Construction of Aggregation Operators
With Noble Reinforcement

Gleb Beliakov and Tomas Calvo

Abstract—This paper examines disjunctive aggregation opera-
tors used in various recommender systems. A specific require-
ment in these systems is the property of noble reinforcement: allowing a
collection of high-valued arguments to reinforce each other while
avoiding reinforcement of low-valued arguments. We present a new
construction of Lipschitz-continuous aggregation operators with
noble reinforcement property and its refinements.

Index Terms—Aggregation operators, fuzzy logic, information
fusion, Lipschitz aggregation operators, noble reinforcement.

I. INTRODUCTION

A G G R E G A T I O N of various pieces of evidence is an im-
portant step in most decision support systems, multicri-
teria decision making, and group decision making. Aggregation
operators are mathematical objects that perform precisely this
type of information fusion. For extensive overviews of different
classes of aggregation operators, see [1]–[4].

In [5], Yager introduced the property of “noble” reinforce-
ment of aggregation operators, which finds its use in various re-
commender systems. Consider an online store that recommends
customers various products, such as movies, music, or books,
based on users’ preferences and past purchases. The system re-
commends a number of products that may be of interest to the
user. The recommendation is based on aggregating the strengths
of justifications. Any justification provides a sufficient reason
to recommend a product, and the more and stronger the jus-
tifications are, the stronger is the recommendation. The avail-
able products are sorted and shortlisted with respect to the total
strength of recommendation.

In this context, aggregation of justifications should satisfy the
following requirements. It must be a disjunctive operator, such
that the aggregated score is no smaller than any of the individual
scores, symmetric, satisfy \( f(1, x) = 1, f(0, x) = x \), and possess
noble reinforcement property, that is, only reinforce sufficiently
high scores (see Section II). The aim of the latter property is to
avoid mutual reinforcement of low scores: if an item has several
very weak justifications, we should not recommend it.

While the first three requirements are easily met by choosing
a triangular conorm, or any other disjunctive aggregation opera-
tor, the requirement of noble reinforcement prompted Yager to
study a number of new constructions.

In particular, after defining disjunctive aggregation opera-
tors with noble reinforcement property for a crisp threshold
(based on ordinal sums of triangular conorms), he applies
Takagi–Sugeno–Kang (TSK) fuzzy methodology [6] to accom-
modate an inherently fuzzy character of the noble reinforce-
ment requirement, namely, fuzzy set sufficiently high score.
Further, he extends noble reinforcement property to include a
minimal number of high scores required for reinforcement.

In this paper, we develop an alternative construction of
Lipschitz-continuous aggregation operators with the above-mentioned properties, which is easy to implement
in a computer program, and which is also easy to extend
to match other requirements, such as cancelative behaviour,
absorbent elements, symmetry (or lack of it), and other types
of interpolatory conditions. Lipschitz continuity is associated
with the stability of aggregation procedure [7], and functions
from the class of \( p \)-stable aggregation operators (i.e., with
the Lipschitz constant \( M = 1 \), which includes popular \( 1 \)-Lipschitz
and kernel operators) do not increase input errors.

We state from the beginning that our construction is of
interpolatory type, i.e., it is based on interpolation (or ap-
proximation) of certain prototypical tuples of arguments and
aggregated values. Such tuples (and their subsets) correspond
to some easily understood prototypical situations, which
characterize the desired aggregation operator. Interpolatory
techniques are very flexible and are used in various contexts,
for instance, fitting parameters of a t-conorm [5], fitting weights
of ordered weighed averaging (OWA) operators [8], [9], fitting
coefficients of a fuzzy measure [10]–[12], fitting generator
functions [13], [14], and fitting general aggregation operators
with desired properties to data [15].

Our approach is based on a general method of monotone in-
terpolation of scattered data developed in [16]. It provides not
only an aggregation operator matching specified requirements
but also an interval of values any such operator can take, and
thus produces the largest, the smallest, and the optimal aggre-
gation operators.

The rest of this paper is organized as follows. Section II pro-
vides the basic definitions and also discusses Yager’s noble rein-
fforcement property and optimal Lipschitz interpolation method.
In Section III, we develop aggregation operators with noble re-
forcement property. In Section IV, we discuss a number of
refinements and generalizations. This paper concludes with a
short summary.

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II. BASIC DEFINITIONS

A. Aggregation Operators

Denote by \( I = [0,1] \) the unit interval. We shall use \( x, y, z \) to denote vectors in \( I^n \), i.e., \( x = (x_1, \ldots, x_n) \).

**Definition 1:** [4] An aggregation operator is a function \( F : \bigcup_{n \in \mathbb{N}} I^n \to I \) such that

i) \( F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n) \) whenever \( x_i \leq y_i \) for all \( i \in \{1, \ldots, n\} \).

ii) \( F(t) = t \) for all \( t \in I \).

iii) \( F(0, \ldots, 0) = 0 \) and \( F(1, \ldots, 1) = 1 \).

Each aggregation operator \( F \) can be represented by a family of \( n \)-ary aggregation operators \( f_n : I^n \to I \) given by \( f_n(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) \). We list a number of important properties and classes of aggregation operators; for a detailed discussion see [1]–[4]. To simplify the notation, we shall use operators \( \min(x) \) and \( \max(x) \) componentwise and will also understand componentwise vector inequalities \( x \leq y \).

- An aggregation operator \( f_n \) is said to have conjunctive behavior on \( I^n \) if \( f_n(x) \leq \min(x) \).
- Similarly, an aggregation operator has disjunctive behavior on \( I^n \) if \( f_n(x) \geq \max(x) \).
- An aggregation operator has a neutral element \( e \in I \) if

\[
\forall n \in \mathbb{N}, \quad \forall x \in I^n : f_n(x, e, \ldots, e) = f_n(x)
\]

- An aggregation operator is idempotent if \( f_n(t, t, \ldots, t) = t \) holds for every \( t \in I \).
- An aggregation operator \( f_n \) is symmetric if \( f_n(x) = f_n(x^P) \), where \( x^P \) is any permutation of the components of \( x \).
- An \( n \)-ary aggregation operator \( f_n \) is called Lipschitz if there is a constant \( M > 0 \), such that

\[
\forall n \in \mathbb{N}, \quad \forall x, y \in I^n : |f_n(x) - f_n(y)| \leq M|x - y|
\]

in some norm \( \| \cdot \| \). The smallest such number \( M \) is called the Lipschitz constant of \( f_n \).
- An aggregation operator \( f_n \) is called \( 1 \)-Lipschitz if its Lipschitz constant in \( \| \cdot \|_1 \) norm is 1.
- An aggregation operator \( f_n \) is called kernel if its Lipschitz constant in \( \| \cdot \|_\infty \) norm is one.

