

Quantum surfaces, special functions, and the tunneling effect

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Abstract

The notion of quantum embedding is considered for two classes of examples: quantum coadjoint orbits in Lie coalgebras and quantum symplectic leaves in spaces with non-Lie permutation relations. A method for constructing irreducible representations of associative algebras and the corresponding trace formulas over leaves with complex polarization are obtained. The noncommutative product on the leaves incorporates a closed 2-form and a measure which (in general) are different from the classical symplectic form and the Liouville measure. The quantum objects are related to some generalized special functions. The difference between classical and quantum geometrical structures could even occur to be exponentially small with respect to the deformation parameter. That is interpreted as a tunneling effect in the quantum geometry.

Dedicated to the memory of Professor M. Flato

1 Introduction

A manifold \mathcal{M} is called a *quantum manifold* if there is an associative non-commutative algebra of functions over \mathcal{M} with unity element 1 and with involution $f \rightarrow \bar{f}$ given by the complex conjugation. More precisely, following the pioneer works [2, 3], we assume that there is a family of algebras

$\mathcal{F}(\mathcal{M})$ parameterized by $\hbar \geq 0$ such that the product \star in \mathcal{F} is a deformation of the usual commutative product of functions:

$$f \star g = fg + \hbar c_1(f, g) + \hbar^2 c_2(f, g) + \dots, \quad \hbar \rightarrow 0, \quad (1.1)$$

on a subspace in \mathcal{F} consisting of \hbar -independent smooth functions. The coefficients c_k in (1.1) are assumed to be bidifferential operators of order k .

Of course, the operation

$$i(c_1(f, g) - c_1(g, f)) \stackrel{\text{def}}{=} \{f, g\}$$

determines the Poisson brackets over \mathcal{M} , and so, the quantum manifold is automatically a Poisson manifold.

Let $\mathfrak{X} \subset \mathcal{M}$ be one of symplectic leaves in \mathcal{M} . Then the Poisson structure generates on \mathfrak{X} a symplectic form ω_0 (see [32, 35]). So, the leaf \mathfrak{X} itself can be considered as a Poisson manifold, and can be quantized. Denote by $*$ a quantum product in a space of functions over \mathfrak{X} , and ask: how this product could be related to the product \star over \mathcal{M} ?

The first idea is to check whether the restriction operation

$$f \mapsto f|_{\mathfrak{X}}, \quad \mathcal{F}_*(\mathcal{M}) \mapsto \mathcal{F}_*(\mathfrak{X}) \quad (1.2)$$

is a homomorphism of algebras. But in [14, 17, 48] it has been proven (in the case where $\mathcal{M} = \mathfrak{g}^*$ is a Lie coalgebra and \mathfrak{X} a coadjoint orbit) that there is no quantum product $*$ on \mathfrak{X} such that the mapping (1.2) is a homomorphism.

A way to avoid this difficulty was suggested in [25]: let us replace the classical restriction operation in (1.2) by a *quantum restriction* operation

$$f \mapsto f|_{\widehat{\mathfrak{X}}} = f|_{\mathfrak{X}} + \hbar e_1(f) + \hbar^2 e_2(f) + \dots \quad (1.3)$$

in order to preserve the homomorphism property on the quantum level:

$$f|_{\widehat{\mathfrak{X}}} * g|_{\widehat{\mathfrak{X}}} = (f \star g)|_{\widehat{\mathfrak{X}}}. \quad (1.4)$$

The differential operators e_j in (1.3) are quantum corrections to the classical restriction operation. They act not only along the leaf \mathfrak{X} but also in transversal directions, so the quantum restriction $f|_{\widehat{\mathfrak{X}}}$ “feels” not only values of f on \mathfrak{X} , but also the germ of f near \mathfrak{X} . Actually, we can describe the

quantum restriction not by the formal \hbar -power series (1.3), but as the action of a pseudodifferential operator composed with the classical restriction:

$$f|_{\widehat{\mathfrak{X}}}(x) = E_{\mathfrak{X}}(\xi(x), -i\hbar \partial/\partial\xi(x))f(\xi(x)). \quad (1.3a)$$

Here the letters ξ and x designate points from \mathcal{M} and \mathfrak{X} , the equation $\xi = \xi(x)$ determines the classical embedding $\mathfrak{X} \subset \mathcal{M}$, and the symbol $E_{\mathfrak{X}}(\xi, \eta)$ is a certain smooth function on $T_{\mathfrak{X}}^*\mathcal{M}$. The operators e_j in (1.3) are obtained from (1.3a) by the Taylor expansion as $\hbar \rightarrow 0$.

An explicit calculation of the homomorphism (1.3), (1.4) was given in [25] for the case of products $\star, *$ generated by (partial) complex structures. When such a homomorphism is fixed, we call \mathfrak{X} a *quantum submanifold* of \mathcal{M} . The procedure (1.2)–(1.4), (1.3a) can be called a *quantum embedding* of \mathfrak{X} into \mathcal{M} .

The simplest submanifolds are two-dimensional surfaces embedded into Euclidean spaces. So, one could ask first of all about *quantum surfaces* homeomorphic to the plane, the sphere, the cylinder, the torus, etc., embedded in the quantum sense into the quantized Euclidean space $\mathcal{M} = \mathbb{R}^m$.

Another interesting class of submanifolds is provided by coadjoint orbits in Lie coalgebras $\mathcal{M} = \mathfrak{g}^*$. The question about quantum coadjoint orbits is very natural and attractive remembering the negative results [14, 17, 48].

Since each surface or orbit admits not only a symplectic structure but also a complex structure, it is natural to consider $*$ -products on them generated by the Kählerian geometry. These are the Wick–Klauder–Berezin products [5, 6, 33, 40, 44] (with some modifications). Namely, let us fix a Kähler form ω and a reproducing measure dm over the symplectic leaf \mathfrak{X} which are certain \hbar -deformations of the classical form ω_0 and the Liouville measure $dm^{\omega_0} = \frac{1}{n!}|\omega_0 \wedge \dots \wedge \omega_0|$. Then the associative product $\psi * \chi$ of two functions ψ and χ on \mathfrak{X} can be defined by the integral formula

$$(\psi * \chi)(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \psi^{\#}(x|y) \chi^{\#}(y|x) \exp \left\{ \frac{i}{\hbar} \int_{\Sigma(x,y)} \omega^{\#} \right\} dm(y), \quad (1.5)$$

where $2n = \dim \mathfrak{X}$, points x, y are running over \mathfrak{X} , the sign $\#$ denotes the holomorphic extension from \mathfrak{X} to the complexification $\mathfrak{X}^{\#} \approx \mathfrak{X} \times \mathfrak{X}$, and $\Sigma(x, y)$ denotes a quadrangle membrane in $\mathfrak{X}^{\#}$ whose boundary consists of paths along fibers of the projections $\mathfrak{X} \xleftarrow{\pi_-} \mathfrak{X}^{\#} \xrightarrow{\pi_+} \mathfrak{X}$ connecting the points $y|y \leftarrow y|x \leftarrow x|x \leftarrow x|y \leftarrow y|y$ (see [22, 23]).

The measure dm and the Kähler form ω are strongly related to each other so that the unity function 1 should be the unity element of the product (1.5); see Section 2.

The function algebra $\mathcal{F}_*(\mathfrak{X})$ with the product (1.5) has a representation $\psi \rightarrow \widehat{\psi}$ by the Wick pseudodifferential operators $\widehat{\psi}$ acting in the Hilbert space $\mathcal{L}(\mathfrak{X})$ of antiholomorphic sections over \mathfrak{X} ; see Section 3.

The homomorphism $f \rightarrow \pi_{\mathfrak{X}}(f)$ determined by

$$\pi_{\mathfrak{X}}(f) \stackrel{\text{def}}{=} \widehat{f|_{\widehat{\mathfrak{X}}}} \quad (1.6)$$

is an *irreducible Hermitian representation* of the original algebra $\mathcal{F}_*(\mathcal{M})$ in the Hilbert space $\mathcal{L}(\mathfrak{X})$. This representation corresponds to the symplectic leaf $\mathfrak{X} \subset \mathcal{M}$. The trace formula for this representation is

$$\text{tr } \pi_{\mathfrak{X}}(f) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} f|_{\widehat{\mathfrak{X}}} dm, \quad \dim \pi_{\mathfrak{X}} = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} dm. \quad (1.7)$$

The operation of quantum restriction onto the leaf can be reconstructed from the irreducible representation using the formula for Wick symbols:

$$f|_{\widehat{\mathfrak{X}}}(x) = \text{tr} (\pi_{\mathfrak{X}}(f) \Pi(x)), \quad x \in \mathfrak{X}. \quad (1.8)$$

Here $\Pi : \mathfrak{X} \rightarrow \text{Hom}(\mathcal{L}(\mathfrak{X}))$ is the *coherent mapping* determined by

$$\Pi(x)^2 = \Pi(x) = \Pi(x)^*, \quad \text{tr } \Pi(x) = 1, \quad \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \Pi dm = I, \quad (1.9)$$

$$\text{tr} (\Pi(x)\Pi(y)) = \exp \left\{ \frac{i}{\hbar} \int_{\Sigma(x,y)} \omega^{\#} \right\}. \quad (1.10)$$

In particular, for any Casimir function K (i.e., the center element in $\mathcal{F}_*(\mathcal{M})$), the operator $\pi_{\mathfrak{X}}(K) = \lambda \cdot I$ is scalar, and from (1.8) we see that the constant λ is equal to $\lambda = K|_{\widehat{\mathfrak{X}}}$. So the eigenvalues of the quantum Casimir elements are calculated by (1.3a):

$$\lambda = E_{\mathfrak{X}} \left(\overset{2}{\xi}, -i\hbar \overset{1}{\partial/\partial\xi} \right) K(\xi) \Big|_{\xi \in \mathfrak{X}}. \quad (1.11)$$

For example, let $\mathcal{M} = \mathfrak{g}^*$ be a Lie coalgebra, and let $\mathfrak{X} \subset \mathfrak{g}^*$ be a coadjoint orbit. The general formula (1.8) allows one to define Wick symbols of *group elements*:

$$e_*^{i\eta/\hbar} \stackrel{\text{def}}{=} e^{i\eta/\hbar} \Big|_{\widehat{\mathfrak{X}}}, \quad \eta \in \mathfrak{g}.$$

Here elements η are considered as linear functions on \mathfrak{g}^* , which can, of course, be restricted to the orbit \mathfrak{X} .

The mapping

$$\exp(\eta) \mapsto e_*^{i\eta/\hbar}, \quad G \supset \exp(\mathfrak{g}) \rightarrow \mathcal{F}_*(\mathfrak{X}),$$

is a realization of the Lie group G (the group corresponding to the Lie algebra \mathfrak{g}) in the Wick function algebra over \mathfrak{X} :

$$e_*^{i\eta/\hbar} * e_*^{i\eta'/\hbar} = e_*^{i\eta \circ \eta'/\hbar}, \quad e_*^{i\eta/\hbar} * e_*^{-i\eta/\hbar} = 1,$$

where $\eta \circ \eta'$ is the group (Campbell–Hausdorff) multiplication on \mathfrak{g} .

Formula (1.3a) applied to the exponential function reads

$$e_*^{i\eta/\hbar} = E_{\mathfrak{X}}(-i\hbar \overset{1}{\partial/\partial\eta}, \overset{2}{\eta}) e^{i\eta/\hbar} \Big|_{\mathfrak{X}}. \quad (1.12)$$

Passing to operators, we obtain the formula for the irreducible representation $\pi_{\mathfrak{X}}$ of the Lie group G in the Hilbert space $\mathcal{L}(\mathfrak{X})$:

$$\pi_{\mathfrak{X}}(\exp(\eta)) = E_{\mathfrak{X}}(-i\hbar \overset{1}{\partial/\partial\eta}, \overset{2}{\eta}) \widehat{e^{i\eta/\hbar}} \Big|_{\mathfrak{X}}. \quad (1.13)$$

At last, using the general trace formula (1.7), we obtain the *character of the representation*:

$$\begin{aligned} \text{tr } \pi_{\mathfrak{X}}(\exp(\eta)) &= \frac{1}{(2\pi\hbar)^{\dim \mathfrak{X}/2}} \int_{\mathfrak{X}} e_*^{i\eta/\hbar} dm \\ &= \frac{1}{(2\pi\hbar)^{\dim \mathfrak{X}/2}} E_{\mathfrak{X}}(-i\hbar \overset{1}{\partial/\partial\eta}, \overset{2}{\eta}) \int_{\mathfrak{X}} e^{i\eta/\hbar} dm. \end{aligned} \quad (1.14)$$

The measure dm in this case is proportional to the classical Liouville measure dm^{ω_0} on the orbit. The last integral looks very similar to the Kirillov character formula [31], but the operator of the quantum restriction onto the orbit \mathfrak{X} (presented by the symbol $E_{\mathfrak{X}}$) and the group elements $e_*^{i\eta/\hbar}$ are certainly new objects in this framework.

Note that if the orbit \mathfrak{X} admits a G -invariant complex structure, then the symbol $E_{\mathfrak{X}}$ and the functions $e_*^{i\eta/\hbar}$ are very easily and explicitly calculated just by solving a certain first order differential equation over \mathfrak{X} or by evaluating the area of certain membranes in $\mathfrak{X} \times \mathfrak{X}$; see Section 4. In this way we obtain the explicit formula (1.13) for irreducible representations of the Lie group by Wick pseudodifferential operators over coadjoint orbits and the expression (1.14) for characters of these representations, as well as the eigenvalues (1.11) of the Casimir elements.

In Section 5 we leave the case of Lie algebras. We consider general function algebras with complex polarization and demonstrate explicit formulas for their Hermitian irreducible representations. A notion of *special functions* is associated with quantum Kähler leaves.

Then we are concentrated on the second example: 2-dimensional surfaces of revolution $\mathfrak{X} \subset \mathbb{R}^m$ endowed with a generic (not group-invariant) complex structure. The quantum realization of such surfaces is described in Section 6 using the results of Section 5, and we focus the attention on the algebraic polynomial case.

Assume that the surface \mathfrak{X} is embedded into $\mathcal{M} = \mathbb{R}^m$ by means of algebraic equations, and moreover, the complex structure on \mathfrak{X} is taken in such a way that all the operators $\pi_{\mathfrak{X}}(f)$ are *differential* (i.e., not generic pseudodifferential) for polynomial f . It is always possible to realize this situation in the case where \mathfrak{X} is homeomorphic to the plane or the sphere. Then the quantum Kähler form ω and the measure dm in (1.5) are very special; namely, in local complex coordinates z we have

$$\omega = i\hbar\bar{\partial}\partial \ln k(|z|^2), \quad dm = k(|z|^2)\ell(|z|^2)d\bar{z}dz, \quad (1.15)$$

where k and ℓ are certain *hypergeometric functions*. And vice versa: any hypergeometric function is related in this way to some quantum surface homeomorphic to the plane or sphere. This was proved in [28]; see Section 6.

In the case where \mathfrak{X} is homeomorphic to the cylinder, the representation $\pi_{\mathfrak{X}}(f)$ cannot act by purely differential operators, but must also include the shift operator $\exp\{\hbar\partial/\partial\bar{z}\}$. The corresponding $*$ -product is given again by (1.5) and (1.15), where the functions k and ℓ are some *theta-functions*, and the argument $|z|^2$ in (1.15) must be replaced by $z + \bar{z}$.

