorithm which combines clustering and simulation methods to efficiently generate a set of non-dominated solutions. This heuristic relies on the assumption that the variance of the demand between any pair of centroids is of low magnitude. This assumption is, to a reasonable extent, satisfied in the transportation network using fine aggregation levels of TAZ. This is because travel behaviour is not an arbitrary flow between zones. Instead it happens due to human-human and human-environment interactions which are commonly aggregated to the zone level in transport modelling. Neighbour zones share many attributes including similarities in socio-demographic attributes of residences and employees, as well as built form characteristics. Thus, it is quite surprising if neighbour zones show totally different travel patterns. If that happens, the heuristic algorithm still isolates the dissimilar zones to a new cluster identified from the surrounding zones.

The data used for the Sydney instance study to some extent supports this hypothesis. Figure 6 shows the distribution of travel productions and attractions in Sydney area surrounding the LRT line. Some neighbour zones appear to have dissimilar production and attraction patterns at the centre of maps of Figure 6. This is because of large areas of parks covering those zones which are located close to Sydney CBD.

The City of Sydney is among the cities with the most expensive real estate in the world. As a result land acquisition and construction is a costly task in constructing public transport projects. To account for this considerably high cost, land value for the proposed potential station sites are acquired from publicly available real estate data sources (Domain Real Estate Website).

![Figure 5](image1.png) Total Ridership against the Total Number of Stations opened. Results for an instance study on a LRT design problem with a uniformly distributed demand centroids and a randomly chosen required budget per station.

![Figure 6](image2.png) Distribution of travel production (right) and attraction (left) in zones surrounding the LRT line.
The results of the implementation of the heuristic algorithm on the City of Sydney network are detailed in Figure 7 which depicts the set of non-dominated and dominated solutions obtained, as well as the number of stations associated to each non-dominated solution. It can be seen from figure 7 that the objective function does not considerably improve as a result of more budget allocation beyond 14 stations.

Given a fixed total budget, after point B₁, relatively little benefit is obtained for project B while project A still can increase the overall benefit obtained from both projects. Therefore, there is no benefit to invest in project B up to point B₂ after which project B can reasonably return the investment. It can also be discerned from Figure 8 that project B has a high return up to point B₁ so the proposed budget allocation shown in the figure seems to be justifiable. Ignoring the pattern of the cost-benefit function may result in an investment with little outcomes.

This behaviour of the cost-benefit function may seem to be quite intuitive but it is commonly neglected in large scale project evaluations.

5 POLICY IMPLICATIONS
This study emphasizes on the importance of considering the properties of optimization of multi-objective functions when policies are evaluated. Specifically regarding large scale projects with practical complications, this study attempts to raise awareness about severe misleading consequences of neglecting these properties. We observed that the benefit (ridership) and cost (budget) do not necessarily present a monotonically increasing pattern. Even if a monotonically increasing pattern is generally and locally observed, the slope (the first derivative) of benefit against cost is not always declining. This is a crucial finding for policy assessment purposes, given limited budget.

Consider the case presented in Figure 8 which for the purpose of discussion exaggerates the fluctuation in the ridership-cost pattern. In this figure, two projects, for example an LRT project and a bus network expansion project, are considered. Having the benefit-cost diagrams for both projects and a given total budget, budget can be allocated to each project such that the total profit is maximized. This should be done with great precision because, for one project, given a specific budget allocation, increasing budget may not considerably increase the benefit, but at the same time for the other project marginally increasing the budget may considerably improve the benefits. This argument is schematically explained in Figure 8.

In short, for policy implementations regarding projects with multiple objectives, which actually includes most of the planning projects in large metropolitan areas, the trade-off between different objectives should be carefully studied, otherwise little benefit might be achieved through a large investment in one project which could have resulted in large benefit by shifting the investment into other projects.

6 SENSITIVITY ANALYSIS
The parameters of access travel time and in-vehicle travel considered in the elastic demand of equation 3 are approximations obtained from the demand model of the city of Sydney. In order to see the sensitivity of the network design to changes in these parameters, 10 and 20 percent positive and negative changes to these parameters are examined in this study. Positive changes mean the quality of service drops meaning that in-vehicle travel time or access travel times are considered by the traveller more cumbersome. Negative changes imply improvements in the
system so that travel times are interpreted less unpleasant by travellers compared to the base case scenario. Figure 9 shows the results for the 8 scenarios and the base scenario.

It can be seen in figure 9 that improvements in the system result in higher levels of ridership. This impact is more severe for changes in access travel time as the coefficient of access travel time is larger in the probability function used in equation 4.

**Figure 9** Dominated Solutions for the City of Sydney instance for 9 scenarios.

Demand in the neighbourhood of different stations varies depending on factors such as built form indicators like population density of the area. This can affect the chance of stations being included in the optimal solution. Figure 10 presents the frequency of the 49 stations being included in the optimal solution for the base scenario. As it can be seen in the figure, stations closer to Sydney CBD are more likely to be included in the optimal solution while in less dense urban areas such as closer to the southern terminal the chance of stations being included in the optimal solution is smaller.

**Figure 10** Frequency of stops being included in the nondominated solutions for the base scenario

7 CONCLUSIONS

In this paper we have introduced a mathematical formulation for bi-criteria LRT-NDP which seeks to find efficient locations for transit stations. Two objective functions, i) maximising the total ridership, ii) minimising the total budget were considered and represented using a MILP formulation. An elastic demand model based on the maximisation of random utility was developed to model the choice between transit service and private transport. The elastic demand function was estimated for all possible network design resulting in a matrix of demand values from every origin to all destination through each pair of stations. This way of using elastic demand in the optimization problem facilitated the solution algorithm.

A heuristic solution algorithm based on k-means clustering was proposed to approximate the Pareto front of the bi-criteria LRT-NDP on large instances. The performance of the heuristic algorithm was demonstrated on two benchmark instances for which all non-dominated solutions could be obtained. The solution algorithm was implemented on the City of Sydney network, in which realistic assumptions were made for the elastic demand function. Specifically, the assumptions included i) the demand for the LRT service between two zones be proportional to the population at the origin and the employment size at the destination, ii) the parameters of the
utility function for LRT were obtained from the strategic mode choice model of the City of Sydney, and iii) the assumptions on the average walking travel time and transit travel time.

In addition, this study explored the trade-off between budget and ridership in the context of designing a LRT network. More specifically, the rate of benefit for different amounts of investments was analysed. It was found that the increasing rate of benefit for more investments does not have a monotonic pattern, which implies that at some levels of budget allocation the rate of benefit gain is higher. The policy implications of this analysis extend to the instance where more than one project is of interest. Although the trade-off between the objective functions is intuitive, the severe consequences of ignoring this concept were emphasized in this paper to raise more awareness especially among decision makers.

Among the future research directions of this work, we seek to investigate the impact relaxing the assumptions of our model. Examining other objective functions including passenger-kilometre and equity is another direction to expand this research. To further realise the problem definition, station capacities should be introduced as restrictions in the mixed integer program which then affect the demand function as well by bringing in waiting time within the utility function.

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