The representation of an aggregation operator \( F \) as a family of \( n \)-ary aggregation operators allows one to define the respective properties of \( F \) through those of \( f_n \). A property is valid for \( F \) if it is valid for all \( f_n, n > 1 \).

Associativity of aggregation operators allows one to construct the whole family \( F \) from a binary operator \( f_2 \). This is a rather technical property, which in many cases does not arise from application requirements (see [5]) and on the other hand is quite restrictive. Prototypical examples of associative aggregation operators are triangular norms and conorms, uninorms, and nullnorms. The severe restrictions imposed by associativity prompted Yager to consider aggregation operators he calls generalized OR (GENOR) and generalized AND (GENAND), which fulfill the properties of triangular conorms and norms except associativity [5].

B. Noble Reinforcement

Yager [5] has indicated a number of requirements on the aggregation operator to be used in recommender systems. In this context, the aggregation procedure should be disjunctive. Any feature (justification) is sufficient to justify a recommendation by itself; however, several justifications provide a stronger recommendation. Hence the aggregation operator is bounded by maximum from below. Further, the aggregation operator needs to be symmetric and have zero as the neutral element.

All these features are obtained by using any t-norm operator. However, there exists one additional property that restricts the choice of \( F \). Yager [5] expresses it as follows.

**Requirement 1:** “If some justifications highly recommend an object, without completely recommending it, we desire to allow these strongly supporting scores to reinforce each other.”

The key element here is that only high values of arguments of \( F \) allow reinforcement. This avoids the situation when a number of low scores provide a strong recommendation because of reinforcement. For example, taking a bounded sum t-conorm \( F(x) = \max\{1, \sum x_i\} \) with ten arguments, ten low values of 0.1 provide a full recommendation. Dual product t-conorm \( F(x) = 1 - \prod (1 - x_i) \) also has a similar effect, although less pronounced. In the context of recommender systems (and also search engines), this behavior is very undesirable because it results in long and inappropriate recommendation lists.

On the other hand, \( \max \) operator does not provide any reinforcement. Yager proposed to use fuzzy TSK methodology [6] to build the required aggregation operators. First take a crisp threshold \( \alpha \) (to characterize high values) and express noble reinforcement as follows. Let \( E \) denote a subset of indexes \( E \subseteq \{1, \ldots, n\} \) and \( \bar{E} \) denote its complement.

**Definition 2:** An aggregation operator \( F_\alpha \) has a noble reinforcement property with respect to a crisp threshold \( \alpha \) if it can be expressed as

\[
F_\alpha(x) = \begin{cases} 
A_{i \in E}(x), & \text{if } \exists E \subseteq \{1, \ldots, n\} \forall i \in E : x_i \geq \alpha \\
\max(x), & \text{otherwise} 
\end{cases}
\]

where \( A_{i \in E}(x) \) is a disjunctive aggregation operator (i.e., greater than or equal to maximum), applied only to the components of \( x \) with the indexes in \( E \).

That is, only the components of \( x \) greater than \( \alpha \) are reinforced.

Next define a fuzzy set high by means of a membership function \( \mu_h(t) \). Note that \( \mu_h(t) \) is monotone, increasing bijection of \( I \). Let us denote by \( \min(E) = \min_{i \in E} x_i \). Calculate the value of the aggregation operator using

\[
F(x) = \max_E \{ \mu_h(\min(E)) A_{i \in E}(x) + (1 - \mu_h(\min(E))) \max(x) \} \\
= \max_E \{ \max(x) + \mu_h(\min(E))(A_{i \in E}(x) - \max(x)) \}. \tag{2}
\]
As a disjunctive operator, A Yager uses a t-conorm \( S \) and obtains an associative function \( F_\alpha \). It is also possible to use other associative disjunctive functions, such as t-supernorms [17]. However, despite associativity of \( F_\alpha \), the aggregation operator \( F \) is no longer associative. But because it is built from associative functions \( \max \) and \( S \), it can be easily computed for any dimensionality \( n \). Further, Yager shows that the maximum with respect to \( E \) is attained when \( E = G_j \) and \( \min(E) = x(j) \) for some index \( j \), where \( x(j) \) is the \( j \)th largest component of \( x \) and \( G_j = \{ i : x_i \geq x(j) \} \) is the subset of \( j \) largest components of \( x \). This greatly simplifies the calculations.

As an example of an associative operator with noble reinforcement property (for a crisp set high) of type (1), Yager defines a t-conorm

\[
F(x) = \begin{cases} 
\max(x), & \text{if } \min(x) < \alpha \\
1, & \text{otherwise},
\end{cases}
\]

It is a generalization of the drastic sum and provides full reinforcement for high values of \( x \). However, it has a limited versatality. It is straightforward to generalize this result.

**Proposition 1:** Define a triangular conorm by means of an ordinal sum

\[
S_\alpha(x_1, x_2) = \begin{cases} 
\alpha + (1 - \alpha)S \left( \frac{x_1}{\max(x_1, x_2)}, \frac{x_2}{\max(x_1, x_2)} \right), & \text{if } x_1, x_2 \geq \alpha \\
\max(x_1, x_2), & \text{otherwise}
\end{cases}
\]

where \( S \neq \max \), \( S_\alpha(x_1, \ldots, x_n) \) (defined by using associativity for any dimension \( n \)) has noble reinforcement property.

**Proof:** First, the ordinal sum is indeed a t-conorm [17], i.e., it is associative (monotonicity, symmetry, and neutral element are straightforward to prove). Then \( S_\alpha(x_1, \ldots, x_n) \) is defined for any \( n \) and \( S_\alpha(x_1, \ldots, x_n) = S_\alpha(S_\alpha(x_1, x_2), S_\alpha(x_3, \ldots, x_n)) \), where \( x_I \) denotes the subset of components of \( x \) such that \( \forall i \in I : x_i \geq \alpha \) and \( x_J \) denotes its complement to \( x \), i.e., \( \forall i \in J : x_i < \alpha \). But \( S_\alpha(x_I) = \max(x_I) < \alpha \). Hence if \( x_I \) is nonempty \( S_\alpha(x_1, \ldots, x_n) = \max(S_\alpha(x_I), \max(x_J)) \) \( = S_\alpha(x_J) \geq \max(x_1, \ldots, x_n) \). If all components of \( x \) are smaller than \( \alpha \), \( S_\alpha(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n) \). Thus \( S_\alpha \) verifies the noble reinforcement property.