Note that our way to relate the theta-function to the cylinder is different from that used in [47] for the construction of the Weyl quantization over

the torus, as well as from the approaches based on discrete subgroups of the Heisenberg group in [15, 36].

What is important and unexpected is the \hbar -expansion of the quantum Kähler form ω and the reproducing measure dm in the cylindric case. We prove in Section 7 that these quantum objects differ from the classical form ω_0 and the classical Liouville measure dm^{ω_0} on the symplectic leaf \mathfrak{X} by exponentially small quantum corrections of order $O(e^{-\pi^2/\hbar})$. These corrections are given precisely by some theta-series, and they control the difference between the topology of the cylinder and the topology of the plane.

2 Reproducing measure

First of all we introduce and discuss some general definitions related to the Wick quantization of Kählerian manifolds.

Let \mathfrak{X} be a Kählerian manifold with the Kähler form ω . Following [16] consider a Hermitian line bundle over \mathfrak{X} whose curvature is $i\omega/\hbar$. The Hermitian bilinear form on each fiber is $(u, v)_\hbar = u\bar{v}\exp(-F/\hbar)$, where F is a local Kähler potential, i.e.,

$$\omega = i\bar{\partial}\partial F. \quad (2.1)$$

If the cohomology class of the curvature is multiple $2\pi i$, i.e.,

$$\frac{1}{2\pi\hbar}[\omega] \in H^2(\mathfrak{X}, \mathbb{Z}), \quad (2.2)$$

then this line bundle admits nontrivial antiholomorphic sections. Each section u has its Hermitian fiberwise norm $\rho_u = (u, u)_\hbar$ called a *density function*.

Denote by $\mathcal{L} = \mathcal{L}(\mathfrak{X})$ the space of antiholomorphic sections with Hilbert norm

$$\|u\|_{\mathcal{L}} = \left(\frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \rho_u dm \right)^{1/2}, \quad (2.3)$$

where $2n = \dim \mathfrak{X}$ and dm is a smooth positive measure on \mathfrak{X} .

The space \mathcal{L} can be characterized by its reproducing kernel [9, 45] $\mathcal{K} = \sum_j |u_j|^2$, $\{u_j\}$ is an orthonormal basis in \mathcal{L} . Of course, \mathcal{K} depends not only on the choice of the Kähler structure over \mathfrak{X} but also on the choice of the measure dm . The reproducing kernel determines a new Kähler form $\omega_m = i\hbar\bar{\partial}\partial \ln \mathcal{K}$. So, we obtain a transform $\omega \rightarrow \omega_m$ of Kählerian structures over \mathfrak{X} .

Definition 2.1 The measure dm on \mathfrak{X} is called a *reproducing measure* corresponding to the Kähler form ω if $\omega_m = \omega$.

In general, the question about the existence and uniqueness of the reproducing measure corresponding to the given form ω is open.

For an arbitrary measure dm let us define the function

$$\eta = \mathcal{K} \exp(-F/\hbar), \quad (2.4)$$

and introduce the new form and the new measure

$$\omega' = \omega + i\hbar\bar{\partial}\partial \ln \eta, \quad dm' = \eta dm. \quad (2.5)$$

Under this replacement the scalar product (2.3) and so the space \mathcal{L} and the reproducing kernel \mathcal{K} are not changed.

Lemma 2.1 *The measure dm' is the reproducing measure corresponding to the Kähler form ω' .*

Thus we see that for each Kähler form there is always another Kähler form (at the same cohomology class) for which the reproducing measure does exist.

Note that the natural choice of the measure on \mathfrak{X} is the Liouville measure $dm^\omega = \frac{1}{n!} |\omega \wedge \dots \wedge \omega|$. For this choice the function η (2.4) was first introduced in [43], and the question of whether $\eta = \text{const}$ or not was raised. If $\eta = \text{const}$ then $\omega' = \omega$ in (2.5) and the reproducing measure corresponding to ω is $dm' = \text{const} \cdot dm^\omega$. For instance, on homogeneous Kählerian manifolds this is the case (see [8, 43]).

In what follows we *assume that the reproducing measure over \mathfrak{X} exists.*

For each $x \in \mathfrak{X}$ denote by $\Pi(x)$ the linear operator in \mathcal{L} defined by

$$(\Pi(x)u, u)_{\mathcal{L}} = \rho_u(x), \quad \forall u \in \mathcal{L}. \quad (2.6)$$

Lemma 2.2 *For the mapping $\Pi : \mathfrak{X} \rightarrow \text{Hom } \mathcal{L}$, all properties (1.9) hold. In particular, we have the estimate*

$$\rho_u(x) \leq \|u\|_{\mathcal{L}}^2, \quad \forall u \in \mathcal{L}, \quad \forall x \in \mathfrak{X}.$$

Using the terminology [22], we call

$$p(x, y) = \text{tr} (\Pi(x)\Pi(y)) \quad (2.7)$$

the *probability function*. Its properties are the following:

$$0 \leq p(x, y) \leq 1, \quad p(x, x) = 1,$$

$$\frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} p(x, y) dm(y) \equiv 1, \quad \forall x \in \mathfrak{X}. \quad (2.8)$$

In the exponential representation $p(x, y) = \exp\{-d(x, y)^2/2\hbar\}$ the function $d(\cdot, \cdot)$ is the Calabi distance between points of \mathfrak{X} (for details see [13]).

An important question about the Calabi distance: does it actually distinguish points of \mathfrak{X} , that is, does it follow from $d(x, y) = 0$ that $x = y$? Or, using another language:

$$p(x, y) = 1 \quad \implies \quad x = y. \quad (2.9)$$

If this property holds, the manifold \mathfrak{X} is called the *probability space* (see in [4, 39]). Some sufficient condition for (2.9) was mentioned in [13]. In what follows, we assume that property (2.9) holds.

Now, following [19, 20, 46], consider the *complexification* $\mathfrak{X}^\# = \mathfrak{X} \times \mathfrak{X}$ of the manifold \mathfrak{X} . There are projections π_+ and π_- from $\mathfrak{X}^\#$ to the multipliers \mathfrak{X} . Points in $\mathfrak{X}^\#$ we denote by $x|y$, so that $\pi_+(x|y) = y$, $\pi_-(x|y) = x$. Also let us identify \mathfrak{X} with the diagonal $\text{diag}(\mathfrak{X} \times \mathfrak{X})$, i.e., $x \equiv x|x$. The holomorphic differential on $\mathfrak{X}^\#$ is defined by $\partial_{x|y}^\# = (\bar{\partial}_x, \partial_y)$, and the tangent space $T_{x|x}\mathfrak{X}^\#$ is identified with the direct product of polarizations $T_x^{(0,1)}\mathfrak{X} \oplus T_x^{(1,0)}\mathfrak{X}$, so that fibers of π_+ and π_- are considered as integral leaves of the complex polarization and its conjugate.

Note that the reproducing kernel \mathcal{K} is naturally extended to the complexification: $\mathcal{K}^\#(x|y) = \sum_j u_j(x)\overline{u_j(y)}$ to be $\partial^\#$ -holomorphic over $\mathfrak{X}^\#$. Also the form $\omega_{x|y}^\# = i\hbar\bar{\partial}_x\partial_y \ln p(x, y) = i\hbar\bar{\partial}\partial \ln \mathcal{K}^\#(x|y)$ is the holomorphic extension of the Kähler form ω from \mathfrak{X} to the complexification $\mathfrak{X}^\#$. The set $\Sigma_\#$ of singularities of $\omega^\#$ consists of all pairs of points $x, y \in \mathfrak{X}$ for which $p(x, y) = 0$ (or the Calabi distance $d(x, y) = \infty$). The dimension of $\Sigma_\#$ does not exceed $2n = \dim \mathfrak{X}$, and intersections of $\Sigma_\#$ with fibers of π_+ and π_- are transversal.

Thus we can integrate the closed form $\omega_\#$ over two-dimensional membranes in $\mathfrak{X} \times \mathfrak{X}$ whose boundaries belong to fibers of π_+ or π_- . Note that

$\omega_{\#}$ vanishes on these fibers and so the integral of $\omega_{\#}$ does not depend on the shape of a boundary-path along fibers of π_+ or π_- . This integral also does not depend on the shape of the membrane itself because of closedness of the form $\omega_{\#}$ and because of the quantization conditions (2.2) around two-dimensional holes.

Let us specify the shape of the membrane. Take two point $x, y \in \mathfrak{X}$ and identify them with points $x|x$ and $y|y$ on $\text{diag}(\mathfrak{X} \times \mathfrak{X})$. Then consider points $x|y$ and $y|x$ in $\mathfrak{X}^{\#}$ which are points of intersection of π_+ -fibers with π_- -fibers over x and y . Take any closed path with pieces running along π_+ - and π_- -fibers $x|x \leftarrow x|y \leftarrow y|y \leftarrow y|x \leftarrow x|x$ and consider a quadrangle membrane $\Sigma(x, y)$ in $\mathfrak{X}^{\#}$ whose boundary coincides with this closed path. Then we obtain formula (1.10)

$$p(x, y) = \exp \left\{ \frac{i}{\hbar} \int_{\Sigma(x, y)} \omega_{\#} \right\}. \quad (2.10)$$

Thus the probability function is determined by the Kähler form only and, in particular, it does not explicitly depend on the reproducing measure. This fact allows us to consider the property (2.8) together with (2.10) as a *linear equation for the reproducing measure* corresponding to the Kähler form.

From this equation it is possible, for instance, to find the formal \hbar -power expansion of the reproducing measure under the *a priori* assumption that the measure is \hbar -smooth as $\hbar \rightarrow 0$. Namely, the integral in (2.8) can be asymptotically evaluated by the stationary phase method as in [43]. Under condition (2.9) there is a unique point $y = x$ where the phase $\int_{\Sigma(x, y)} \omega_{\#}$ takes its minimum (zero) value, and this minimum is not degenerate. So, if we represent the unknown measure dm in (2.8) by some function σ as follows:

$$dm = \sigma dm^{\omega}, \quad \text{where} \quad dm^{\omega} = \frac{1}{n!} |\omega \wedge \cdots \wedge \omega|, \quad (2.11)$$

and introduce Hermitian matrices $((\omega_{\nu\mu}))$ and $((\omega^{-1\mu\nu}))$ of the Kähler form and of the Poisson tensor with respect to local complex coordinates on \mathfrak{X} :

$$\omega = i\omega_{\nu\mu}(x) d\bar{z}^{\nu}(x) \wedge dz^{\mu}(x), \quad \{z^{\mu}, \bar{z}^{\nu}\} = i\omega^{-1\mu\nu},$$

then equation (2.8) becomes asymptotically equivalent to

$$\frac{1}{(2\pi\hbar)^n} \int_{\mathbb{C}^n} d\bar{v}dv \exp \left\{ -\frac{1}{\hbar} \Omega v \bar{v} + (\bar{v} \bar{\partial} + v \partial) \right\} (\sigma \det \omega) = 1. \quad (2.12)$$

Here $\partial = \partial/\partial z(x)$, the matrix Ω is the deformation of ω :

$$\Omega_{\nu\mu}(x, V) = |\varphi(v\partial)|^2 \omega_{\nu\mu}(x),$$

where $\varphi(\xi) \stackrel{\text{def}}{=} (e^\xi - 1)/\xi$, and v are complex coordinates of vectors $V \in T_x \mathfrak{X} \approx \mathbb{C}^n$. The explicit derivation of the Gaussian integral in (2.12) transforms this equation to the following one:

$$f_{\hbar}(x, \nabla)\sigma = 1, \quad \nabla \stackrel{\text{def}}{=} \frac{1}{\det \omega(x)} \cdot d_x \cdot \det \omega(x), \quad (2.13)$$

where f_{\hbar} is the function on $T^* \mathfrak{X}$ given by

$$f_{\hbar}(x, p) = \det \omega(x) \left[\frac{\exp\{\hbar \Omega^{-1}(x, \cdot) p p\}}{\det \Omega(x, \cdot)} \right]^{\flat}. \quad (2.14)$$

Here the contraction operation \flat applied to any function $g(V, p)$ on $T_x \mathfrak{X} \times T_x^* \mathfrak{X}$ polynomial in p produces a polynomial function on $T_x^* \mathfrak{X}$ by the formula

$$g(\cdot, p)^{\flat} \stackrel{\text{def}}{=} g(d_p, p)1, \quad p \in T_x^* \mathfrak{X}.$$

Explicit expression for function (2.14) is

$$\begin{aligned} f_{\hbar} = & \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{|\alpha_0|, |\beta_0| \geq 0 \\ |\alpha_1| + |\beta_1| \geq 1 \\ |\alpha_k| + |\beta_k| \geq 1}} \frac{\hbar^{|\alpha_0| + \dots + |\alpha_k|}}{(|\alpha_1| + 1) \dots (|\alpha_k| + 1) (|\beta_1| + 1) \dots (|\beta_k| + 1) \alpha! \beta!} \\ & \cdot \det \omega \cdot \bar{\partial}^{\alpha_1} \partial^{\beta_1} \omega_{\nu_1 \mu_1} \dots \bar{\partial}^{\alpha_k} \partial^{\beta_k} \omega_{\nu_k \mu_k} \frac{\partial^k}{\partial \omega_{\nu_1 \mu_1} \dots \partial \omega_{\nu_k \mu_k}} \left(\frac{\mathcal{F}_{\beta_0 + \dots + \beta_k}^{\alpha_0 + \dots + \alpha_k}(\omega^{-1})}{\det \omega} \right) \bar{p}^{\alpha_0} p^{\beta_0}. \end{aligned} \quad (2.15)$$

Here we use the following notations: for arbitrary matrix A and multi-indices α, β the polynomial $\mathcal{F}_{\beta}^{\alpha}(A)$ in matrix elements of A is given by $\mathcal{F}_{\beta}^{\alpha}(A) \stackrel{\text{def}}{=} (A \partial / \partial v)^{\alpha} v^{\beta} \Big|_{v=0}$.

As it follows from (2.14), (2.15):

$$f_{\hbar}(x, p) = 1 + \hbar f^{(1)}(x, p) + \hbar^2 f^{(2)}(x, p) + \dots, \quad (2.16)$$

where $f^{(k)}$ are explicitly given symbols on $T^*\mathfrak{X}$ polynomial in p of degree $2k$. The corresponding differential operators $f^{(k)}(x, \nabla)$ are interesting geometric invariants of the Kählerian structure ω (see [25]). For instance,

$$f^{(1)}(x, \nabla) = \frac{1}{2}(\Delta + \omega^{-1\mu\nu} \rho_{\nu\mu}),$$

where Δ is the Laplace–Beltrami operator on \mathfrak{X} and $\rho_{\nu\mu} = \bar{\partial}_\nu \partial_\mu \ln \det \omega$ is the matrix of the Ricci form

$$\rho = i\rho_{\nu\mu} d\bar{z}^\nu \wedge dz^\mu.$$

The formal asymptotic solution of equation (2.13) is the following

$$\sigma = 1 + \sum_{s \geq 1} \hbar^s \sigma_s, \quad \sigma_s = \sum_{l=1}^s \sum_{\substack{k_1, \dots, k_l \geq 1 \\ |k|=s}} (-1)^l f^{(k_1)}(x, \nabla) \dots f^{(k_l)}(x, \nabla) 1. \quad (2.17)$$

In particular,

$$\sigma_1 = -\frac{1}{2} \omega^{-1\mu\nu} \rho_{\nu\mu}$$

is half the scalar curvature of the Kähler manifold \mathfrak{X} .

