Pairings and Signed Permutations

Valerio De Angelis

1. A COMBINATORIAL IDENTITY. This note is motivated by the identity

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2n-2i}{n-i} = 4^n. \tag{1}$$

Identity (1) is easy to prove: note that the left side is a convolution, so multiply it by x^n, sum, and recognize the square of a binomial series. But a combinatorial interpretation of the same identity is not so easy to find. In [**3**], M. Sved recounts the story of the identity and its combinatorial proofs. She relates how, after she challenged her readers (in a previous article) to find a combinatorial proof, Paul Erdős "was quick to point out that ... Hungarian mathematicians tackled it in the thirties: P. Veress proposing it, and G. Hajos solving it" [**3**, p. 44]. In the same article, she outlines more than one combinatorial proof supplied by her readers.

Identity (1) has been mentioned by many other authors. We refer readers to [**2**, Exercise 2c, p. 44], [**1**] (in the foreword by D. Knuth) for recent citations. In this note, we describe a new combinatorial construction from which (1) is readily derived.

2. PAIRINGS AND GRAPHS. Using the identity $(2i)! = 2^i i!\,(2i-1)!!$ (where $(2i-1)!! = (2i-1)(2i-3)\cdots 5\cdot 3\cdot 1$), (1) becomes

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2n-2i-1)!! = 2^n n!. \tag{2}$$

We now describe a combinatorial proof of (2). Its right-side motivates the following construction.

Let n be a positive integer, and let $[n] = \{1, 2, \ldots, n\}$. If Y is a subset of $[n]$, we write $\pm Y = \{k \in \mathbb{Z} : |k| \in Y\}$. We construct a directed graph with vertex set $\pm[n]$ consisting of disjoint, simple cycles such that there is an edge between the elements of each pair $\{k, -k\}$. Choose for each k a direction for the edge between k and $-k$. This can be done in 2^n ways. Let $s(k)$ and $t(k)$ be the starting and ending vertices of the edge, respectively. Given any permutation π of $[n]$, place an edge starting at $t(k)$ and ending at $s(\pi(k))$. This defines the graph. An example is shown in Figure 1.

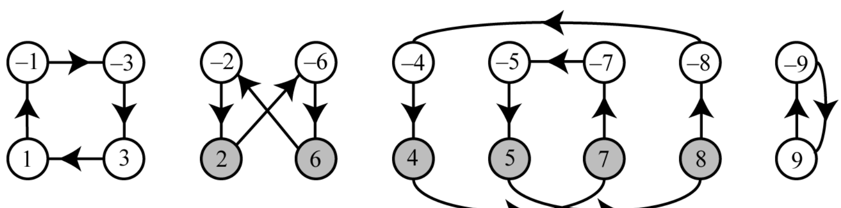

Figure 1.

By construction, there are $2^n n!$ such graphs. We now count the number of these graphs in a different way. A *pairing* on a subset Y of $[n]$ is a partition of $\pm Y$ into two-element sets. A pairing σ on $[n]$ is *compatible* with a subset Y of $[n]$ if $|k|$ belongs to Y whenever $\{j, k\}$ is in σ and $|j|$ is in Y. If $|Y| = i$, clearly there are $(2i - 1)!!\,(2n - 2i - 1)!!$ pairings on $[n]$ compatible with Y. If γ is a cycle on the graph, we use $m(\gamma)$ to signify $\min\{|k| : k \text{ is a vertex on } \gamma\}$. Let i be the number of positive vertices belonging to cycles γ such that the edge between $m(\gamma)$ and $-m(\gamma)$ terminates at $m(\gamma)$. There are $\binom{n}{k}$ ways to choose subsets Y of $[n]$ consisting of such vertices. After all edges connecting k and $-k$ (for each k in $[n]$) have been removed, the remaining edges define a pairing on $[n]$ compatible with Y. For the example in Figure 1, we have $Y = \{2, 4, 5, 6, 7, 8\}$ (the shaded vertices), and $\sigma = \{\{1, 3\}, \{-2, 6\}, \{-1, -3\}, \{-5, -7\}, \{-4, -8\}, \{2, -6\}, \{5, 8\}, \{4, 7\}, \{-9, 9\}\}$. Conversely, each such pair (σ, Y) determines the cycle structure of the graph, as follows: For the vertices in $\pm Y$, begin by placing an edge starting at $\min Y$ and ending at the other element j of the pair containing $\min Y$. Then place an edge starting at j and ending at $-j$, then an edge starting at $-j$ and ending at the other element of the pair containing $-j$, and so on until a cycle γ is formed. Repeat this procedure for the set $\pm Y \setminus \gamma$, and so forth, until all vertices in $\pm Y$ are accounted for. For the vertices in $\pm[n] \setminus \pm Y$ do the same, but start at $-\min([n] \setminus Y)$. This process yields the left side of (2).