**Remark 1:** As opposed to using a t-conorm \( S \) in (1) directly, we use a scaled version of a t-conorm \( \tau \) in (3), which guarantees continuity of \( S_\alpha \) as long as \( S \) itself is continuous. In fact one can also use a t-supernorm [17] (which is a commutative and associative aggregation operator greater than \( \max \)) rather than a triangular conorm \( \tau \). T-supernorms need not have a neutral element, which implies that the operator \( S_\alpha \) will be discontinuous.

Using Proposition 1 and construction (2), we can build aggregation operators with noble reinforcement with respect to a fuzzy set high. However, such an operator may not be exactly what is required for recommender systems.

Yager refines user requirements further and introduces conditions related to the cardinality of subsets of mutually reinforcing arguments. A refinement of noble reinforcement follows.

**Requirement 2:** Provide reinforcement if at least \( k \) arguments are high.

For a crisp threshold \( \alpha \), we give the following definition.

**Definition 3:** An aggregation operator \( F_\alpha \) provides noble reinforcement of at least \( k \) arguments if it can be expressed as

\[
F_\alpha(x) = \begin{cases} 
A_{i \in E}(x), & \text{if } \exists E \subseteq \{1, \ldots, n\}, |E| \geq k, \\
\forall i \in E : x_i \geq \alpha, \text{ and } \forall i \in \bar{E} : x_i < \alpha, & \text{otherwise}
\end{cases}
\]

where \( A_{i \in E}(x) \) is a disjunctive aggregation operator applied to the components of \( x \) with the indexes in \( E \).

To fuzzify the crisp threshold \( \alpha \), we apply (2). Now we cannot obtain an associative aggregation operator \( F_\alpha \) verifying (4); see the discussion in [5]. This is one of the reasons for exploring other constructions, as in [5]. Yager’s construction involves a nondecreasing permutation of components of \( x \) denoted by \( x(\cdot) \) and an aggregation operator defined as

\[
F(x) = \max(x) + (1 - \max(x)) \max_{j=1}^{\min(x)} \mu_j(x(j))
\]

with \( \mu_j(x) \) denoting the membership function of the fuzzy set the minimal number of components. The higher the number of high components of \( x \), starting from some minimal number, the stronger is the reinforcement.

The goal of the subsequent sections is to develop a general and simple approach to construction of Lipschitz-continuous aggregation operators that exhibit noble reinforcement property expressed in (1) or (4). Such operators can be found for any crisp threshold \( \alpha \). Fuzzification of the threshold is performed by applying (2). A similar fuzzification procedure is performed with respect to the minimal number of components required for reinforcement.

**C. Optimal Lipschitz Interpolation**

First we note that when expressed in words, noble reinforcement does not require \( A_{i \in E} \) in (1) or (4) to be exactly a triangular conorm; we can use any symmetric disjunctive aggregation operator. We are also interested in Lipschitz-continuous aggregation operators because they are stable with respect to input inaccuracies [7]. We shall need the following results on optimal interpolation.

Suppose that we have a set of desired values of the aggregation operator \( D = \{(x, f_n(x)) : x \in \Omega \} \) and the Lipschitz condition

\[
\exists M > 0, \text{ such that } \forall x, z \in \Omega : |f_n(x) - f_n(z)| \leq M|x - z|.
\]

We denote the set of functions whose Lipschitz constant is no greater than \( M \) by \( \text{Lip}(M) \) and the set of monotone functions by \( \text{Mon} \). We assume that \( \Omega \) is a compact set.

The data are consistent with the Lipschitz condition and monotonicity if \( f_n \in \text{Lip}(M) \cap \text{Mon} \). Then tight upper and lower bounds on any function from the set \( \text{Lip}(M) \cap \text{Mon} \) that interpolate the data are given (see [16]) by

\[
\sigma_u(x) = \min_{z \in \Omega} \{|f_n(x) + M|(x - z)_+|\}
\]

\[
\sigma_l(x) = \max_{z \in \Omega} \{|f_n(x) - M|(z - x)_+|\}
\]

(5)
where $\xi_k$ denotes the positive part of vector $\xi : \xi_+ = (\xi_1, \ldots, \xi_n)$, with

$$\bar{\xi}_i = \max\{\xi_i, 0\}.$$

The optimal interpolant is the one that minimizes the largest error in the worst case scenario

$$g(x) = \min_{\mathcal{V}} \max_{\nu \in \mathcal{V}} \|u - f\|_{\infty},$$

where $\mathcal{V} = \{v \forall x : v(x) = f_n(x)\}$ is the set of all functions that interpolate the data. The solution is given by the central scheme [18] as

$$g(x) = \frac{1}{2}(\sigma_f(x) + \sigma_u(x)). \quad (6)$$

The monotone interpolation method developed in [16] is based on resolving (5) explicitly for various choices of the set $\Omega$.

In the case of aggregation operators, we immediately obtain the general bounds

$$B_u(x) = \min\{M||x||, 1\}$$

$$B_l(x) = \max\{1 - M||1 - x||, 0\}$$

which follow from the interpolation conditions $f_u(0, \ldots, 0) = 0$ and $f_l(1, \ldots, 1) = 1$. For $p$-stable aggregation operators with $M = 1$ [7], we obtain Yager triangular norms and conorms as the bounds (the parameter depends on the norm used in the Lipschitz condition)

$$T_1(x) \leq f_n(x) \leq S_1(x),$$

The construction method we develop in the sequel is based on identifying the subset $\Omega$ from the noble reinforcement requirements (1) or (4), and resolving (5) to obtain the bounds $\sigma_u, \sigma_l$ which will be applied in conjunction with the general bounds $B_u, B_l$ as

$$\min\{\sigma_u(x), B_u(x)\}$$

$$\max\{\sigma_l(x), B_l(x)\}.$$  

It is often required to obtain a symmetric function that interpolates the data. This is the case in the context of recommender systems, hence it was one of Yager's requirements [5]. A simple technique is to consider the simplex $D = \{x \in I^n | x_1 \geq x_2 \geq \ldots \geq x_n\}$ as the basic domain (i.e., apply the interpolation method on $D$) and then to extend the function to $I^n$ by using $f(x) = f_D(x_1)$, where $f_D$ is the interpolant defined on $D$ and $x_1$ is a nonincreasing permutation of the components of $x$ (see, e.g., [15]). Of course, the interpolation conditions should be restricted to $D$ as well.