Theorem 2.1 *Let the Kähler form ω over \mathfrak{X} satisfy conditions (2.2), (2.9). Then the formal \hbar -expansion for the reproducing measure over \mathfrak{X} is given by*

$$dm \sim (1 + \hbar \sigma_1 + \hbar^2 \sigma_2 + \dots) dm^\omega, \quad (2.18)$$

where dm^ω is the Liouville measure (2.11), functions σ_s are determined by (2.17), and symbols $f^{(k)}$ are taken from (2.15), (2.16).

Now from (1.9) we obtain a useful corollary.

Corollary 2.1 *Under conditions (2.2), (2.9), the dimension of the space of antiholomorphic sections over the compact Kähler manifolds \mathfrak{X} can be calculated by the formula*

$$\dim \mathcal{L}(\mathfrak{X}) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} (1 + \hbar \sigma_1 + \dots + \hbar^n \sigma_n) dm^\omega, \quad (2.19)$$

where the functions σ_s are given in (2.17).

Note that (2.19) is a precise formula, not asymptotic in \hbar , and it contains only the first n coefficients σ_s ($1 \leq s \leq n$). Indeed, in all higher terms (for $s > n$) the integrals $\frac{\hbar^s}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \sigma_s dm^\omega = O(\hbar^{s-n})$ are asymptotically small, and so, they should be just zero, since the left-hand side of (2.19) is integer.

Also note that each σ_s in (2.19) is a functional of the Kähler metric ω homogeneous of degree $-s$, and so, if we include \hbar into the definition of the Kähler form on \mathfrak{X} , then \hbar disappears from (2.19). This means that one can set $\hbar = 1$ simultaneously in (2.19) and in (2.2).

An analogous way to derive formulas of type (2.19) (but not in this explicit form) was suggested in [41]; for instance, in [41] there is a discussion about relations between $*$ -product and the Riemann–Roch–Hirzebruch theorem. Our formulas for σ_s on the right-hand side of (2.19) just represent in some way the Riemann–Roch number and Hilbert–Samuel polynomial [11, 12, 38].

In particular, for compact 2-dimensional surfaces \mathfrak{X} we conclude from the Gauss–Bonnet theorem that the integral $\frac{1}{\pi} \int_{\mathfrak{X}} \sigma_1 dm^\omega$ is the Euler number $\chi(\mathfrak{X}) = c_1(\mathfrak{X})$ of the surface. Thus in this case formula (2.19) reads: $\dim \mathcal{L}(\mathfrak{X}) = N + \frac{1}{2}\chi(\mathfrak{X})$, where $N = \frac{1}{2\pi\hbar} \int_{\mathfrak{X}} |\omega| \in \mathbb{Z}_+$.

3 Quantization by complexification

Over any Kählerian manifold \mathfrak{X} the probability function (2.10) determines the probability operator \mathcal{P} acting by the formula

$$(\mathcal{P}\psi)(x) \stackrel{\text{def}}{=} \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} p(x, y) \psi(y) dm(y). \quad (3.1)$$

Lemma 3.1 *The probability operator is a positive self-adjoint contraction in the space $L^2(\mathfrak{X}, dm)$.*

We denote by $M = M(\mathfrak{X})$ the range of the operator \mathcal{P} . Then $L^2 = M \oplus \text{Ker } \mathcal{P}$. The space M is \mathcal{P} -invariant and endowed with the norm

$$\|\psi\|_M = (\mathcal{P}\psi, \psi)_{L^2}^{1/2}. \quad (3.2)$$

Each function $\psi \in M$ can be represented as $\psi = \mathcal{P}\varphi$, where $\varphi \in M$, and so we can define

$$\psi^\#(x|x') \stackrel{\text{def}}{=} \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \exp \left\{ \frac{i}{\hbar} \int_{\Sigma(x|x', y)} \omega^\# \right\} \varphi(y) dm(y), \quad (3.3)$$

where $\Sigma(x|x', y)$ is a membrane in $\mathfrak{X}^\#$ with the boundary $y|y \leftarrow y|x' \leftarrow x|x' \leftarrow x|y \leftarrow y|y$. The function $\psi^\#$ (3.3) is the $\partial^\#$ -holomorphic extension of ψ to the complexification $\mathfrak{X}^\#$. Of course, $\psi^\#(x|x) = \psi(x)$.

Now let us introduce the Hilbert norm

$$\|\psi\|_W = \left(\frac{1}{(2\pi\hbar)^{2n}} \iint_{\mathfrak{X} \times \mathfrak{X}} |\psi^\#(x|x')|^2 p(x, x') dm(x) dm(x') \right)^{1/2} \quad (3.4)$$

and denote by $W = W(\mathfrak{X})$ the completion of the space M by this norm. The probability operator is an isometry $M \xrightarrow{\mathcal{P}} W$.

For each $\psi \in W$ one can define the *Wick pseudodifferential operator* $\widehat{\psi}$ in the space \mathcal{L} by the bilinear form

$$(\widehat{\psi}u, u) \stackrel{\text{def}}{=} (\psi, \rho_u)_W, \quad \forall u \in \mathcal{L}. \quad (3.5)$$

Explicit formula is the following:

$$(\widehat{\psi}u)(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \mathcal{K}^\#(x|y) \psi^\#(x|y) u(y) e^{-F(y|y)/\hbar} dm(y). \quad (3.6)$$

The function ψ is called the low symbol [33], or the Wick symbol, or the covariant symbol [6] of the operator $\widehat{\psi}$. The symbol is reconstructed by the formula

$$\psi(x) = \text{tr}(\widehat{\psi}\Pi(x)), \quad (3.7)$$

and so, the correspondence $\psi \rightarrow \widehat{\psi}$ is one-to-one. Of course, the complex conjugate function corresponds to the adjoint operator: $\overline{\psi} \rightarrow \widehat{\psi}^*$. Pure states are exactly Wick operators corresponding to density functions:

$$\widehat{\rho}_u v = (v, u)_{\mathcal{L}} u.$$

Positive operators have positive symbols.

The reproducing kernel is the “eigenfunction” of all Wick operators:

$$\widehat{\psi}\mathcal{K}^\# = \psi^\#\mathcal{K}^\#$$

(where $\widehat{\psi}$ acts by the first argument, i.e., $\widehat{\psi} \approx \widehat{\psi} \otimes I$).

Now let us introduce an associative multiplication to the space of Wick symbols. Note that there is a natural “matrix” or groupoid multiplication of sections generated by the scalar product (2.3), namely,

$$(\psi \times \chi)(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \psi^\#(x|y) \chi^\#(y|x) \exp\{-F(y|y)/\hbar\} dm(y).$$

But the unity element of this multiplication is the section \mathcal{K} . To make this operation to be defined on functions, and to make the unity element to be the unity function 1, one must normalize this product in the following way: $\psi * \chi \stackrel{\text{def}}{=} (\mathcal{K}\psi \times \mathcal{K}\chi)/\mathcal{K}$. An explicit formula for the final product is (1.5), that is,

$$(\psi * \chi)(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} p(x, y) \psi^\#(x|y) \chi^\#(y|x) dm(y). \quad (3.8)$$

Another version of (3.8) is the formula for the left multiplication

$$\psi * = \frac{1}{\mathcal{K}} \cdot \widehat{\psi} \cdot \mathcal{K}. \quad (3.9)$$

The probability function is the “eigenfunction” of the multiplication operator:

$$\psi(x) * p(x, y) = p(x, y) * \psi(y) = \psi^\#(x|y) p(x, y),$$

and, in particular, the following reproducing property holds:

$$\psi(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} p(x, y) * \psi(y) dm(y), \quad \forall x \in \mathfrak{X}.$$

Note that if $\psi \in L^1(\mathfrak{X}, dm)$ then the operator $\widehat{\psi}$ is of trace class:

$$\text{tr } \widehat{\psi} = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \psi dm \stackrel{\text{def}}{=} \text{tr } \psi. \quad (3.10)$$

For the product of two Wick operators we have

$$\text{tr}(\widehat{\psi} \widehat{\chi}^*) = (\psi, \chi)_W \quad (3.11)$$

and so the function space $W(\mathfrak{X})$ is *isomorphic to the space of Hilbert–Schmidt operators in $\mathcal{L}(\mathfrak{X})$* . In particular, from (3.7) it follows

$$|\psi(x)| \leq \|\psi\|_W, \quad \forall x \in \mathfrak{X},$$

and from (3.10), (3.11) we have

$$\mathrm{tr}(\overline{\chi} * \psi) = \mathrm{tr}(\psi * \overline{\chi}) = (\psi, \chi)_W, \quad (3.12)$$

where the trace is defined by the integral (3.10).

Theorem 3.1 *The space $W(\mathfrak{X})$ endowed with the product $*$ (3.8) is an associative algebra with unity element 1 and with involution $\psi \rightarrow \overline{\psi}$. Because of (3.12) $W(\mathfrak{X})$ is a Frobenius algebra, and moreover:*

$$|(\psi * \chi)(x)| \leq \|\psi * \chi\|_W \leq \|\psi\|_W \cdot \|\chi\|_W, \quad \forall x \in \mathfrak{X}.$$

The mapping $\psi \rightarrow \widehat{\psi}$ given by (3.6) is an isomorphism of this algebra to the algebra of Hilbert–Schmidt operators in $\mathcal{L}(\mathfrak{X})$, i.e.,

$$\widehat{\psi} \cdot \widehat{\chi} = \widehat{\psi * \chi}. \quad (3.13)$$

As usual, one can extend the algebra $W(\mathfrak{X})$ in order to include not only the Hilbert–Schmidt operators. Anyway, the product $*$ in any extended algebra $\mathcal{F}_*(\mathfrak{X})$ is given by the same formula (3.8) until the integral in (3.8) makes sense (may be, as a distribution), the same is about the representation of this algebra by Wick pseudodifferential operators given by (3.6).

Note that the Wick product (3.8) is represented by the probability operator \mathcal{P} (3.1):

$$(\psi * \chi)(x) = \mathcal{P}_{y \rightarrow x}(\psi^\#(x|y)\chi^\#(y|x)), \quad (3.14)$$

where the subscript $y \rightarrow x$ indicates that the operator \mathcal{P} acts by the variable y and the result is a functions of the variable x .

If both the functions ψ, χ are smooth and do not depend on the deformation parameter \hbar (or depend smoothly as $\hbar \rightarrow 0$) and the reproducing measure also has the regular \hbar -expansion (2.18), then it is possible to derive from (3.14) the formal \hbar -power series expansion for the product $\psi * \chi$. Indeed, when solving equation (2.8) for the reproducing measure (2.11), we have already obtained the expression:

$$\mathcal{P} \sim I + \sum_{k \geq 1} \hbar^k \mathcal{P}^{(k)}, \quad \mathcal{P}^{(k)} = \sum_{s=0}^k f^{(k-s)}(x, \nabla) \circ \sigma_s, \quad (3.15)$$

where the symbols $f^{(k)}$ and functions σ_s are given by (2.16), (2.17) (we denote $f^{(0)} \equiv 1$, $\sigma_0 \equiv 1$). The coefficients $\mathcal{P}^{(k)}$ of this expansion are differential operators of order $2k$ determined by the Kähler form ω only. For instance,

$$\mathcal{P}^{(1)} = \frac{1}{2}\Delta \equiv \omega^{-1\mu\nu}\partial_\mu\bar{\partial}_\nu, \quad \mathcal{P}^{(2)} = \frac{1}{8}\Delta^2 + \frac{1}{2}\rho^{\mu\nu}\partial_\mu\bar{\partial}_\nu, \quad (3.16)$$

where $((\rho^{\mu\nu}))$ is the Ricci tensor. After the substitution of (3.15) to (3.14), we obtain the following formal \hbar -power expansion for the Wick product:

$$(\psi * \chi)(x) \sim \psi(x)\chi(x) + \sum_{k \geq 1} \hbar^k \psi(x) \underset{\leftrightarrow}{\mathcal{P}}^{(k)} \chi(x), \quad x \in \mathfrak{X}, \quad (3.17)$$

where the differential operators $\mathcal{P}^{(k)} = \underset{\leftrightarrow}{\mathcal{P}}^{(k)}$ are defined by (3.15); they act to the left by holomorphic coordinates $z(x)$ and act to the right by antiholomorphic coordinates $\bar{z}(x)$. A different way of calculation of the coefficients $\mathcal{P}^{(k)}$ in (3.17), based on the quantum tensor calculus, was suggested in [25]; other interesting approaches and a list of references can be found in [10, 42].

Finally, we remark that, in general, the choice of the reproducing measure dm is not unique. One can replace it by the measure $(1 + \psi_\hbar) dm$, where ψ_\hbar is any function from the null space $\text{Ker } \mathcal{P}$ such that $|\psi_\hbar| < 1$. This null space is certainly very big for the compact \mathfrak{X} , but all such functions ψ_\hbar are highly oscillating as $\hbar \rightarrow 0$ (see Example 4.1 below). So the assumption about the regular dependence of dm as $\hbar \rightarrow 0$ is not redundant for the expansion (3.17).

4 Quantum restriction onto coadjoint orbits

First of all, we consider the relationship between Wick pseudodifferential operators and operators given by the *geometric quantization theory*.

Let us introduce the Poisson subalgebra $\mathcal{F}^{(1)}(\mathfrak{X})$ consisting of smooth functions over \mathfrak{X} whose Hamiltonian flow preserves the complex polarization. We denote by $\text{ad}(\psi)$ the Hamiltonian field corresponding to the function ψ and split this field in components along the complex polarization and along the conjugate one:

$$\text{ad}(\psi) = \text{ad}_+(\psi) + \text{ad}_-(\psi).$$

If $\psi \in \mathcal{F}^{(1)}(\mathfrak{X})$ then the complex vector field $\text{ad}_-(\psi)$ transfers any antiholomorphic section to an antiholomorphic section.

Lemma 4.1 *If $\psi \in \mathcal{F}^{(1)}(\mathfrak{X})$ then the Wick pseudodifferential operator $\widehat{\psi}$ (3.6) is the following first order differential operator:*

$$\widehat{\psi} = \psi + i\hbar \text{ad}_-(\psi)(F) - i\hbar \text{ad}_-(\psi). \quad (4.1)$$

Moreover, the Dirac axiom holds:

$$\frac{i}{\hbar}[\widehat{\psi}, \widehat{\chi}] = \widehat{\{\psi, \chi\}}, \quad \forall \psi, \chi \in \mathcal{F}^{(1)}(\mathfrak{X}). \quad (4.2)$$

Note that (4.1) is exactly the construction of the geometric quantization over Kählerian manifolds (see [34]). So, the Wick pseudodifferential calculus extends the geometric quantization scheme from the Lie algebra level (4.2) to the associative algebra level (3.13). In particular, for generic $\psi \notin \mathcal{F}^{(1)}(\mathfrak{X})$, the operator $\widehat{\psi}$ ceases to be the first order differential operator.