3. SIGNED PERMUTATIONS. The construction outlined in section 2 can be reformulated without referring to a graph. A *signed permutation* on $[n]$ is a function $\pi : [n] \to \pm[n]$ such that the function $|\pi| : [n] \to [n]$ obtained by dropping all negative signs (that is, $|\pi|(i) = |\pi(i)|$) is bijective. Writing the permutation $|\pi|$ in disjoint cycle represention and then replacing each entry i with $-i$ if $-i$ is in the range of π, we obtain a cycle representation for π.

Example. The signed permutation π on [9] such that $\pi(1) = 3$, $\pi(2) = 6$, $\pi(3) = -1$, $\pi(4) = -7$, $\pi(5) = -8$, $\pi(6) = 2$, $\pi(7) = 5$, $\pi(8) = 4$, $\pi(9) = -9$ is represented as $\pi = (-1, 3)(2, 6)(4, -7, 5, -8)(-9)$.

If $C = (i_1, i_2, \ldots, i_k)$ is a cycle of π, let $s(C) = \min\{|i_j| : 1 \le j \le k\}$, and for i in $[n]$ let $C(i)$ be the unique cycle containing either i or $-i$. Now extend π to a function $\pi : \pm[n] \to \pm[n]$ by defining $\pi(-i) = \pi(i)$. It is then easy to check that $\sigma_\pi = \{\{\pi(i), -\pi^2(i)\} : i \in [n]\}$ is a pairing on $[n]$. An explicit description of σ_π is as follows: if $(i_1, i_2, \ldots, i_k)$ is a cycle of π, then for each j with $1 \le j \le k$ include the pair $\{i_j, -i_{j+1}\}$ (with subscripts taken modulo k). For i in $[n]$ precisely one of $s(C(i))$ or $-s(C(i))$ is in the range of π (equivalently, it appears in the cycle representation of π). Define $Y_\pi = \{i \in [n] : s(C(i)) \text{ is in the range of } \pi\}$. Then σ_π is compatible with Y_π. If π is as in the previous example, then σ_π coincides with the pairing constructed from the graph of Figure 1 and Y_π coincides with the subset Y of shaded vertices in the same figure.

If we now let

$$A_n = \{\pi : \pi \text{ is a signed permutation on } [n]\}$$

and

$$B_n = \{(\sigma, Y) : Y \subset [n] \text{ and } \sigma \text{ is a pairing on } [n] \text{ compatible with } Y\},$$

we clearly have $|A_n| = 2^n n!$ and

$$|B_n| = \sum_{i=0}^{n} \binom{n}{i} (2i - 1)!!\,(2n - 2i - 1)!!.$$

Moreover, a simple description of a bijection between A_n and B_n is given by the map $\pi \mapsto (\sigma_\pi, Y_\pi)$.

ACKNOWLEDGMENTS. I thank the referee for a careful reading and several constructive suggestions, including the explicit description of the pairing σ_π in the last section, and Victor Moll for getting me interested in combinatorics.

REFERENCES

1. M. Petkovsek, H. S. Wilf, and D. Zeilberger, $A = B$, AK Peters, Wellesley, MA, 1996.
2. R. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, Cambridge, England, 1997.
3. M. Sved, Counting and recounting: The aftermath, *Math. Intelligencer* **6** (1984) 44–45.

Mathematics Department, Xavier University of Louisiana, 1 Drexel Drive, New Orleans, LA 70125
vdeangel@xula.edu

Iterated Products of Projections in Hilbert Space

Anupan Netyanun and Donald C. Solmon

1. INTRODUCTION. The theorem on convergence of the iterated product of orthogonal projections in Hilbert space has an interesting history. As pointed out by Deutsch [**6**], convergence was first established for two projections by von Neumann [**13**] in 1933, and this result was rediscovered independently by Aronszajn [**2**] in 1950, Nakano [**11**] in 1953, and Wiener [**14**] in 1955. The first proof for an arbitrary finite number of projections was given by Halperin [**7**] in 1962. The result is the following:

Theorem 1. *Let P_j $(1 \le j \le r)$ be the orthogonal projection onto the closed subspace M_j of the Hilbert space $\mathcal{H}$, and let P_M be the orthogonal projection onto the intersection $M = M_1 \cap \cdots \cap M_r$. If $T = P_r \cdots P_1$, then $T^k \to P_M$ strongly as $k \to \infty$, that is, $\|T^k x - P_M x\| \to 0$ for each x in $\mathcal{H}$.*

An elementary proof in the case $r = 2$ appeared recently in [**4**]. Short proofs of a more general result (which we cover later) appeared in [**1**] and [**3**]. An important special case is the iterative procedure of Kaczmarz [**8**] for solving large linear systems. In this light the result provides a theoretical basis for one of the early algorithms in computed tomography [**12**].

We were curious as to why so many well-known mathematicians came upon this result and how they proved it. Nakano's book [**11**] contains a proof in the case $r = 2$, but no application or reference to Theorem 1 save a citation to his 1940 paper [**10**] (written in Japanese). With hopes that we could interpolate, if not translate, Nakano's paper, we requested a copy. (Through the intercession of St. Jerome) we also obtained a related paper by Nakano's student S. Kakutani [**9**]. That paper is central to this note.