III. CONSTRUCTION OF THE AGGREGATION OPERATOR

Consider (1) with a fixed $\alpha$. Denote by $\bar{E}$ a subset of indexes $\{1, \ldots, n\}$ and by $\hat{E}$ its complement. For $k = 0, \ldots, n$, denote by $E_k$ the set of points in $I^n$ that have exactly $k$ coordinates greater than $\alpha$, i.e.,

$$E_k = \{x \in I^n | x \in E, \text{such that } |E| = k, \forall i \in E : \alpha < x_i \leq 1 \text{ and } \forall j \in \hat{E} : x_j \leq \alpha\}.$$

The subsets $E_k$ form a nonintersecting partition of $I^n$. Further, $E_0 \cup E_1 \cup \cdots \cup E_n$ is a compact set.

Definition 2 of noble reinforcement implies that $f_n(x) = \max\{x(x)\}$ on $E_0$ and $f_n(x) \geq \max\{x\}$ on the rest of the domain, and further $f_n(x) \geq f_n(y)$ for all $x \in E_k, y \in E_m$, $k > m$. The latter is due to monotonicity of disjunctive aggregation operators with respect to argument cardinality. Also, since no reinforcement can happen on the subset $E_1$, we have $f_n(x) = \max\{x\}$ on $E_1 \cup E_0$. This expresses the essence of noble reinforcement requirement.

Let us now take some Lipschitz constant $M \geq 1$ and determine the upper and lower bounds $\sigma_u, \sigma_l$ on $f_n$ from (5). We use $\Omega = E_1 \cup E_0 \cup \{(1, \ldots, 1)\}$, as this is the set on which the values of the aggregation operator are specified.

The datum $f_n(1, \ldots, 1) = 1$ implies the upper bound $f_n(x) \leq 1$. The general lower bound is $f_n(x) \geq \max\{x\}$ due to the disjunctive character of the operator. Now we need to find the upper bound $\sigma_u$ that results from the condition $f_n(x) = \max\{x\}$ on $E_1 \cup E_0$.

Thus for any fixed $x$, we need to compute

$$\sigma_u(x) = \min_{z \in E_1 \cup E_0} \{\max\{z\} + M||(x - z)_{+}||_p\}.$$

Our technique is to reduce the $n$-variate minimization problem to $n$ univariate problems. Consider $x \in E_k$, for a fixed $k$, $1 \leq k \leq n$, which means that $k$ components of $x$ are greater than $\alpha$. Let $j \in E$ be some index such that $x_j > \alpha$. Next we show that the minimum is achieved at $z^*$ whose $j$th component $z^*_j \in [\alpha, x_j]$ is given below, and the rest of the components are fixed at $z^*_i = \alpha, i \neq j$. That is, we only need to find the optimal value of the component $z_j$ and then take the minimum over all $j \in E$.

To show this, note that $||x - z||_+ = \min\{z, |x - z|\}_{+}$ is a decreasing function of $z_j$ for $0 \leq z_j \leq x_j$ and nonincreasing for $x_j \leq z_j \leq \alpha$, if $x_j < \alpha$. Thus the minimum with respect to those components $z_{i \neq j}, i \in \hat{E}$, such that $x_i \leq \alpha$ is achieved at any $z^*_j \in [x_j, \alpha]$, and the contribution of the terms $(x_i - z^*_i)_{+}, i \in \hat{E}$ is null. The expression for $\sigma_u$ becomes

$$\sigma_u(x) = \min_{\mathcal{V}} \min_{\nu \in \mathcal{V}} \max\{\nu \in E, \nu_i \leq z_i \}
\left\{\max\{z_i\} + M \left(\sum_{i \in E} (x_i - z_i)^p\right)^{1/p}\right\}.$$

Note that only one component of $z$ is allowed to be greater than $\alpha$ when $z$ ranges over $\Omega = E_1 \cup E_0$. Denote this component by $j \in E$ and denote by $\gamma_j = \sum_{i \in E, i \neq j} (x_i - z_i)^p$. Note that $\max_{i \in E} z_i = z_j$. Then the minimum of $\gamma_j$ with respect to $z_j, i \in E, i \neq j$ is achieved at $z^*_j = \alpha$. Denote it by $\gamma^*_j = \sum_{i \in E, i \neq j} (x_i - z_j)^p$. Hence we have $k = |E|$ univariate problems

$$\sigma_u(x) = \min_{j \in E} \min_{\alpha \leq z_j \leq x_j} \{z_j + M(\gamma^*_j + (x_j - z_j)^p)^{1/p}\}. \quad (7)$$
Consider the expression under the minimum over $j$. It is a convex function of $z_j$ and hence will have a unique minimum (possibly many minimizers). The following proposition [19] establishes this minimum.

**Proposition 2:** Let $\gamma \geq 0$, $M \geq 1$, $p \geq 1$, $\alpha, \beta \in I$, $\alpha \leq \beta$, and

$$f_\beta(t) = t + M((\beta - t)^p + \gamma)^{1/p}.$$  

The minimum of $f_\beta(t)$ on $[\alpha, \beta]$ is achieved at

- $t^* = \alpha$, if $M = 1$;
- $t^* = \beta$, if $p = 1$ and $M > 1$;
- $t^* = \text{mod}(\alpha, \beta - ((\gamma)/(M(1/(p-1))+(1/(p-1))(\beta-\gamma))$, otherwise,

and its value is

$$\min f_\beta(t) = \begin{cases} M(\gamma + (\beta - \alpha)^p)^{\frac{1}{p}}, & \text{if } t^* = \alpha \\ \beta + M\gamma^{\frac{1}{p}}, & \text{if } t^* = \beta \\ \beta + \left(\frac{\gamma}{(1/(p-1))+(1/(p-1))}\right)^{\frac{1}{p}}, & \text{otherwise}, \end{cases}$$  

(8)

The proof is based on examining the critical points of $f_\beta(t)$ on the specified interval and handling the special cases $M = 1$ and $p = 1$. The main lines are as follows.

The critical points of $f_\beta(t)$ on $[\alpha, \beta]$ are $\alpha, \beta$ and possibly the solution to

$$1 = M\frac{(\beta - t)^{p-1}}{\gamma + (\beta - t)^{p}} = M\frac{(\beta - t)^{p-1}}{\gamma + \gamma^{\frac{1}{p}}},$$  

$p > 1$.