Example 4.1 Let \mathfrak{X} be the unit sphere

$$\mathbb{S}^2 = \{\xi \cdot \xi = 1\} \subset \mathbb{R}^3, \quad \xi = (\xi^1, \xi^2, \xi^3), \quad \xi^j \text{ are Cartesian coordinates.}$$

The complex structure is standard: $z = \frac{\xi^1 + i\xi^2}{1 - \xi^3}$, $\omega = \omega_0 = \frac{2i d\bar{z} \wedge dz}{(1 + |z|^2)^2}$. The quantization condition (2.2) implies $\hbar = 2/N$, where $N \in \mathbb{Z}_+$. The reproducing measure is $dm = (1 + \hbar/2) dm^\omega$, and the probability function is $p = \left(\frac{1 + \xi \cdot \xi'}{2}\right)^N$, where $\xi, \xi' \in \mathbb{S}^2$. The probability operator is given by the formula

$$\mathcal{P} = \sum_{k=0}^N \frac{(N+1)!N!}{(N+1+k)!(N-k)!} P^{(k)},$$

where $P^{(k)}$ is the projection to the space $\mathcal{F}^{(k)}$ of k th spherical harmonics, that is to the eigenspace of the Laplacian $\Delta_{\mathbb{S}^2}$ corresponding to the eigenvalue $k(k+1)$. In this case

$$\text{Ker } \mathcal{P} = \bigoplus_{k \geq N+1} \mathcal{F}^{(k)}, \quad \mathcal{F}_*(\mathbb{S}^2) = \bigoplus_{0 \leq k \leq N} \mathcal{F}^{(k)}, \quad \dim \mathcal{F}_*(\mathbb{S}^2) = (N+1)^2.$$

For instance, if $N = 1$ (that is, $\hbar = 2$), then $\dim \mathcal{F}_*(\mathbb{S}^2) = 4$. In this case let us take the following basis in $\mathcal{F}_*(\mathbb{S}^2)$: $1, x^j \stackrel{\text{def}}{=} \xi^j|_{\mathbb{S}^2}$ ($j = 1, 2, 3$). Then

the Wick algebra structure (3.8) on $\mathcal{F}_*(\mathbb{S}^2)$ is generated by relations between basis functions:

$$\begin{aligned} x^j * x^j &= 1 \quad (j = 1, 2, 3), \\ x^1 * x^2 &= -x^2 * x^1 = -ix^3, \\ x^2 * x^3 &= -x^3 * x^2 = -ix^1, \\ x^3 * x^1 &= -x^1 * x^3 = -ix^2. \end{aligned}$$

So, in this case $\mathcal{F}_*(\mathbb{S}^2)$ is *isomorphic to the quaternion algebra*.

Note that for arbitrary N all functions x^j belong to the Poisson subalgebra $\mathcal{F}^{(1)}(\mathbb{S}^2) \approx \mathfrak{su}(2)$, and the first order differential operators \widehat{x}^j (4.1) represent this Lie algebra in the $(N + 1)$ -dimensional Hilbert space $\mathcal{L}(\mathbb{S}^2)$:

$$\frac{i}{\hbar}[\widehat{x}^1, \widehat{x}^2] = \widehat{x}^3, \quad \frac{i}{\hbar}[\widehat{x}^2, \widehat{x}^3] = \widehat{x}^1, \quad \frac{i}{\hbar}[\widehat{x}^3, \widehat{x}^1] = \widehat{x}^2, \quad \widehat{x}^{j*} = \widehat{x}^j \quad (j = 1, 2, 3).$$

The explicit formulas for \widehat{x}^j are the following:

$$\widehat{x}^1 - i\widehat{x}^2 = \hbar\bar{z}(\bar{z}\bar{\partial} - N), \quad \widehat{x}^1 + i\widehat{x}^2 = -\hbar\bar{\partial}, \quad \widehat{x}^3 = \hbar(\bar{z}\bar{\partial} - N/2).$$

The direct derivation of the Casimir operator gives

$$(\widehat{x}^1)^2 + (\widehat{x}^2)^2 + (\widehat{x}^3)^2 = (1 + \hbar) \cdot I \neq I. \quad (4.3)$$

But the classical Casimir is $(x^1)^2 + (x^2)^2 + (x^3)^2 = \xi \cdot \xi|_{\mathbb{S}^2} = 1$. The difference between classical and quantum values of the Casimir elements is one of the reasons which gives rise to the question about the quantum restriction operation (1.3).

Now let us consider a general Lie algebra \mathfrak{g} and its coadjoint orbits [31] $\mathfrak{X} \subset \mathfrak{g}^*$ with standard symplectic form $\omega = \omega_0$ generated on \mathfrak{X} by the linear Lie–Poisson brackets from \mathfrak{g}^* . We assume that $\dim \mathfrak{X}$ is maximal and there is a \mathfrak{g} -invariant complex structure on \mathfrak{X} with respect to which the form ω_0 is Kählerian (for instance, it is true if \mathfrak{g} is compact). Assume that the quantization condition (2.2) holds and construct the space $\mathcal{L}(\mathfrak{X})$ using Kähler potentials $F = F_0$ of the form ω_0 and the measure $dm = \text{const} \cdot dm^{\omega_0}$.

Any vector $\eta \in \mathfrak{g}$ is identified with the linear function over \mathfrak{g}^* by the formula $\eta(\xi) \stackrel{\text{def}}{=} \langle \eta, \xi \rangle \quad \forall \xi \in \mathfrak{g}^*$. Consider the classical restriction $\eta|_{\mathfrak{X}}$ of the function η to the orbit \mathfrak{X} . Then $\eta|_{\mathfrak{X}} \in \mathcal{F}^{(1)}(\mathfrak{X})$, and one can apply

the construction of the geometric quantization from Lemma 4.1. So, the first order differential operator $\widehat{\eta}|_{\mathfrak{X}}$ appears in the Hilbert space $\mathcal{L}(\mathfrak{X})$ of antiholomorphic sections over \mathfrak{X} .

Let us fix any basis η^1, \dots, η^m in \mathfrak{g} and denote $x^j = \eta^j|_{\mathfrak{X}}$. Then we have the set of operators \widehat{x}^j in $\mathcal{L}(\mathfrak{X})$.

Each polynomial f on $\mathfrak{g}^* \approx \mathbb{R}^m$ is given by a polynomial function of the basis elements $f = f(\eta^1, \dots, \eta^m)$. Its classical restriction onto the coadjoint orbit is given by $f|_{\mathfrak{X}} = f(x^1, \dots, x^m)$.

The *quantum restriction* $f|_{\widehat{\mathfrak{X}}}$ we define in such a way that

$$\widehat{f}|_{\widehat{\mathfrak{X}}} = f(\widehat{x}^1, \dots, \widehat{x}^m), \quad (4.4)$$

where on the right-hand side the operators \widehat{x}^j are Weyl-symmetrized.

The quantum product of two Weyl-symmetrized polynomials on \mathfrak{g}^* we denote by \star . So, in view of (4.4), we have

$$\widehat{f}|_{\widehat{\mathfrak{X}}} \cdot \widehat{g}|_{\widehat{\mathfrak{X}}} = \widehat{(f \star g)}|_{\widehat{\mathfrak{X}}}.$$

Using the Wick product $*$ of symbols over the Kählerian manifolds \mathfrak{X} , we obtain the desirable formula (1.4), i.e., the homomorphism $f \rightarrow f|_{\widehat{\mathfrak{X}}}$ from the algebra of polynomials $\mathcal{F}_*(\mathfrak{g}^*)$ to the Wick algebra $\mathcal{F}_*(\mathfrak{X})$.

To calculate the quantum restriction $f|_{\widehat{\mathfrak{X}}}$ explicitly, let us apply formula (3.7) to the relation (4.4):

$$f|_{\widehat{\mathfrak{X}}} = \text{tr} (f(\widehat{x}^1, \dots, \widehat{x}^m)\Pi) = \frac{1}{\mathcal{K}} f(\widehat{x}^1, \dots, \widehat{x}^m)\mathcal{K}.$$

Since $\mathcal{K} = \exp(F_0/\hbar)$ and \widehat{x}^j are defined by (4.1) with $F = F_0$, we obtain the following statement.

Theorem 4.1 *The quantum restriction of the function $f \in \mathcal{F}(\mathfrak{g}^*)$ onto the coadjoint orbit \mathfrak{X} with invariant complex structure is given by the formula*

$$f|_{\widehat{\mathfrak{X}}} = f(x - i\hbar \text{ad}_-(x)) 1. \quad (4.5)$$

On the right-hand side of (4.5) the first order differential operators, which are arguments of the function f , are Weyl symmetrized, and the resulting operator is applied to the unity function 1 over \mathfrak{X} .

Let us calculate the leading quantum correction of order \hbar in the formal \hbar -expansion (1.3) for the quantum restriction. From (4.5) it follows that

$$e_1 = \frac{1}{2} \mathbf{g}_-^{j\ell} D_j D_\ell = \frac{1}{4} (\Delta - \Delta(x^j) D_j). \quad (4.6)$$

Here D_j are partial derivatives by the coordinates $\xi^j = \langle \eta^j, \xi \rangle$ in \mathfrak{g}^* ; by Δ we denote the Laplace–Beltrami operator on the Kähler orbit $\mathfrak{X} \subset \mathfrak{g}^*$; and $\mathbf{g}_-^{j\ell}$ is the “polarized part” of the Lie–Poisson tensor, i.e.,

$$\mathbf{g}_-^{j\ell} \stackrel{\text{def}}{=} \omega_0^{-1\mu\nu} \partial_\mu x^j \cdot \bar{\partial}_\nu x^\ell = -i \operatorname{ad}_-(x^j)(x^\ell). \quad (4.7)$$

The tensor (4.7) is Hermitian and degenerate, rank $\mathbf{g}_- = \dim \mathfrak{X}$.

For instance, if f is a *quadratic function* on \mathfrak{g}^* , then its quantum restriction to the orbit \mathfrak{X} is precisely given by

$$f|_{\hat{\mathfrak{X}}} = f|_{\mathfrak{X}} + \frac{\hbar}{2} \operatorname{tr}(\mathbf{g}_- D^2 f).$$

Now, denote by $\{K_s\}$ a basis of Casimir functions of Lie–Poisson algebra $\mathcal{F}(\mathfrak{g}^*)$; then coadjoint orbits are determined by equations $\{K_s = \text{const}\}$, and $\{\partial/\partial K_s\}$ is a basis of normal vector fields over each orbit. Then we can split the operator e_1 (4.6) in components tangent and transversal to \mathfrak{X} :

$$e_1 = e_1^\parallel + e_1^\perp,$$

where the transversal component is the vector field given by

$$e_1^\perp = -\frac{1}{4} \Delta(x^j) \cdot D_j K_s \frac{\partial}{\partial K_s} = \frac{1}{2} \mathbf{g}_-^{j\ell} D_{j\ell}^2 K_s \frac{\partial}{\partial K_s}. \quad (4.8)$$

Note that the operator on the right-hand side of (4.8) does not depend on the choice of Casimir functions K_s .

Corollary 4.1 *The leading quantum correction e_1 in the \hbar -expansion (1.3) of the quantum restriction onto the coadjoint orbit $\mathfrak{X} \subset \mathfrak{g}^*$ is the second order differential operator given by (4.6). The component normal to the orbit is the first order operator given by (4.8). This normal vector field e_1^\perp is correctly defined on the domain \mathcal{N} of regular points in \mathfrak{g}^* , where the rank of the Lie–Poisson tensor is maximal. This field is a Poisson vector field, i.e., has*

zero Schouten brackets with the Lie–Poisson tensor, and so [26, 30, 35] it generates a Poisson cohomology class $[e_1^\perp] \in \mathcal{H}^1(\mathcal{N})$.

The field e_1^\perp differs only by a constant multiplier from the mean curvature vector field determined by the Levi–Civita connection on Kähler leaves $\mathfrak{X} \subset \mathfrak{g}^*$ and by the affine connection on \mathfrak{g}^* .

The last statement of this Corollary was conjectured by A. Weinstein in the discussion about formulas (4.6), (4.8) presented in the author’s lecture at the conference “Poisson–2000” (Luminy, June 2000).

The cohomology class $[e_1^\perp]$ is an important characteristic of the quantum embedding of quantized symplectic leaves to quantized Poisson manifolds.

Example 4.1 (continuation). In the case $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{X} = \mathbb{S}^2$ we have

$$e_1 = \frac{1}{4}\Delta_{\mathbb{S}^2} + \frac{\partial}{\partial K}, \quad K \stackrel{\text{def}}{=} (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2. \quad (4.9)$$

Since K itself is a quadratic function, we obtain the precise formula:

$$K|_{\widehat{\mathfrak{X}}} = K|_{\mathfrak{X}} + \hbar e_1(K) = 1 + \hbar.$$

This result exactly correlates with the derivation (4.3).

Now we calculate the quantum restriction (4.5) without any \hbar -expansions by using expression (1.3a). The symbol $E_{\mathfrak{X}}$ of the operator of quantum restriction is given by

$$E_{\mathfrak{X}}(\cdot, \eta) = e^{-i\eta/\hbar}|_{\mathfrak{X}} \cdot e^{i\eta/\hbar}|_{\widehat{\mathfrak{X}}}. \quad (4.10)$$

On the right-hand side we consider the vector $\eta \in \mathfrak{g}$ as a linear function on \mathfrak{g}^* and apply both types of restriction onto the orbit \mathfrak{X} : classical and quantum. For the quantum one let us apply Theorem 4.1, that is, take $f = \exp(i\eta/\hbar)$ in (4.5):

$$e^{i\eta/\hbar}|_{\widehat{\mathfrak{X}}} = \exp\left\{\frac{i}{\hbar}\langle\eta, x - i\hbar \text{ad}_-(x)\rangle\right\}1.$$

The operators $\text{ad}_-(x)$ are of the first order, and so the action of the last exponent can be evaluated by the method of characteristics. Thus we obtain from(4.10):

$$E_{\mathfrak{X}}(\xi, \eta) = \exp\left\{\frac{i}{\hbar}\left(\int_0^1 \langle\eta, \Xi_-(t, \xi, \eta)\rangle dt - \langle\eta, \xi\rangle\right)\right\} = \exp\left\{-\frac{\eta\widetilde{\mathfrak{g}}\eta}{\hbar}\right\}. \quad (4.11)$$

Here Ξ_- is the characteristic:

$$\frac{d}{dt}\Xi_- = i\eta\mathbf{g}_-(\Xi_-), \quad \Xi_- \Big|_{t=0} = \xi \in \mathfrak{X}, \quad (4.12)$$

the tensor \mathbf{g}_- is defined by (4.7), and

$$\tilde{\mathbf{g}}(\xi, \eta) \stackrel{\text{def}}{=} \int_0^1 (1-t)\mathbf{g}_-(\Xi_-(t, \xi, \eta)) dt.$$

Corollary 4.2 (a) *The quantum restriction onto the coadjoint orbit \mathfrak{X} is given by formula (1.3a), where the symbol $E_{\mathfrak{X}}$ is determined by (4.11) via the trajectories Ξ_- of the “polarized” Lie–Hamilton system (4.12).*

(b) *The group elements (1.12) are given by the formula*

$$e_*^{i\eta/\hbar}(\xi) = \exp \left\{ -\frac{i}{\hbar}\eta\tilde{\mathbf{g}}(\xi, \eta)\eta + \frac{i}{\hbar}\langle \eta, \xi \rangle \right\}, \quad \xi \in \mathfrak{X}. \quad (4.13)$$

Some other way to calculate the exponential function from Corollary 4.2 (b) was presented in [23, 24] using the areas of dynamical membranes in $\mathfrak{X}^\#$.

Note that the trajectories Ξ_- in (4.12) leave the real coadjoint orbit \mathfrak{X} and belong to its complexification. Nevertheless, the action functions in the exponents (4.11), (4.13) have a nonnegative imaginary part. The group elements $e_*^{i\eta/\hbar}$ are asymptotically (as $\hbar \rightarrow 0$) concentrated at the points where this imaginary part vanishes, namely, at fixed points of the coadjoint action of $\exp(\eta)$ on \mathfrak{X} . This *effect of concentration* is purely quantum one. The quantum group element $e_*^{i\eta/\hbar}$ is exponentially small as $\hbar \rightarrow 0$ outside the set of fixed points. Thus we observe the appearance of the *tunneling* under a certain “energy barrier”, and we obviously see the difference between the classical and quantum restrictions of the function $e^{i\eta/\hbar}$ to the orbit.

Example 4.1 (continuation). In the case $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{X} = \mathbb{S}^2 = \{\xi \cdot \xi = 1\}$ we have

$$\mathbf{g}_-^{jk}(\xi) = \frac{1}{2}(\delta^{j\ell} - \xi^j\xi^\ell).$$

Instead of the vector characteristic Ξ_- we consider the scalar $X = \langle \eta, \Xi_- \rangle$. Then equations (4.12) are reduced to

$$\frac{d}{dt}X = \frac{i}{2}(|\eta|^2 - X^2), \quad X \Big|_{t=0} = \langle \eta, \xi \rangle.$$

The solution is

$$X = |\eta| \frac{(\langle \eta, \xi \rangle + |\eta|) \exp\{i|\eta|t\} + \langle \eta, \xi \rangle - |\eta|}{(\langle \eta, \xi \rangle + |\eta|) \exp\{i|\eta|t\} - \langle \eta, \xi \rangle + |\eta|}.$$

Thus from formula (4.11) we obtain the symbol $E_{\mathbb{S}^2}$ of the quantum restriction operation (1.3a):

$$\begin{aligned} E_{\mathbb{S}^2}(\xi, \eta) &= \exp \left\{ \frac{i}{\hbar} \left(\int_0^1 X(t, \xi, \eta) dt - \langle \eta, \xi \rangle \right) \right\} \\ &= \left(\cos \frac{|\eta|}{2} + \frac{i}{|\eta|} \sin \frac{|\eta|}{2} \cdot \langle \eta, \xi \rangle \right)^{2/\hbar} e^{-i\langle \eta, \xi \rangle / \hbar} \end{aligned} \quad (4.14)$$

(we recall that $2/\hbar = N$ is an integer number, and $\xi \in \mathbb{S}^2$). The approximation of $E_{\mathbb{S}^2}(\xi, \eta)$ near $\eta = 0$ is the following:

$$E_{\mathbb{S}^2}(\xi, \eta) = \exp \left\{ -\frac{1}{4\hbar} (|\eta|^2 - \langle \eta, \xi \rangle^2) + O(\eta^3/\hbar) \right\}. \quad (4.15)$$

So, the quantum restriction operator $f|_{\widehat{\mathbb{S}^2}} = \left(E_{\mathbb{S}^2}(\xi, -i\hbar \partial/\partial\xi) f(\xi) \right)|_{\mathbb{S}^2}$ is given precisely by formula (4.14), and its first approximation (on \hbar -independent functions) is evaluated by (4.15):

$$E_{\mathbb{S}^2}(\xi, -i\hbar \partial/\partial\xi) = I + \frac{\hbar}{4} \sum_{s,\ell=1}^3 (\delta^{s\ell} - \xi^s \xi^\ell) \frac{\partial^2}{\partial \xi^s \partial \xi^\ell} + O(\hbar^2).$$

From this formula we obtain, of course, the same expression (4.9) for the first quantum correction e_1 . All other corrections e_j ($j \geq 2$) in expansion (1.3) are easily extracted from the precise formula (4.14).

The group elements in this example are derived from (4.14):

$$e_*^{i\eta/\hbar} = E_{\mathbb{S}^2} e^{i\eta/\hbar} = \left(\frac{\cos(|\eta|/2)}{\cos(S_\eta/2)} \right)^{2/\hbar} e^{iS_\eta/\hbar}, \quad (4.16)$$

where $S_\eta(\xi) = 2 \arctan(\langle \frac{\eta}{|\eta|}, \xi \rangle \tan \frac{|\eta|}{2})$ and $2/\hbar = N$. This formula presents a realization of the Lie group $G = \text{SU}(2)$ in the function algebra $\mathcal{F}_*(\mathbb{S}^2)$ (of spherical harmonics of order $\leq N$). The corresponding Wick differential

operators make up the Hermitian irreducible representation $\pi_{\mathbb{S}^2}$ of the group $SU(2)$ in the space $\mathcal{L}(\mathbb{S}^2)$ (of all polynomials of degree $\leq N$).

Let the vector $\eta \neq 0$ belong to the domain $\{|\eta| < 2\pi\}$ where the exponential mapping $\exp : \mathfrak{su}(2) \rightarrow SU(2)$ is one-to-one. Then the function (4.16) is concentrated at two points $\xi_{\pm} = \pm\eta/|\eta|$, and it is exponentially small (as $\hbar \rightarrow 0$) out of this set. The points ξ_{\pm} are fixed points of the coadjoint action of the element $\exp(\eta) \in SU(2)$ on the orbit $\mathfrak{X} = \mathbb{S}^2$. Of course, the exceptional value $\eta = 0$ corresponds to the unity element of the group: in this case the function $e_*^{i\eta/\hbar}$ is equal to 1 identically. On the boundary circle $|\eta| = 2\pi$ the function $e_*^{i\eta/\hbar}$ is equal to $(-1)^N$, and so, if N is even (what corresponds to representations of $SO(3)$) then $e_*^{i\eta/\hbar}$ is the same unity function 1. But if N is odd then one has to go to the second sheet of the universal covering, that is, to the domain $2\pi < |\eta| < 4\pi$, in order to obtain the realization of the whole group $SU(2)$ in the function space $\mathcal{F}_*(\mathbb{S}^2)$.

5 Irreducible representations and special functions corresponding to complex polarizations

As we saw in Section 4, the construction of quantum restriction to coadjoint orbits is actually equivalent to the construction of irreducible representations corresponding to these orbits (formulas (1.8), (4.4) demonstrate this relationship). Until we are in the case of Lie algebras, we can use the geometric quantization theory to produce these irreducible representations. But if we deal with nonlinear Poisson brackets and with algebras $\mathcal{F}_*(\mathcal{M})$ generated by non-Lie permutation relations, then the problem of quantum restriction onto symplectic leaves $\mathfrak{X} \subset \mathcal{M}$ becomes rather difficult, since in this case there is no general construction of irreducible representations.

In the given section, following [25], we describe one possible construction based on the notion of complex polarization of quantum algebra. In what follows, we modify the approach [25] in order to avoid the use of the complexified phase space (complexified symplectic groupoid) over \mathcal{M} .

Let $\mathcal{F} = \mathcal{F}(\mathcal{M})$ be a quantized algebra of functions over \mathcal{M} with an associative multiplication \star satisfying (1.1), with the unity element, and the involution $f \rightarrow \bar{f}$. For any two subspaces $\mathcal{R}, \mathcal{T} \subset \mathcal{F}$ we denote by $\mathcal{R} \otimes \mathcal{T}$ the subspace obtained by the composition of tensor product and multiplication \star .

Let \mathcal{F}^+ be a complex polarization in \mathcal{F} , i.e., a subalgebra with the following properties:

- (i) the subspace $\overline{\mathcal{F}^+} \star \mathcal{F}^+$ coincides with \mathcal{F} ,
- (ii) the subalgebra $\mathcal{F}^0 \stackrel{\text{def}}{=} \mathcal{F}^+ \cap \overline{\mathcal{F}^+}$ is commutative.

A point $a \in \mathcal{M}$ is called a *vacuum point* with respect to the polarization if

- (iii) $(f \star g)(a) = f(a)g(a) \quad \forall f \in \mathcal{F}, \quad \forall g \in \mathcal{F}^+$.

Under these conditions we call \star the *normal product*.

Let us choose a commutative functional basis Z^1, \dots, Z^n in $\mathcal{F}^+ \setminus \mathcal{F}_0$ and denote by P_1, \dots, P_n the Darboux dual subset of functions in \mathcal{F} :

$$[Z^s, P_j]_{\star} = \hbar \delta_j^s, \quad [Z^s, Z^j] = [P_s, P_j]_{\star} = 0 \quad (s, j = 1, \dots, n). \quad (5.1)$$

To simplify the construction, we assume that Z^s and P_j satisfy the boundary conditions

$$Z^s(a) = 0, \quad P_j(a) = 0 \quad (s, j = 1, \dots, n). \quad (5.2)$$

Also assume that the following \star -exponential exists in the algebra \mathcal{F} :

$$U_z \stackrel{\text{def}}{=} \exp_{\star}(zP/\hbar), \quad z = (z^1, \dots, z^n) \in \mathcal{D}. \quad (5.3)$$

Here $\mathcal{D} \subset \mathbb{C}^n$ is a domain containing the point $z = 0$. The function U_z is the solution of the Cauchy problem

$$\hbar \partial U_z / \partial z = P \star U_z, \quad U_0 = 1 \quad (5.4)$$

(the general \star -exponential was considered in [3]; see the discussion in [18]). We denote

$$\mathcal{K}(\bar{z}|z) \stackrel{\text{def}}{=} (\overline{U_z} \star U_z)(a) = \sum_{|\alpha|, |\beta| \geq 0} \frac{\bar{z}^{\alpha} z^{\beta}}{\hbar^{|\alpha|+|\beta|}} k_{\alpha\beta}, \quad (5.5)$$

where

$$k_{\alpha\beta} = \frac{1}{\alpha! \beta!} (\overline{P^{\star\alpha}} \star P^{\star\beta})(a).$$

The sign \star in notation of powers means that these powers are taken in the algebra \mathcal{F} (i.e., in the sense of the normal product \star). In addition to the above properties (i)–(iii), we introduce the quantum Kählerian condition:

(iv) the matrix $((k_{\alpha\beta}))$ is positive definite (maybe, not strictly).

By \mathcal{L}_a we denote the space of antiholomorphic distributions $u(\bar{z}) = \sum \bar{z}^\alpha u_\alpha$ generated by vectors $((u_\alpha))$ orthogonal to the null kernel of the matrix $((k_{\alpha\beta}))$, and on \mathcal{L}_a introduce the Hilbert norm

$$\|u\| \stackrel{\text{def}}{=} \left(\sum \hbar^{|\alpha|+|\beta|} k^{-1\alpha\beta} u_\alpha \bar{u}_\beta \right)^{1/2}. \quad (5.6)$$

Obviously, the function \mathcal{K} (5.4) is just the reproducing kernel of the space \mathcal{L}_a , that is, the integral kernel of the unitary operator in \mathcal{L}_a . One can ask about the reproducing measure dm for the space \mathcal{L}_a :

$$\|u\| = \left(\frac{1}{(2\pi\hbar)^n} \int \frac{|u|^2}{\mathcal{K}} dm \right)^{1/2}, \quad (5.7)$$

where the integral is taken over the domain \mathcal{D} . Then, in the space \mathcal{L}_a we can define the Wick pseudodifferential operators using the kernel \mathcal{K} and the Hilbert structure (5.6) or (5.7). We denote these operators again by the hat sign: $\widehat{\psi}$, where ψ are symbols on \mathcal{D} extended to symbols $\psi^\#(\bar{z}|z)$ on $\mathcal{D} \times \mathcal{D}$ holomorphic in z and in \bar{z} .

Let us take $f \in \mathcal{F}(\mathcal{M})$ and determine the following Wick symbol

$$f_a(\bar{z}|z) = \frac{(\bar{U}_z \star f \star U_z)(a)}{(\bar{U}_z \star U_z)(a)}. \quad (5.8)$$

Theorem 5.1 *The correspondence*

$$f \rightarrow \widehat{f}_a \quad (5.9)$$

is an irreducible Hermitian representation of the algebra $\mathcal{F}(\mathcal{M})$ (with the normal product \star) in the Hilbert space \mathcal{L}_a .

Explicit formulas for operators \widehat{f}_a were given in [25] via symbols of left and right quantum reduction mappings [21, 26] in the complexified symplectic groupoid over \mathcal{M} .

Consider what happens with coordinate functions $Z^s \in \mathcal{F}^+$ under the correspondence (5.9). From (5.8) and (5.1) it follows that

$$(Z^s)_a = z^s, \quad s = 1, \dots, n. \quad (5.10)$$

Thus the representation (5.9) transforms each function \overline{Z}^s to the operator of multiplication by \overline{z}^s , and each function Z^s to the adjoint operator $\widehat{z}^s = (\overline{z}^s)^*$ in the Hilbert space \mathcal{L}_a .

Now let us take a basis of functions A^1, \dots, A^k in the subalgebra \mathcal{F}^0 (from condition (ii)) and pass to the classical limit:

$$\mathcal{A}^j(\overline{z}|z) \stackrel{\text{def}}{=} \lim_{\hbar \rightarrow 0} (A^j)_a(\overline{z}|z), \quad j = 1, \dots, k.$$

Explicit formulas for functions \mathcal{A}^j are given in [25].

Note that Z^1, \dots, Z^n are complex coordinates, and A^1, \dots, A^k are real coordinates on \mathcal{M} , and $2n + k = \dim \mathcal{M}$, $2n = \dim \mathfrak{X}_a$.

Corollary 5.1 *In the Poisson manifold \mathcal{M} the symplectic leaf \mathfrak{X}_a (containing the vacuum point a) is given by the following parametrization of coordinates:*

$$Z = z, \quad A = \mathcal{A}(\overline{z}|z). \quad (5.11)$$

Here z is running over a domain $\mathcal{D} \subset \mathbb{C}^n$. Formula (5.11) introduces the complex structure to the leaf \mathfrak{X}_a from the local chart \mathcal{D} , and transfers each symbol (5.8) onto \mathfrak{X}_a . The symbols f_a are quantum restrictions of functions f to the leaf:

$$f|_{\widehat{\mathfrak{X}}_a} = f_a, \quad (f \star g)|_{\widehat{\mathfrak{X}}_a} = f|_{\widehat{\mathfrak{X}}_a} * g|_{\widehat{\mathfrak{X}}_a},$$

where $*$ is the Wick product over \mathfrak{X}_a generated by the Kählerian form

$$\omega = i\hbar \overline{\partial} \partial \ln \mathcal{K}. \quad (5.12)$$

If the reproducing measure dm (5.7) exists, then the space \mathcal{L}_a is identified with the space $\mathcal{L}(\mathfrak{X}_a)$ of antiholomorphic sections. The quantum Kählerian objects on \mathfrak{X}_a in the classical limit $\hbar \rightarrow 0$ admit the asymptotics

$$\omega = \omega_0 + O(\hbar), \quad dm = dm^{\omega_0} + O(\hbar), \quad (5.13)$$

where the symplectic form ω_0 on \mathfrak{X}_a is generated from \mathcal{M} by the Poisson structure.