We resolve it for $t$ and ensure it falls within $[\alpha, \beta]$ (hence we take the median). If $M = 1$, the derivative is nonnegative on $[\alpha, \beta]$ and the minimum is achieved at $t = \alpha$. Finally, if $p = 1$, $f_\beta(t) = t(1 - M) + \gamma + M\beta$ on $[\alpha, \beta]$ and is decreasing for $M > 1$.

Thus for each $j \in E$, we obtain the optimum by replacing $\beta$ with $x_j$ and $\gamma$ with $\gamma_j^\alpha$ in (8). The final expression for $\sigma_u(x)$ is obtained by taking the minimum of these optima over all $j \in E$ as in (7).

Function $\sigma_u(x)$ is obviously a Lipschitz continuous function with the Lipschitz constant $M \geq 1$. $M$ serves as a parameter that controls the degree of reinforcement. If $M = 1$, $p = \infty$ (kernel aggregation operators), no reinforcement takes place as $\sigma_u = \max$ on the whole domain.

The largest and the smallest aggregation operators with the desired behavior are given by

$$U(x) = \min\{1, \sigma_u(x)\}$$

and

$$L(x) = \max(x)$$

and the optimal aggregation operator is $F_\alpha(x) = (1/2)(U(x) + L(x))$. Interestingly, when $M = 1$, we obtain that $U(x) = S_\alpha(x)$, the ordinal sum given in Proposition 1, with $S$ being Yager t-conorm with parameter $p$.

**Example 1:** Consider the case of 1-Lipschitz aggregation operators $M = 1, p = 1$. Define the subset $E \subseteq \{1, \ldots, n\}$ as in Definition 2. The minimum in (7) is achieved at $z_j = \alpha$ for every $j \in E$. The upper bound $U(x)$ is given as

$$U(x) = \min\left(1, \alpha + \sum_{i|x_i > \alpha} (x_i - \alpha)\right).$$

The largest 1-Lipschitz aggregation operator with no reinforcement with threshold $\alpha$ is given as

$$F_\alpha(x) = \begin{cases} \min\{1, \alpha + \sum_{i|x_i > \alpha} (x_i - \alpha)\}, & \text{if } \exists x_i > \alpha \\ \max(x), & \text{otherwise} \end{cases}$$  

(9)

which is the ordinal sum of Lukasiewicz t-conorm and max.

The optimal aggregation operator is

$$f_n(x) = \frac{1}{2}(\max(x), F_\alpha(x))$$

which is no longer a t-conorm.

We would like to mention that even though we did not use associativity, the aggregation operator $F_\alpha$ is defined for any number of arguments in a consistent way, so that the neutral element is zero. To pass from a crisp threshold $\alpha$ to a fuzzy set $\text{high}$, we apply as earlier (2), with operator $A$ replaced by $F_\alpha$ and $\alpha$ taking values in the discrete set $\{x_1, \ldots, x_n\}$.

Now consider the requirement (4), which involves cardinality of $E$. It reads that $F_\alpha(x)$ is maximum whenever less than $k$ components of $x$ are greater than or equal to $\alpha$. Therefore we use the interpolation condition $f_n(x) = \max(x)$ on $\Omega = E_0 \cup E_1 \cup \cdots \cup E_{k-1}$. As earlier, $\sigma_u$ is given by

$$\sigma_u(x) = \min_{z \in \Omega = E_0 \cup E_1 \cup \cdots \cup E_{k-1}} \{\max(z) + M\|(z - x)\|_p\}.$$  

(10)

Let us compute this bound explicitly. We have an $n$-variate minimization problem, which we intend to simplify. As earlier, $x$ is fixed and $E$ denotes the subset of components of $x$ greater than $\alpha$, $\tilde{E}$ denotes its complement, and $|E| \geq k$. The minimum with respect to those components of $z$ whose indexes are in $\tilde{E}$ is achieved at any $z^*_i \in [x_i, \alpha]$, $i \in \tilde{E}$. So we fix these components, say, at $z^*_i = \alpha$ and concentrate on the remaining part of $z$.

At most $k-1$ of the remaining components of $z$ are allowed to be greater than $\alpha$ when $z$ ranges over $\Omega$; we denote them by $z_{K_1}, \ldots, z_{k-1} \in K \subseteq E$, $|K| = k - 1$. The minimum with respect to the remaining components is achieved at $z^*_i = \alpha$, $i \notin K$. Now take all possible subsets $K \subseteq E$ and reduce the $n$-variate minimization problem to a number of $k$-1-variate problems with respect to $z_{K_1}, \ldots, z_{k-1}$.

$$\sigma_u(x) = \min_{K \subseteq E, |K| = k-1} \min_{z \in K} \max(z_{K})$$

$$+ M\left(\sum_{i \in K} (x_i - \alpha)\right)^{1/p} + \sum_{i \in K} (x_i - z_i^\alpha)^1,$$

\(1/p\)
Denote by $\gamma^e_K = \sum_{i \in E \setminus K} (x_i - \alpha)^p$

$$\sigma_u(x) = \min_{K \subseteq E, |K| = k-1} \min_{z_i \in \mathbb{R}} \left\{ \max_{i \in K} (z_i) + \frac{M}{1/p} \left( \gamma^e_K + \sum_{i \in K} (x_i - z_i)_+^p \right) \right\}. \tag{11}$$

Now we show that the minimum for a fixed $K$ is achieved when all the variables $z_i, i \in K$ are equal, and hence obtain a univariate minimization problem. Let us arrange the components of $x, x_i, i \in K$, in decreasing order, so that $x_{K_1} \geq \ldots \geq x_{K_{k-1}}$. Next we show that we can rewrite the previous expression as

$$\sigma_u(x) = \min_{K \subseteq E, |K| = k-1} \min_{t \in [\alpha, x_{K_1}]} \left\{ t + M \left( \gamma^e_K + \sum_{i \in K} (x_i - t)_+^p \right) \right\}. \tag{11}$$

Let us consider a fixed value of $z_{K_1} \in [\alpha, x_{K_1}]$. The minimum

$$\min_{z_i \in [\alpha, x_{K_1}]} \sum_{i \in K} (x_i - z_i)_+^p = \min_{z_i \in [\alpha, x_{K_1}]} \sum_{i \in K} (x_i - z_i)_+^p$$

because values of $z_i$ larger than $x_i$ do not augment the sum. The minimum of this expression is achieved at all $z_i^* \in [x_{K_1}, x_{K_1}]$ since the terms $(x_{K_1} - z_i)_+^p$ are null. On the other hand, the function $\max_{i \in K} (z_i) + M \left( \gamma^e_K + \sum_{i \in K} (x_i - z_i)_+^p \right)$ is increasing in $z_i$ for $z_i \geq x_{K_1}$ but is constant for $z_i \in [x_{K_1}, x_{K_1}]$. Thus the point $z_i^* = z_{K_1}, i \in K$ is a minimizer for any fixed $z_{K_1} \in [\alpha, x_{K_1}]$. Therefore, we only need to consider minimization with respect to the component $z_{K_1}$ on $[\alpha, x_{K_1}]$, as all the other components $z_i^*$ are determined automatically at an optimum value for any $z_{K_1}$. Hence we need to solve (11) (with $t = z_{K_1}$).