The reproducing kernel \mathcal{K} (5.5) is the key object of the construction just described. This kernel determines the quantum Kähler structure (5.12) on the symplectic leaf \mathfrak{X}_a satisfying the quantization condition $\frac{1}{2\pi\hbar}[\omega] \in$

$H^2(\mathfrak{X}_a, \mathbb{Z})$. Note that \mathcal{K} , as a holomorphic section of the complex line bundle over $\mathfrak{X}_a^\#$, does not depend on the choice of the bases $\{Z^j\}$ and $\{P_j\}$ (modulo changes of variables and transfers to equivalent bundles). That is why we call the kernel (5.5) a *special function* corresponding to the quantum complex polarization over the symplectic leaf \mathfrak{X}_a with the vacuum point a .

In the next section we demonstrate some examples where \mathcal{K} turns out to be a hypergeometric function.

6 Quantum surfaces of revolution and hypergeometric functions

As an example we consider two-dimensional leaves (surfaces) of simplest topology but with arbitrary complex structure. Namely, let us consider the algebraic surface

$$\rho(t) - (S_1^2 + S_2^2) = K = \text{const}, \quad (6.1)$$

where S_1, S_2, t are coordinates in \mathbb{R}^3 and the function ρ is a polynomial. For each t Eq. (6.1) describes a circle, so (6.1) is a surface of revolution with the axis t . The topology depends on values of the constant K . If $\max \rho > K > \min \rho$, then the surface is homeomorphic to the plane or to the sphere, if $K < \min \rho$, then the surface is homeomorphic to the cylinder or the torus.

It is more convenient to deal with surfaces embedded in \mathbb{R}^{k+2} (with $k \geq 1$) by equations

$$\mathfrak{X} = \begin{cases} \rho(A) - (S_1^2 + S_2^2) = K = \text{const}, \\ \varkappa_j(A) = \text{const}_j \quad (j = 1, \dots, k-1), \end{cases} \quad (6.2)$$

where $A = (A_1, \dots, A_k)$, A_μ are coordinates in \mathbb{R}^k . Additional independent polynomial functions $\varkappa_1, \dots, \varkappa_{k-1}$ in (6.2) determine an algebraic curve in \mathbb{R}^k playing the role of the axis of revolution. Note that somewhere we need to consider not the whole \mathbb{R}^k but only a suitable domain in \mathbb{R}^k ; we shall do this without making additional notations.

Our goal is to introduce a quantum version of the surfaces (6.2). First of all, one has to quantize the space \mathbb{R}^{k+2} to where these surfaces are embedded. To do this, let us introduce a Poisson structure to \mathbb{R}^{k+2} such that surfaces (6.2) are symplectic leaves (i.e., K and \varkappa_j are Casimir functions).

Let v be a vector field on \mathbb{R}^k annulling all \varkappa_j ; that is, \varkappa_j are integrals of motion for the dynamical flow

$$\gamma^t : \mathbb{R}^k \rightarrow \mathbb{R}^k, \quad \frac{d}{dt}\gamma^t = v(\gamma^t), \quad \gamma^0 = \text{id}. \quad (6.3)$$

If we like to live inside the algebraic case, we have to assume that this flow preserves polynomials. It is certainly true if all the components $v_\mu(A)$ of the field $v = v_\mu(A)\partial/\partial A_\mu$ are polynomial, and also the Jacobi matrices $((\partial v_\mu/\partial A_\nu))$ are upper (or lower) triangle for all $A \in \mathbb{R}^k$; then the trajectories $\gamma^t(A)$ are polynomial in A and in t .

Denote $\lambda = v(\rho)$, and introduce the following Poisson brackets on \mathbb{R}^{k+2} :

$$\begin{aligned} \{S_2, S_1\} &= \frac{1}{2}\lambda(A), & \{A_\mu, S_1\} &= v_\mu(A)S_2, & \{A_\mu, S_2\} &= -v_\mu(A)S_2, \\ \{A_\mu, A_\nu\} &= 0, & \mu, \nu &= 1, \dots, k. \end{aligned} \quad (6.4)$$

Lemma 6.1 *Symplectic leaves of the Poisson structure (6.4) are given by Eqs. (6.2).*

If we denote $C = \overline{B} = S_1 + iS_2$, then the nontrivial relations (6.4) take the form

$$\{C, B\} = i\lambda(A), \quad \{C, A\} = iv(A)C. \quad (6.4a)$$

To define quantum permutation relations we will use the dynamical flow (6.3) as a *deforming flow* considering the time variable as a quantum deformation parameter.

Denote $\lambda^{\hbar}(A) = \rho(A) - \rho(\gamma^{-\hbar}(A))$, where $\hbar > 0$, and set the permutation relations between quantum generators:

$$[\mathbf{C}, \mathbf{B}] = \lambda^{\hbar}(\mathbf{A}), \quad \mathbf{C}\mathbf{A} = \gamma^{\hbar}(\mathbf{A})\mathbf{C}, \quad [\mathbf{A}_\mu, \mathbf{A}_\mu] = 0, \quad \mathbf{C} = \mathbf{B}^*, \quad \mathbf{A} = \mathbf{A}^*. \quad (6.5)$$

Note that $\rho(\mathbf{A}) - \mathbf{C}\mathbf{B}$ and $\varkappa_j(\mathbf{A})$ are the Casimir elements (the center elements) of the algebra generated by relations (6.5). Also note that in the limit as $\hbar \rightarrow 0$, under the assumption that $\frac{i}{\hbar}[\cdot, \cdot] \rightarrow \{\cdot, \cdot\}$, relations (6.5) are transferred to (6.4) or (6.4a).

Let us introduce an associative quantum product \star over \mathbb{R}^{k+2} corresponding to algebra (6.5):

$$[f(\overset{3}{\mathbf{B}}, \overset{2}{\mathbf{A}}, \overset{1}{\mathbf{C}})] \cdot [g(\overset{3}{\mathbf{B}}, \overset{2}{\mathbf{A}}, \overset{1}{\mathbf{C}})] = k(\overset{3}{\mathbf{B}}, \overset{2}{\mathbf{A}}, \overset{1}{\mathbf{C}}), \quad k = f \star g. \quad (6.6)$$

Here f, g are arbitrary polynomial. We use the normal ordering of generators and pose indices $1, 2, \dots$ indicating their order from right to left as in [37]. The product \star can be explicitly calculated using the technique of “regular representation” (see, for example, [26], Appendix 2). Namely,

$$(f \star g)(B, A, C) = f(L_B^3, L_A^2, L_C^1) g(B, A, C), \quad (6.7)$$

where $L_B = B\star$, $L_A = A\star$, $L_C = C\star$ are operators of the left regular representation of permutation relations. In the case of relations (6.5) it is rather easy to derive (see [28]):

$$L_B = B, \quad L_A = \gamma^{\hbar B \partial / \partial B}(A), \quad L_C = C \gamma^{\hbar \star} + \Lambda^{\hbar}(A, \overset{2}{B} \partial / \overset{1}{\partial B}) \partial / \partial B. \quad (6.8)$$

Here the function Λ^{\hbar} is defined by

$$\Lambda^{\hbar}(A, t) = \frac{\exp\{t(\gamma^{\hbar \star} - I)\} - I}{t(\gamma^{\hbar \star} - I)} \lambda^{\hbar}(A),$$

and $\gamma^{\hbar \star}$ denotes the shift operator by the variable A , i.e., $(\gamma^{\hbar \star} f)(A) = f(\gamma^{\hbar}(A))$.

Lemma 6.2 *Formulas (6.7), (6.8) determine the associative product \star over \mathbb{R}^{k+2} satisfying (6.6). This is the normal product with respect to the polarization $\mathcal{F}^+ = \{f(A, C)\}$ and the vacuum point $S_1 = S_2 = 0$, $A = a$, where $a \in \mathbb{R}^k$ is arbitrarily fixed.*

Later on we denote the vacuum point $(0, 0, a) \in \mathbb{R}^{k+2}$ by the same letter a , as we did in Section 5.

Example 6.1 The Lie algebra $\mathfrak{su}(1, 1)$ is generated by the relations

$$[\mathbf{S}_1, \mathbf{S}_2] = i\hbar \mathbf{S}_3, \quad [\mathbf{S}_2, \mathbf{S}_3] = -i\hbar \mathbf{S}_1, \quad [\mathbf{S}_3, \mathbf{S}_1] = -i\hbar \mathbf{S}_2.$$

Denote $\mathbf{C} = \mathbf{S}_1 + i\mathbf{S}_2 = \mathbf{B}^*$, $\mathbf{A} = \mathbf{S}_3 + \hbar/2$. Then we obtain relations of the type (6.5):

$$\mathbf{CB} = \mathbf{BC} + 2\hbar \mathbf{A} - \hbar^2, \quad \mathbf{CA} = (\mathbf{A} + \hbar)\mathbf{C}.$$

The deforming flow $\gamma^t : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\gamma^t(A) = A + t$, and

$$\lambda^{\hbar}(A) = 2\hbar A - \hbar^2, \quad \rho(A) = A^2, \quad \Lambda^{\hbar}(A, t) = 2\hbar A + \hbar^2 t - \hbar^2.$$

So, by formulas (6.8) we obtain

$$L_B = B, \quad L_A = A + \hbar \frac{\partial}{\partial B}, \quad L_C = Ce^{\hbar \partial / \partial A} + 2\hbar A \frac{\partial}{\partial B} + \hbar^2 B \frac{\partial^2}{\partial B^2} - \hbar^2 \frac{\partial}{\partial B}.$$

Thus the normal \star -product corresponding to the polarization $\mathcal{F}^+ = \{(A, C)\}$ of the Lie algebra $\mathfrak{su}(1, 1)$ is determined by formula (6.7) with operators L_B, L_A, L_C given above.

Now let us return to the general algebra (6.5). We define the quantum Wick product $*$ on symplectic leaves in \mathbb{R}^{k+2} , the operation on the quantum restriction to leaves, and describe the corresponding special functions.

The symplectic leaf $\mathfrak{X}_a \subset \mathbb{R}^{k+2}$ passing through the vacuum point (see Lemma 6.2) is determined by the following values of constants in (6.2): $K = \rho(a)$, $\text{const}_j = \varkappa_j(a)$; that is,

$$\mathfrak{X}_a = \{S_1^2 + S_2^2 = \rho(A) - \rho(a), \varkappa_j(A) = \varkappa_j(a)\}. \quad (6.9)$$

Here we assume that $a \in \mathbb{R}^k$ is not a point of local minimum or maximum of the function ρ . Also we consider only the connected component of the set determined by equations (6.9).

In order to introduce a complex structure on the leaf \mathfrak{X}_a following the approach of Section 5, one has to fix a function $Z = Z(A, C) \in \mathcal{F}^+ \setminus \mathcal{F}^0$. Let us choose a polynomial g which is an integral of motion for the flow γ^t (6.3), and $g(a) = \rho(a)$. Assume that $\rho(A) > g(A)$ for all $A \neq a$ from a neighborhood of the point a . Now we split the polynomial $\rho(A) - g(A)$ in two polynomial multipliers:

$$\rho(A) - g(A) = D(A) \cdot E(A) \quad (6.10)$$

in such a way that

$$D(a) \neq 0, \quad E(a) = 0. \quad (6.11)$$

Then in the neighborhood of the point $A = a$ we define $Z(A, C) = C/\overline{D}(A)$.

One can parameterize points A along the leaf (6.9) by the time on the trajectory: $A = \gamma^t(a)$. Then we have:

$$\begin{aligned} \rho(\gamma^t(a)) &> g(\gamma^t(a)) && \text{for } 0 < t < t^*, \\ \rho(\gamma^{t^*}(a)) &= g(\gamma^{t^*}(a)). \end{aligned} \quad (6.12)$$

The value of t^* can be $t^* = \infty$, or $t^* < \infty$.

In the last case denote $a^* = \gamma^{t^*}(a)$. The point $S_1 = S_2 = 0$, $A = a^*$ is “polar” with respect to the vacuum point on the leaf \mathfrak{X}_a . In this case, in addition to (6.11), we assume that the polynomial D has zero at a^* :

$$D(a^*) = 0. \quad (6.13)$$

The equation

$$z = C/\overline{D}(A) \quad (6.14)$$

determines the complex coordinate z all over the leaf \mathfrak{X}_a except the polar point. The polar point corresponds to $z = \infty$. In a neighborhood of the polar point one has to make the usual change of variables taking the new complex coordinate $z' = 1/z$.

Thus we introduce the global complex structure on the symplectic leaf \mathfrak{X}_a in both cases: $t^* = \infty$ and $t^* < \infty$.

Note that in the case $t^* < \infty$ the leaf \mathfrak{X}_a is compact and diffeomorphic to \mathbb{S}^2 . The value $t^* = t^*(a)$ depends on a . We assume that a is chosen in such a way that

$$t^*(a) = (N + 1)\hbar, \quad N \in \mathbb{Z}_+. \quad (6.15)$$

Now let us denote

$$\mathcal{D}(t) = D(\gamma^t(a)), \quad \mathcal{E}(t) = E(\gamma^t(a))$$

and consider the following differential equation for the function $k = k(r)$, $r \geq 0$:

$$\overline{\mathcal{E}}\left(\hbar r \frac{d}{dr}\right)k = r\mathcal{D}\left(\hbar r \frac{d}{dr} + \hbar\right)k, \quad k(0) = 1. \quad (6.16)$$

The solution is given by the series

$$k(r) = 1 + \sum_{n=1}^N \frac{r^n}{\mathcal{H}(\hbar)\mathcal{H}(2\hbar)\dots\mathcal{H}(n\hbar)}, \quad \mathcal{H} \stackrel{\text{def}}{=} \frac{\overline{\mathcal{E}}}{\mathcal{D}} \quad (6.17)$$

Here $N = \infty$ if $t^* = \infty$, and we assume that the multipliers D , E are taken in such a way that the series (6.17) has a convergency domain $\{|z| < \infty\}$.

Since \mathcal{D} and \mathcal{E} are polynomial, the function k (6.17) is of hypergeometric type. Also $k > 0$ everywhere. So one can define the Kähler form on the leaf \mathfrak{X}_a by the formula:

$$\omega \stackrel{\text{def}}{=} i\hbar\bar{\partial}\partial \ln k(|z|^2). \quad (6.18)$$

We call ω the *hypergeometric Kähler form* on the surface of revolution \mathfrak{X}_a (6.9) with the complex structure (6.14).

Lemma 6.3 (a) *The form (6.17) is globally defined on \mathfrak{X}_a , and condition (6.15) is equivalent to (2.2): $\frac{1}{2\pi\hbar} \int_{\mathfrak{X}_a} \omega = N$.*

(b) *The function $\mathcal{K}(\bar{z}|z) = k(|z|^2)$ is the reproducing kernel for the space \mathcal{L}_a of antiholomorphic distributions with the Hilbert norm*

$$\|u\| = \left(\sum_{n \geq 0} \mathcal{H}(\hbar) \dots \mathcal{H}(n\hbar) |u_n|^2 \right)^{1/2}, \quad \text{where } u(\bar{z}) = \sum_{n \geq 0} \bar{z}^n u_n.$$

This is the same norm and the same reproducing kernel as could be obtained by general formulas (5.6), (5.5); the Darboux coordinate P dual to $Z = C/\bar{D}(A)$ (see (5.1),(5.2)) in this case is given by the formula $P = Bt(A)/\bar{E}(A)$, where $t = t(A)$ is time along trajectories of the flow (6.3) with the initial data $t(a) = 0$.

(c) *The asymptotics of \mathcal{K} as $\hbar \rightarrow 0$ is the following:*

$$\mathcal{K}(\bar{z}|z) = \text{const} \sqrt{g_0(|z|^2)} e^{F_0(|z|^2)/\hbar} (1 + O(\hbar)), \quad g_0(r) = (rF_0'(r))',$$

where F_0 is the solution of the problem

$$\mathcal{H}(rF_0'(r)) = 0, \quad F_0(0) = 0,$$

and $\text{const} = \lambda(a)^{1/2}/|D(a)|$. The hypergeometric form (6.18) has the asymptotics:

$$\omega = \omega_0 + O(\hbar), \quad \omega_0 \stackrel{\text{def}}{=} ig_0(|z|^2) d\bar{z} \wedge dz,$$

where ω_0 is the classical symplectic form on the symplectic leaf \mathfrak{X}_a generated from \mathbb{R}^{k+2} by the Poisson structure (6.4).

(d) *The Hilbert norm in \mathcal{L}_a can be represented in the integral form (5.7) via the reproducing measure*

$$dm = k(|z|^2) \ell(|z|^2) d\bar{z} dz \quad (6.19)$$

if there exists a positive solution of the hypergeometric equation:

$$\bar{\mathcal{E}}\left(-\hbar r \frac{d}{dr}\right)\ell = r\mathcal{D}\left(-\hbar r \frac{d}{dr} - \hbar\right)\ell, \quad \frac{1}{\hbar} \int_0^\infty \ell(r) dr = 1. \quad (6.20)$$

The asymptotics is the following: $dm = dm^{\omega_0} + O(\hbar)$.

These statements were proved in [28]. Pay attention to equation (6.20), which differs from (6.16) by changing the sign near the parameter \hbar only. But the properties of the function ℓ in (6.20) are certainly not the same as of the function k in (6.16); for instance, ℓ can have a weak singularity at $r = 0$, and it is decreasing as $r \rightarrow \infty$. A series of examples was considered in [28].

Now let us come back to the general Theorem 5.1. In the given example the irreducible representation (5.9) of the algebra (6.5) can be realized in the space $\mathcal{L}_a = \mathcal{L}(\mathfrak{X}_a)$ by the following operators:

$$\hat{A}_a = \gamma^{\hbar z \bar{\partial} + \hbar}(a), \quad \hat{B}_a = \mathcal{D}(\hbar z \bar{\partial}) \cdot \bar{z}, \quad \hat{C}_a = \frac{1}{\bar{z}} \cdot \mathcal{E}(\hbar z \bar{\partial}). \quad (6.21)$$

Symbols A_a, B_a, C_a are the quantum restriction of the coordinate functions A, B, C to the leaf \mathfrak{X}_a . They can be calculated, say, by (3.9):

$$\begin{aligned} A_a &= \frac{1}{\mathcal{K}} \hat{A}_a(\mathcal{K}) = \gamma^{\hbar r d/dr + r F'(r) + \hbar}(a) 1(r), \\ C_a &= \frac{1}{\mathcal{K}} \hat{C}_a(\mathcal{K}) = \frac{1}{\bar{z}} \mathcal{E}(\hbar r d/dr + r F'(r)) 1(r) = \bar{B}_a, \end{aligned}$$

where $r = |z|^2$ and F is the quantum Kähler potential $F(r) = \hbar \ln k(r)$.

Theorem 6.1 *Let \star be the normal product over \mathbb{R}^{k+2} defined by (6.7), (6.8), and let $*$ be the Wick product over the surface of revolution $\mathfrak{X}_a \subset \mathbb{R}^{k+2}$ generated by the hypergeometric Kähler form (6.18). Then for any polynomial $f = f(B, A, C)$ on \mathbb{R}^{k+2} the quantum restriction to the surface \mathfrak{X}_a is given by*

$$f|_{\hat{\mathfrak{X}}_a} \equiv f_a = \frac{1}{\mathcal{K}} f\left(\overset{3}{\hat{B}}_a, \overset{2}{\hat{A}}_a, \overset{1}{\hat{C}}_a\right)(\mathcal{K}),$$

where the differential operators $\hat{B}_a, \hat{A}_a, \hat{C}_a$ are defined in (6.21) and $\mathcal{K} = k(\bar{z}z)$ is the hypergeometric function (6.17). The conditions (1.3), (1.4) are satisfied.

Note that the key role in this construction is played by the hypergeometric reproducing kernel \mathcal{K} which can be also considered as the coherent state in $\mathcal{L}(\mathfrak{X}_a)$ corresponding to the irreducible representation (6.21) of the algebra (6.5):

$$\mathcal{K}(\bar{z}|z) = \exp \left\{ \frac{z\widehat{P}_a}{\hbar} \right\} 1(\bar{z}) = \left(I + \sum_{n \geq 1} \frac{z^n}{\overline{\mathcal{E}(n\hbar)} \dots \overline{\mathcal{E}(\hbar)}} \widehat{B}_a^n \right) 1(\bar{z}). \quad (6.22)$$

Here $1 = 1(\bar{z})$ is the “vacuum” element in $\mathcal{L}(\mathfrak{X}_a)$, the operator \widehat{B}_a is given in (6.21), and

$$\widehat{P}_a = \widehat{B}_a \overline{E}^{-1}(\widehat{A}_a) t(\widehat{A}_a) = \hbar \bar{z} \mathcal{H}(\hbar \bar{z} \bar{\partial} + \hbar)^{-1} (\bar{z} \bar{\partial} + 1).$$

The operator acting to the vacuum element in (6.22) is a hypergeometric function in the creation operator \widehat{B}_a , or just the exponential function in the creation operator \widehat{P}_a . This second creation operator \widehat{P}_a is not very convenient since, in general, it is pseudodifferential. So, staying in the class of representations of algebra (6.5) by differential operators, we have to deal with hypergeometric functions instead of the standard exponential function.

Example 6.1 (continuation). In the case of the Lie algebra $\mathfrak{su}(1,1)$, the symplectic leaves (6.9) are hyperboloids: $\mathfrak{X}_a = \{A^2 - BC = a^2, A \geq a\}$, where $a = \text{const} > 0$. Let us fix $g(A) = \text{const} = \rho(a) = a^2$. There are two possible choices of the multipliers D, E (6.10) which factorize the difference $\rho(A) - \rho(a) = (A - a)(A + a)$.

Variante I. First, we can take $E(A) = A - a$ and $D(A) = A + a$. Then $\mathcal{E}(t) = t$, $\mathcal{D}(t) = t + 2a$, and equations (6.16), (6.20) read

$$\hbar(1 - r)dk/dr = (2a + \hbar)k, \quad \hbar(1 - r)d\ell/dr = (\hbar - 2a)\ell. \quad (6.23)$$

Since $k(0) = 1$, then $k(r) = (1 - r)^{-(2a+\hbar)/\hbar}$. This function is singular at $r = 1$, what means that the convergency radius of the series (6.17) in this variante is equal to 1 (but not to ∞). Hence the normalization condition for the function ℓ is $\frac{1}{\hbar} \int_0^1 \ell(r) dr = 1$. Solving the second equation (6.23), we obtain $\ell(r) = 2a(1 - r)^{(2a-\hbar)/\hbar}$. Thus the quantum symplectic form and the reproducing measure in this variante are given by

$$\omega = i(2a + \hbar) \frac{d\bar{z} \wedge dz}{(1 - |z|^2)^2}, \quad dm = 2a \frac{d\bar{z} dz}{(1 - |z|^2)^2}.$$

They are transported to the hyperboloid \mathfrak{X}_a by means of the $\mathfrak{su}(1, 1)$ -invariant complex structure (6.14):

$$z = \frac{C}{A+a} \quad \text{or} \quad A = \frac{a(1+|z|^2)}{1-|z|^2}, \quad C = \frac{2az}{1-|z|^2}.$$

Of course, the quantum geometrical data ω , dm are \hbar -deformations of the classical data ω_0 , dm^{ω_0} , where ω_0 is the $\mathfrak{su}(1, 1)$ -invariant Kählerian form on the hyperboloid: $\omega_0 = 2ia(1-|z|^2)^{-2}d\bar{z} \wedge dz = \frac{i}{2}A^{-1}dB \wedge dC|_{\mathfrak{X}_a}$. In this variant, all the operators (6.21) are of the first order and generate the standard irreducible representation of the Lie algebra $\mathfrak{su}(1, 1)$.

Variante II. Let us make another choice: $E(A) = A^2 - a^2$ and $D(A) = 1$. Then $\mathcal{E}(t) = t(2a+t)$ and, instead of (6.23), we have the following equations for the functions k and ℓ :

$$\begin{aligned} \hbar^2 r d^2 k / dr^2 + (2a\hbar + \hbar^2) dk / dr - k &= 0, \\ \hbar^2 r d^2 \ell / dr^2 - (2a\hbar - \hbar^2) d\ell / dr - \ell &= 0. \end{aligned}$$

The normalization conditions are the same as in (6.16), (6.20); so the solution is

$$k(r) = \tilde{I}_{2a/\hbar} \left(\frac{2}{\hbar} \sqrt{r} \right), \quad \ell(r) = \tilde{M}_{2a/\hbar} \left(\frac{2}{\hbar} \sqrt{r} \right). \quad (6.24)$$

Here \tilde{I}_ν and \tilde{M}_ν are modified Bessel and MacDonal functions:

$$\begin{aligned} \tilde{I}_\nu(y) &\stackrel{\text{def}}{=} \sum_{n \geq 0} \left(\frac{y}{2} \right)^{2n} \frac{\Gamma(\nu+1)}{n! \Gamma(\nu+n+1)}, \\ \tilde{M}_\nu(y) &\stackrel{\text{def}}{=} \frac{(y/2)^\nu}{\hbar \Gamma(\nu+1)} \int_{-\infty}^{\infty} \exp\{-y \cosh t - \nu t\} dt. \end{aligned}$$

Thus the quantum symplectic form and the reproducing measure in this variant are given by

$$\omega = i\hbar \bar{\partial} \partial \ln \tilde{I}_{2a/\hbar}(2|z|/\hbar), \quad dm = (\tilde{I}_{2a/\hbar} \tilde{M}_{2a/\hbar})(2|z|/\hbar) d\bar{z} dz. \quad (6.25)$$

They are transported to the hyperboloid \mathfrak{X}_a by means of the complex structure (6.14):

$$z = C, \quad A = (a^2 + |z|^2)^{1/2}. \quad (6.26)$$

The operators (6.21) in this variant are the following:

$$\widehat{A}_a = a + \hbar + \hbar\bar{z}\bar{\partial}, \quad \widehat{B}_a = \bar{z}, \quad \widehat{C}_a = \hbar\bar{\partial} \cdot (2a + \hbar\bar{z}\bar{\partial}). \quad (6.27)$$

Since the complex structure (6.26) is not $\mathfrak{su}(1,1)$ -invariant, these operators are not all of the first order; they generate the irreducible representation of $\mathfrak{su}(1,1)$ given in [1]. The corresponding ‘‘Bessel’’ geometrical data (6.25) on the hyperboloid are quantum \hbar -deformations of the classical data ω_0 , dm^{ω_0} , where $\omega_0 = \frac{i}{2}(a^2 + |z|^2)^{-1/2} d\bar{z} \wedge dz = \frac{i}{2}A^{-1}dB \wedge dC \Big|_{\mathfrak{X}_a}$.

Example 6.2 Quadratic algebra of the Zeeman effect. The Hydrogen atom in a homogeneous magnetic field is described by the Hamiltonian: $\mathbf{H} = (\widehat{p} - \mathcal{A}(q))^2 - |q|^{-1}$, where $\widehat{p} = -i\hbar\partial/\partial q$, $q \in \mathbb{R}^3$, and the magnetic potential \mathcal{A} has the following components $\mathcal{A}_1 = -\frac{1}{2}\varepsilon q_2$, $\mathcal{A}_2 = \frac{1}{2}\varepsilon q_1$, $\mathcal{A}_3 = 0$. Here we assume that the magnetic field is directed along the third coordinate axis, and the values of ε characterize the strength of the field. Applying the quantum averaging procedure, we can transform this Hamiltonian with arbitrary accuracy $O(\varepsilon^{n+2})$ to the following form

$$\mathbf{H} \sim \mathbf{H}_0 + \varepsilon\mathbf{M}_3 + \varepsilon^2 f^{(n)}(\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3; \mathbf{H}_0, \mathbf{M}_3) + O(\varepsilon^{n+2}). \quad (6.28)$$

Here $\mathbf{H}_0 = |q|(\widehat{p}^2 + \frac{1}{4})$ describes the Hydrogen atom itself, \mathbf{M}_3 is the third component of the angular momentum $\mathbf{M} = q \times \widehat{p}$, and the operators \mathbf{S}_j generate the *algebra of joint symmetries* of \mathbf{H}_0 and \mathbf{M}_3 , that is, $[\mathbf{H}_0, \mathbf{S}_j] = [\mathbf{M}_3, \mathbf{S}_j] = 0$, $j = 0, \dots, 4$. More precisely, formulas for \mathbf{S}_j are

$$\mathbf{S}_0 = \mathbf{L}_3 - \mathbf{R}_3, \quad \mathbf{S}_1 = \mathbf{L}_1\mathbf{R}_2 - \mathbf{L}_2\mathbf{R}_1, \quad \mathbf{S}_2 = \mathbf{L}_1\mathbf{R}_1 + \mathbf{L}_2\mathbf{R}_2, \quad \mathbf{S}_3 = \mathbf{L}_3\mathbf{R}_3 + \mathbf{L}^2,$$

where $\mathbf{L} = (\mathbf{M} + \mathbf{N})/2$, $\mathbf{R} = (\mathbf{M} - \mathbf{N})/2$, and $\mathbf{N} = q(\widehat{p}^2 + \frac{1}{4}) - \widehat{p} \times \widehat{\mathbf{M}} + \widehat{\mathbf{M}} \times \widehat{p}$.

On the right-hand side of (6.28) we assume that generators \mathbf{S}_j are ordered in some way. The function $f^{(n)}$ is a polynomial of degree n in the variables S_j ; details and explicit formulas for $f^{(2)}$ see in [27].