The minimum over all subsets $K$ in (11) has to be computed exhaustively. For the inner problem (for a fixed $K$), we have

$$\min_{t \in [\alpha, x_{K_1}]} \left\{ t + M \left( \gamma^e_K + \sum_{i \in K} (x_i - t)_+^p \right) \right\}. \tag{12}$$

Note that the objective function is convex and piecewise smooth. Consider a partition of the interval $[\alpha, x_{K_1}] : [\alpha, x_{K_{m-1}}], [x_{K_{m-1}}, x_{K_{m-2}}], \ldots, [x_{K_{k-2}}, x_{K_{k-1}}].$ Taking the derivative on each subinterval and equalling it to zero, we have the following generic equation, $m = 2, \ldots, k - 1:

$$M \left( \gamma^e_K + \sum_{i = m, m+1, \ldots, k_1} (x_i - t)_+^p \right) \times \sum_{i = m, m+1, \ldots, k_1} (x_i - t)_+^{p-1} = 1. \tag{13}$$

If the solution falls within $[x_{K_{m}}, x_{K_{m-1}}]$, we obtain a critical point (in addition to critical points $x_{K_{m}}$). We find the minimum by comparing function values at all critical points.

For the special case $p = 1, M \geq 1$, we have $t = x_{K_1}$ and

$$\sigma_u(x) = \min_{K \subseteq E, |K| = k-1} \left( x_{K_1} + M \sum_{i \in E \setminus K} (x_i - \alpha) \right). \tag{14}$$

For $p = 2$, we obtain a quadratic equation in $t$, which we can solve explicitly. Otherwise, we can use the method of bisection to solve (13) or solve (12) directly, e.g., using golden section method.

In summary, the upper bound on a Lipschitz noble reinforcement aggregation operator of level $k$ is given by (10). This complicated optimization problem is simplified and takes the form of a number of univariate minimization problems with piecewise smooth convex objective function (12). We need to find the minimum in (12) for all subsets $K \subseteq E, |K| = k-1,$ where $E$ is the subset of indexes of the components of $x$ that are greater than or equal to $\alpha$.

**Example 2:** Consider again the case of 1-Lipschitz aggregation operators $M = 1, p = 1,$ for $n = 4$ and the requirement that at least $k = 3$ high arguments reinforce each other. We reorder the inputs as $x(1) \geq x(2) \geq x(3) \geq x(4)$. Applying (14), we have the largest 1-Lipschitz operator

$$F_\alpha(x) = \begin{cases} \min\{1, x(1) + x(3) - \alpha\}, & \text{if } x(4) \leq \alpha \text{ and } x(1), x(2), x(3) > \alpha \smallsetminus \min\{1, x(1) + x(3) + x(4) - 2\alpha\}, & \text{if all } x(1), \ldots, x(4) > \alpha \smallsetminus \max(\alpha), & \text{otherwise.} \end{cases} \tag{15}$$

The optimal aggregation operator is

$$f_d(x) = \frac{1}{2}(\max(x), F_\alpha(x)).$$

Finally, if fuzzification is required (to accommodate a fuzzy set high or a fuzzy set at least $k$ components), we proceed as in [5] and apply

$$F(x) = \max_{E \setminus k} \left\{ \max_{E \setminus k} \{\mu_h(\min(E))\mu_q(k)(U(x) - \max(x))\} \right\} \tag{16}$$

where $U(x) = \min\{\sigma_u(x), 1\}$ is computed for fixed $k$ and $E$, such that $\min(E) = \alpha_i$ for some $i$.

The algorithmic implementation of (7) or (11) is straightforward (the former is a special case of the latter).

**Algorithm 1**

**Purpose:** Find the upper bound $\sigma_u(x)$ given by (10).

**Input:** Vector $x$, threshold $\alpha$, subset $E$, cardinality $k$.

**Lipschitz constant $M$, norm $\| \cdot \|_p$.**

**Output:** The value $\sigma_u(x)$.
Step 1 For every subset $K \in E, |K| = k \rightarrow 1$
do
Step 1.1 Compute $\gamma^K_i = \sum_{x \in K} (x_i - \alpha)^p$.\nStep 1.2 Compute the largest component\n$\sigma_K = \max_{x \in K} x_i$.\nStep 1.3 Find the minimum\n$\sigma_K = \min_{x \in \alpha, x_K} \{(t + M(\gamma^K_i + \sum_{x \in K} (x_i - \alpha))^{1/p})\}$\nby using golden section method.
Step 2 Compute $\sigma_u = \min_{K \subseteq E} \sigma_K$.
Step 3 Return $\sigma_u$.

Algorithm 2

Purpose: Compute the value of an aggregation operator with noble reinforcement of at least $k$ components (4).

Input: Vector $x$, threshold $\alpha$, the minimum cardinality $k$ of the subset of reinforcing arguments, Lipschitz constant $M$, norm $||\cdot||_p$.

Output: The value $F(x)$.

Step 1 Compute the subset of indexes $E = \{i | x_i > \alpha\}$.
Step 2 Call Algorithm 1 $(x, \alpha, E, k, M, p)$ and save the output in $\sigma_w$.
Step 3 Compute $F = (\min\{1, \sigma_u\} + \max(x))/2$.
Step 4 Return $F$.

Note 1: When the noble reinforcement property does not involve the minimum cardinality $k$ of the set of high arguments$ [property (1)], use $k = 2$.

Note 2: In the special cases $k = 2$ or $p = 1$ or $p = 2$, one may use explicit formulas (8) and (14) or examine critical points by solving (13), which are faster than the golden section method.

IV. Extensions and Refinements

A. Independent Criteria

The requirement of noble reinforcement can be further refined to accommodate users’ expectations and also the relations between the aggregated criteria. Yager [5] proposes one such refinement, which accommodates the situation with correlated criteria. In many systems, notably the recommender systems, some criteria may not be independent, e.g., when various justifications measure essentially the same concept. It is clear that mutual reinforcement of correlated criteria should be smaller than reinforcement of truly independent criteria.

Yager uses a monotonically decreasing measure on the set of criteria $\mu_f : 2^N \Rightarrow [0, 1]$, such that the value $\mu_f(E)$ represents the degree of mutual independence of the criteria in the set $E \subseteq \{1, 2, \ldots, n\}$. Note that having a larger subset does not increase the degree of independence $A \subseteq B \Rightarrow \mu_f(A) \geq \mu_f(B)$.

Example 3: Consider a simplified recommender system for online movie sales, which recommends movies based on the following criteria: the user likes 1) mystery movies; 2) detectives; 3) special effects; 4) science fiction. One could use the following membership function $\mu_f(E)$:

$$\mu_f(\{i\}) = 1, i = 1, \ldots, 4$$

$$\mu_f(\{2, 2\}) = \mu_f(\{1, 2, 3\}) = \mu_f(\{1, 2, 4\}) = 0.7$$

$$\mu_f(\{1, 3\}) = \mu_f(\{1, 4\}) = \mu_f(\{2, 3\}) = \mu_f(\{2, 4\}) = 1$$

$$\mu_f(\{3, 4\}) = \mu_f(\{2, 3, 4\}) = \mu_f(\{1, 3, 4\})$$

$$= \mu_f(\{1, 2, 3, 4\}) = 0.5.$$.

Yager expresses the noble reinforcement requirement as follows.

Requirement 3: “We desire $k > 1$ independent high scores reinforce each other.”[5]

Let us assume that the function $\mu_f(E)$ is given. Our goal is to revise (16) to accommodate the independence of subsets of criteria $E$. Yager achieves this goal by taking the maximum over all possible subsets of criteria $E$ in which reinforcement takes place. In our notation, we obtain

$$F(x) = \max_{E \in k} (\max(x)$$

$$+ \mu_h(\min(E)) \mu_f(k) \mu_f(E) (U_E(x) - \max(x)))$$

(17)

where $U_E(x)$ denotes the upper bound $U_E(x) = \min\{1, \sigma_w, U(x)\}$ and $\sigma_w, U(x)$ is computed as in (11) but only with respect to criteria from the subset $E$, as explained below.

Remark 2: Note that the maximum over all $E$ (not over $G_j$ as in Section II-B) is essential: because the function $\mu_f$ is nonincreasing, the function $F(x)$ may fail to be monotone. For example, consider the problem with three linearly dependent but pairwise independent criteria $\mu_f(\{1, 2\}) = \mu_f(\{2, 3\}) = \mu_f(\{1, 3\}) = 1, \mu_f(\{1, 2, 3\}) = 0$. Now if all components of $x$ are high, we have no reinforcement. But if two components are high and the remaining is small, we have reinforcement. We ensure monotonicity by taking maximum reinforcement over all possible combinations of the criteria.

Remark 3: The function $\mu_h(\{E\})$ is redundant, as all information about reinforcement by subsets of criteria can be absorbed in $\mu_f$. However, it may be useful to include this factor explicitly to facilitate expressing users’ requirements.

It remains to establish the bounds $\sigma_w, U(x)$ to be used in (17). A straightforward approach is to use (11), as it already contains $E$ as a parameter [in (11) $E$ was determined by the components of $x$; now we explicitly supply $E$ as a parameter].

B. Excluding Other Undesired Reinforcement

We now extend noble reinforcement into another direction. Consider the following.

Requirement 4: Provide reinforcement of at least $k$ high scores, if at least $m$ of these scores are very high.

We proceed as earlier and define two crisp thresholds $\alpha, \beta, \beta > \alpha$; the interval $[\alpha, 1]$ will denote high scores and the interval $[3, 1]$ will denote very high scores. We will subsequently fuzzify the expression we obtain for this crisp case similarly to (2).
Translating the above requirement into mathematical terms, we obtain the following.

**Definition 4:** An aggregation operator \( F_{\alpha, \beta} \) provides noble reinforcement of at least \( k \) high values, with at least \( m \) very high values, with respect to thresholds \( \alpha, \beta \) if it can be expressed as

\[
F_{\alpha, \beta}(x) = \begin{cases}
A_{\epsilon \in E}(x), & \text{if } \exists E \subseteq \{1, \ldots, n\} \mid |E| = k,
\forall i \in E : x_i \geq \alpha, \forall i \notin E : x_i < \alpha \\
\max(x), & \text{otherwise}
\end{cases}
\]

where \( A_{\epsilon \in E}(x) \) is any disjunctive aggregation operator, applied only to the components of \( x \) with the indexes in \( E \).

As earlier, to obtain a Lipschitz continuous aggregation operator, we use the expression

\[
\sigma_{\alpha}(x) = \min_{z \in \Omega} \{\max(z) + M\|\langle x - z \rangle_p\|_p\}
\]

to compute the upper bound and then take \( U(x) = \min\{\sigma_{\alpha}(x), 1\} \) as the greatest aggregation operator with the specified properties.

Note, however, that the definition of the subset \( \Omega \) where the value of \( F \) is restricted to max has changed. Fortunately, we can still use the algorithms from the previous section as follows. We can write \( \Omega = \Omega_1 \cup \Omega_2 \), where \( \Omega_1 = E_0 \cup E_1 \cup \ldots \cup E_{k-1} \) as in (11), and \( \Omega_2 = \{x \in I^n \mid \exists D \text{ such that } |D| = m, \forall i \in \{1, \ldots, n\} \setminus D : x_i \leq \beta\} \) i.e., \( \Omega_2 \) is the set of points that have less than \( m \) coordinates greater than \( \beta \). According to Definition 4, \( F_{\alpha, \beta} \) is restricted to \( \max \) on that subset.

Next we show that the bound \( \sigma_{\alpha} \) can be easily computed by adapting our previous results. First, note that

\[
\sigma_{\alpha}(x) = \min_{z \in \Omega_1} \{\max(z) + M\|\langle x - z \rangle_p\|_p\}
= \min_{z \in \Omega_1} \{\min_{z \in \Omega_2} \{\max(z) + M\|\langle x - z \rangle_p\|_p\}\}
\]

We already know how to compute the minimum over \( \Omega_1 \) by using (11).

Consider a partition of \( I^n \) into subsets

\[
D_j = \{x \in I^n \mid \exists D \text{ such that } |D| = j, \forall i \in D, x_i < 1, \forall j \in \hat{D}, x_j \leq \beta\}
\]

for \( j = 0, \ldots, n \). It is analogous to the partition given by \( E_k \) in Section III, with \( \beta \) replacing \( \alpha \). \( D_j \) is the set of input vectors with \( j \) very high scores.

Now \( \Omega_2 = D_0 \cup \ldots \cup D_{m-1} \). Thus computation of the minimum over \( \Omega_2 \) is completely analogous to the minimum over \( \Omega_1 \) [see (10)]; the only difference is that we take \( m < k \) instead of \( k \) and \( \beta \) rather than \( \alpha \). Hence we apply the solution given in (11) for \( m > 1 \).

The special case \( m = 1 \), i.e., the requirement “at least one score should be very high” is treated differently. In this case, \( \Omega_2 = D_0 \) and solution (11) is not applicable. But in this case an optimal solution is \( z^* = (\beta, \beta, \ldots, \beta) \). To see this, note that \( D_0 \) and the objective function are both convex, and unrestricted minimum \( z = x \) is outside of \( D_0 \). The optimal \( z^* \) is always on the boundary of \( D_0 \), i.e., at least one component of \( z^* \) is \( \beta \). But \( \max(z) = \beta \) in this case. The function \( \beta + M\|\langle x - z \rangle_p\|_p \) is nonincreasing in \( z \); hence we can take the largest feasible value for each \( z \) to obtain a minimizer \( z^* = (\beta, \beta, \ldots, \beta) \). The value of the minimum in this case is

\[
\min_{z \in D_0} \{\max(z) + M\|\langle x - z \rangle_p\|_p\} = \beta + M \left( \sum_{i \in \Omega_0} (x_i - \beta)^p \right)^{1/p}. \]

Fuzzification with respect to \( \alpha \) and \( \beta \) is performed similarly to (2) by using

\[
F(x) = \max_{E, D} \{\max(x)
+ \mu_h(\min(E), \mu_{\alpha, \beta}(\min(D), \max(x)))\}
\]

and, if necessary, also with respect to \( k \) as in (17), with the parameters \( \alpha = \min(E), \beta = \min(D) \).

Let us now formulate a different requirement.

**Requirement 5:** Provide reinforcement of at least \( k \) high scores if we have at most \( m \) low scores.

That is, we desire to have reinforcement when the scores are high or medium, and explicitly prohibit reinforcement if some of the scores are low.

In the above, \( 1 < k \leq n \) and \( 0 \leq m \leq n - k \); when \( m = 0 \), we prohibit reinforcement when at least one low score is present. We proceed as earlier and define two crisp thresholds \( \alpha, \gamma, \gamma < \alpha \); the interval \([0, \gamma]\) will denote low scores and the interval \([\alpha, 1]\) will denote high scores. We will subsequently fuzzify the expression we obtain for this crisp case similarly to (2).

Translating the above requirement into mathematical terms, we obtain the following.

**Definition 5:** An aggregation operator \( F_{\alpha, \gamma} \) provides noble reinforcement of at least \( k \) high values, with at most \( m \) low values, with respect to thresholds \( \alpha, \gamma \) if it can be expressed as

\[
F_{\alpha, \gamma}(x) = \begin{cases}
A_{\epsilon \in E}(x), & \text{if } \exists E \subseteq \{1, \ldots, n\} \mid |E| = k, \\
\forall i \in E : x_i \geq \alpha, \forall i \notin E : x_i < \alpha, & \text{otherwise}
\end{cases}
\]

where \( A_{\epsilon \in E}(x) \) is any disjunctive aggregation operator, applied only to the components of \( x \) with the indexes in \( E \).

We proceed similarly to the previous case. Form a partition of \( I^n \): for \( j = 0, \ldots, n \), define

\[
D_j = \{x \in I^n \mid \exists D \subseteq \{1, \ldots, n\} \text{ such that } |D| = j, \\
\forall i \in D, x_i < 1, \forall j \in \hat{D}, x_j \leq \gamma\}
\]

\( D_j \) is the set of points with \( n - j \) small coordinates, and the aggregation operator \( F_{\alpha, \gamma} \) should be restricted to maximum on
where the minimum over $\Omega_3$ is computed by using (11) and the minimum over $\Omega_3$ is computed analogously (by replacing $\alpha$ with $\gamma$ and $k$ with $n - m$).

Fuzzification is performed by using

$$F(x) = \max_{E:D}(\max(x) + \mu_h(\min(E))) \times (1 - \mu_{\text{min}}(\min(D)))(U(x) - \max(x)).$$

Both requirements discussed in this section led to redefining the subset $\Omega$ on which the aggregation operator coincides with the maximum. In both cases we used the basic (5) and reduced the $\eta$-variate minimization problems to a number of univariate problems, of the same type as (11), with different parameters. Therefore, we can use the same generic algorithms as in Section III to efficiently compute these aggregation operators.

V. CONCLUSION

Noble reinforcement is a useful property that allows reinforcement of high values within an aggregation procedure and prohibits reinforcement of low values. It eliminates undesirable tail effect, by which a sufficiently large number of small scores leads to an almost perfect total score. Further, in some applications, only a certain minimal number of independent arguments with high score should lead to reinforcement.

It can be argued that aggregation operators with both disjunctive and conjunctive behavior (such as uninorms) can also serve the same purpose. Indeed, in this case high values are reinforced and low values are lowered. However, in the specific context of recommender systems, Yager argues that the aggregation procedure must be disjunctive, as possession of one relevant feature (justification) should be enough to recommend the product. The second required feature of an aggregation operator is monotonicity with respect to argument cardinality (which holds for disjunctive operators): the more justifications one has, the stronger is the recommendation.

We have shown in Proposition 1 that an ordinal sum of max operator and another t-conorm delivers the required noble reinforcement. However, the refinements of this property, such as inclusion of the minimal number of reinforcing arguments, treatment of correlated inputs, and others require more sophisticated features, not readily available in standard aggregation operators.

In this paper, we designed a Lipschitz continuous aggregation operator with noble reinforcement, which is stable with respect to input inaccuracies. The Lipschitz constant of the operator is a parameter that controls the degree of reinforcement. Lipschitz continuity holds in both cases: when we use crisp thresholds to define the sets of high, very high, and low argument values and when we use respective fuzzy sets. In Yager’s construction continuity holds only after fuzzifying the thresholds. The advantage of our method is that in practice the user may specify the noble reinforcement requirement in either way.

We provided explicit formulas that allow a straightforward implementation of such an aggregation procedure and also examined a few interesting extensions and refinements. Our construction will find applications in recommender systems and search engines, where various forms of noble reinforcement requirement could arise.

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