The most important fact is that the generators \mathbf{S}_j satisfy the following quadratic permutation relations:

$$\begin{aligned} [\mathbf{S}_1, \mathbf{S}_2] &= \frac{i\hbar}{2}(\mathbf{S}_0\mathbf{S}_3 + \mathbf{S}_3\mathbf{S}_0), & [\mathbf{S}_0, \mathbf{S}_1] &= 2i\hbar\mathbf{S}_2, \\ [\mathbf{S}_2, \mathbf{S}_3] &= -\frac{i\hbar}{2}(\mathbf{S}_0\mathbf{S}_1 + \mathbf{S}_1\mathbf{S}_0), & [\mathbf{S}_0, \mathbf{S}_2] &= -2i\hbar\mathbf{S}_1, \\ [\mathbf{S}_3, \mathbf{S}_1] &= -\frac{i\hbar}{2}(\mathbf{S}_0\mathbf{S}_2 + \mathbf{S}_2\mathbf{S}_0), & [\mathbf{S}_0, \mathbf{S}_3] &= 0. \end{aligned} \quad (6.29)$$

Let us introduce $\mathbf{C} = \mathbf{S}_1 + i\mathbf{S}_2$, $\mathbf{B} = \mathbf{S}_1 - i\mathbf{S}_2$, and also denote $\mathbf{A}_1 = \mathbf{S}_0 - \hbar$, $\mathbf{A}_2 = \mathbf{S}_3 + \frac{\hbar}{2}\mathbf{S}_0$. Then the relations (6.29) can be written in the form (6.5), where $\rho(A) \stackrel{\text{def}}{=} A_2^2$ and the dynamical flow (6.3) in \mathbb{R}^2 is generated by the vector field $v = 2\partial/\partial A_1 - A_1\partial/\partial A_2$. The integral of motion of this field is $\varkappa(A) = A_1^2 + 4A_2$. Thus the symplectic leaf (6.9) is defined as follows:

$$\mathfrak{X}_a = \{A_2^2 - BC = a_2^2, A_1^2 + 4A_2 = a_1^2 + 4a_2\}. \quad (6.30)$$

The representation of the algebra (6.29) related to the Zeeman effect selects the values of the parameters a_1, a_2 : namely, $a_1 < 0, a_2 > 0$. Then the leaf \mathfrak{X}_a (6.30) is topologically a sphere.

Evaluating $\rho(\gamma^t(a)) - \rho(a) = t(t - |a_1|)(t - t_+)(t - t_-)$, where $t_{\pm} = \frac{1}{2}|a_1| \pm \frac{1}{2}(a_1^2 + 8a_2)^{1/2}$, we conclude that the inequality (6.12) is satisfied on the interval $0 < t < t^* = |a_1|$. Taking into account (4.15), we obtain the quantization condition

$$a_1 = -(N + 1)\hbar, \quad N \in \mathbb{Z}_+.$$

The polar point $a^* = \gamma^{t^*}(a)$ is the following: $a^* = (-a_1, a_2)$.

Now let us choose $g(A) = \frac{1}{4}t_+^2(A_1^2 + 4A_2 - t_+^2)$. Then the polynomial $\rho(A) - g(A)$ can be factorized, as in (6.10), taking

$$D(A) = A_2 + \frac{1}{2}t_+A_1 - a_2 + \frac{1}{2}t_+a_1, \quad E(A) = A_2 - \frac{1}{2}t_+A_1 - a_2 + \frac{1}{2}t_+a_1.$$

So, we have

$$\mathcal{D}(t) = D(\gamma^t(a)) = (a_1 + t)(t_+ - t), \quad \mathcal{E}(t) = E(\gamma^t(a)) = t(t_- - t). \quad (6.31)$$

Thus the hypergeometric equations (6.16), (6.20) for k and for ℓ are the following:

$$\begin{aligned} r(1-r)\frac{d^2k}{dr^2} + (\alpha_+ - (\alpha_- - N + 1)r)\frac{dk}{dr} + N\alpha_-k &= 0, \\ r(1-r)\frac{d^2\ell}{dr^2} + (\beta_- - (\beta_+ + N + 3)r)\frac{d\ell}{dr} - (N + 2)\beta_+\ell &= 0, \end{aligned}$$

where $\alpha_{\pm} = 1 - t_{\mp}/\hbar$, $\beta_{\pm} = 1 + t_{\pm}/\hbar$. The solution of these equations could be expressed via the Gauss hypergeometric functions ${}_2F_1(-N, \alpha_-, \alpha_+; r)$ and

${}_2F_1(N+2, \beta_+, \beta_-; r)$. But we prefer to demonstrate explicit formulas:

$$k(r) = \sum_{n=0}^N \frac{N!}{n!(N-n)!} \frac{(t_+ - \hbar)(t_+ - 2\hbar) \dots (t_+ - n\hbar)}{(|t_-| + \hbar)(|t_-| + 2\hbar) \dots (|t_-| + n\hbar)} r^n, \quad (6.32)$$

$$\ell(r) = \frac{\hbar(N+1)\Gamma(1 + (t_+ + |t_-|)/\hbar)}{\Gamma(t_+/\hbar)\Gamma(1 + |t_-|/\hbar)} \int_0^\infty \frac{\lambda^{t_+/\hbar} d\lambda}{(1 + \lambda r)^{N+2}(1 + \lambda)^{1+(t_+ + |t_-|)/\hbar}}.$$

Here Γ is the gamma-function, and in the last integral representation the Euler formula for hypergeometric functions was used.

We see that in this example the reproducing kernel $\mathcal{K}(\bar{z}|z) = k(\bar{z}z)$ is a *Jacobi polynomial* (6.32) of degree N . Functions (6.32) generate the quantum Kählerian form and the reproducing measure on the “sphere” \mathfrak{X}_a (6.30) by the general formulas (6.18), (6.19). The complex structure on \mathfrak{X}_a is determined by (6.14): $z = C/(A_2 + \frac{1}{2}t_+A_1 - a_2 + \frac{1}{2}t_+a_1)$. The corresponding irreducible representation of the algebra (6.29) in the space $\mathcal{L}(\mathfrak{X}_a)$ is given by the operators:

$$\widehat{S}_1 = \frac{1}{2}(\widehat{C} + \widehat{B}), \quad \widehat{S}_2 = \frac{1}{2i}(\widehat{C} - \widehat{B}), \quad \widehat{S}_3 = \widehat{A}_2 - \frac{\hbar}{2}\widehat{A}_1 - \frac{\hbar^2}{2}, \quad \widehat{S}_0 = \widehat{A}_1 + \hbar,$$

where \widehat{C} , \widehat{B} , \widehat{A} are defined via (6.21), (6.31):

$$\begin{aligned} \widehat{A}_1 &= a_1 + 2\hbar + 2\hbar\bar{z}\bar{\partial}, & \widehat{A}_2 &= a_2 - (a_1\hbar + a_1\hbar\bar{z}\bar{\partial}) - (\hbar\bar{z}\bar{\partial} + \hbar)^2, \\ \widehat{B} &= (a_1 + \hbar\bar{z}\bar{\partial})(t_+ - \hbar\bar{z}\bar{\partial}) \cdot \bar{z}, & \widehat{C} &= \hbar\bar{\partial} \cdot (t_- - \hbar\bar{z}\bar{\partial}). \end{aligned}$$

This is the realization of relations (6.29) by the second order differential operators. The equivalent representations can be obtained by three other possible variants of the factorization (6.31) (or choices of complex structures):

$$\begin{aligned} \mathcal{D} &= (a_1 + t)(t_- - t) & \mathcal{D} &= (a_1 + t)(t_+ - t)(t_- - t) & \mathcal{D} &= a_1 + t \\ \mathcal{E} &= t(t_+ - t), & \mathcal{E} &= t, & \mathcal{E} &= t(t_+ - t)(t_- - t). \end{aligned}$$

So, together with the variant considered above, totally we have four different algebraic complex structures on the symplectic leaf $\mathfrak{X}_a \subset \mathbb{R}^4$. They generate four quantum Kähler structures and four irreducible equivalent representations of the algebra (6.29) by differential operators (up to order six) [28].

Other examples and a list of references around algebras of the type (6.5) see in [28].

7 Quantum cylinder and theta-functions

Finally, we consider surfaces of revolution (6.2)

$$\mathfrak{X} = \{\rho(A) - BC = K, \varkappa_j(A) = \text{const}_j (j = 1, \dots, k-1)\} \quad (7.1)$$

under the condition $K < \min \rho$. In this case, the values $BC = |C|^2$ are strictly positive on \mathfrak{X} , and so, no vacuum point exists. Instead of that there are noncontractible 1-cycles: sections of \mathfrak{X} by the planes $A = \text{const}$. Thus the surface \mathfrak{X} topologically is the cylinder or the torus. Actually, the axis of revolution for the surface \mathfrak{X} is determined by the trajectory $\{\gamma^t(a_0)\}$, starting from a point a_0 , where $\varkappa_j(a_0) = \text{const}_j (j = 1, \dots, k-1)$. If this trajectory is periodic, then \mathfrak{X} is homeomorphic to the torus \mathbb{T}^2 ; if the trajectory is not periodic, then \mathfrak{X} is homeomorphic to the cylinder $\mathbb{S} \times \mathbb{R}$. Here we consider only the cylinder case.

As above, let us take some polynomial $g(A)$ such that $g(\gamma^t(a_0)) = g(a_0) = K$ for any t , and $\rho(A) > g(A)$ in a neighborhood of \mathfrak{X} . Moreover, we assume that there is a (complex) polynomial $M(A)$ such that

$$\rho(A) - g(A) = |M(A)|^2. \quad (7.2)$$

Then one can choose the following multipliers in the factorization (6.10): $D(A) = M(A)e^{-t(A)}$, $E(A) = \overline{M}(A)e^{t(A)}$, where $t(A)$ is the time along trajectories of the flow (6.3).

Since instead of the vacuum point on \mathfrak{X} there is a noncontractible circle, it is natural to consider a ring in the complex plane as a coordinate chart. That is why in all formulas (6.14), (6.18), (6.21), (6.22) we now replace the complex variable z by e^z , and the real variable $r = z\bar{z}$ we replace by $r = z + \bar{z}$. Then the operator $\hbar r d/dr$ played the role of the “quantum time” in (6.16), (6.20) must be replaced by the operator $\hbar d/dr$.

In order to obtain an analog of the equation (6.16) in the cylinder case, let us first remark that (6.16) is equivalent to the equations

$$\widehat{A}\mathcal{K} = \overline{\widehat{A}}\mathcal{K}, \quad \widehat{B}\mathcal{K} = \overline{\widehat{C}}\mathcal{K}. \quad (7.3)$$

Here the bar $\overline{\cdot}$ denotes the complex conjugation of an operator; after this conjugation the operator acts by z (not by \bar{z}). The operators \widehat{A} , \widehat{B} , \widehat{C} in (7.3) are generators of the irreducible representation of algebra (6.5) in the

space of antiholomorphic sections. Now we have to use not the operators of type (6.21) but the following ones:

$$\widehat{A} = \gamma^{\hbar\bar{\partial}+\hbar}(a_0), \quad \widehat{B} = \mathcal{D}(\hbar\bar{\partial}) \cdot e^{\bar{z}}, \quad \widehat{C} = e^{-\bar{z}} \cdot \mathcal{E}(\hbar\bar{\partial}).$$

Thus equations (7.3) imply

$$\mathcal{K}(\bar{z}|z) = k(\bar{z} + z), \quad M(\hbar\bar{\partial})e^{-\hbar\bar{\partial}}e^{\bar{z}}k(\bar{z} + z) = e^{-z}M(\hbar\partial)e^{\hbar\partial}k(\bar{z} + z).$$

Since $M \neq 0$, we obtain the equations which do not depend on the polynomial M at all:

$$k(r + 2\hbar) = e^r k(r), \quad k(r + 2\pi i) = k(r). \quad (7.4)$$

The last condition follows from the $2\pi i$ -periodicity of $\mathcal{K}(\bar{z}|z)$ by both \bar{z} and z .

The equations for the function ℓ (the density of the reproducing measure) in general are the following:

$$\widehat{A}' \ell = \overline{\widehat{A}}' \ell, \quad \widehat{B}' \ell = \overline{\widehat{C}}' \ell,$$

where the prime $'$ denotes the transposition with respect to the standard pairing of functions: $\langle \varphi, \psi \rangle = \int \varphi \psi d\bar{z} dz$. From these equations, we obtain $\ell = \ell(r)$, where $r = \bar{z} + z$, and

$$\ell(r + 2\hbar) = e^{-r} \ell(r), \quad \frac{1}{\hbar} \int_{-\infty}^{\infty} \ell(r) dr = 1. \quad (7.5)$$

The solution of the last problem is just the Gaussian function

$$\ell(r) = (\hbar/4\pi)^{1/2} e^{-(r-\hbar)^2/4\hbar}. \quad (7.6)$$

The solution of equations (7.4) is given by the series

$$k(r) = \sum_{n \in \mathbb{Z}} e^{-n^2 \hbar + n(r-\hbar)} = \theta(r - \hbar, e^{-\hbar}). \quad (7.7)$$

Here we denote by θ the following theta-function:

$$\theta(\alpha, q) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} q^{-n^2} e^{n\alpha}, \quad q < 1. \quad (7.8)$$

Theorem 7.1 *Assume that the trajectory $\{\gamma^t(a_0)\}$ is not periodic and thus the surface \mathfrak{X} (7.1) is homeomorphic to the cylinder. Then*

(a) *In the space \mathcal{L} of antiholomorphic $2\pi i$ -periodic functions with the Hilbert norm*

$$\|u\| = \left(\frac{1}{4\pi\sqrt{\pi\hbar}} \int_{0 < \text{Im } z < 2\pi} |u(\bar{z})|^2 e^{-(\bar{z}+z-\hbar)^2/4\hbar} d\bar{z}dz \right)^{1/2},$$

the irreducible representation of algebra (6.5) acts by the following operators:

$$\hat{A} = \gamma^{\hbar\bar{\partial}+\hbar}(a_0), \quad \hat{B} = e^{\bar{z}-\hbar\bar{\partial}-\hbar/2}M(\hat{A}), \quad \hat{C} = \overline{M}(\hat{A})e^{\hbar\bar{\partial}-\bar{z}+\hbar/2}, \quad (7.9)$$

where the polynomial M is taken from (7.2). This representation, up to equivalence, does not depend on the choice of the point a_0 , but is parametrized by values of the Casimir functions K, \varkappa_j in (7.1).

(b) *The space \mathcal{L} is identified with the space $\mathcal{L}(\mathfrak{X})$ of antiholomorphic sections over the symplectic leaf (7.1) supplied with the complex structure $e^z = Ce^{t(A)}/\overline{M(A)}$ and with the Kählerian form*

$$\omega = i\hbar\bar{\partial}\partial \ln \mathcal{K}, \quad \mathcal{K}(\bar{z}|z) = \theta(\bar{z} + z - \hbar, e^{-\hbar}), \quad (7.10)$$

where the theta-function θ is defined by (7.8).

(c) *The reproducing measure on \mathfrak{X} corresponding to the Kählerian form (7.10) is given by*

$$dm = \frac{1}{2}\theta\left(\frac{i\pi}{\hbar}(\bar{z} + z - \hbar), e^{-\pi^2/\hbar}\right) d\bar{z}dz. \quad (7.11)$$

The last statement about the measure $dm = k(\bar{z} + z)\ell(\bar{z} + z) d\bar{z}dz$ is derived from (7.6), (7.7) and from the Jacobi transform of theta-functions which reads in our case

$$\begin{aligned} \mathcal{K}(\bar{z}|z) &\equiv k(\bar{z} + z) = (\pi/\hbar)^{1/2} e^{(\bar{z}+z-\hbar)^2/4\hbar} \theta\left(\frac{i\pi(\bar{z} + z - \hbar)}{\hbar}, e^{-\pi^2/\hbar}\right) \\ &= (\pi/\hbar)^{1/2} e^{(\bar{z}+z-\hbar)^2/4\hbar} (1 + O(e^{-\pi^2/\hbar})). \end{aligned} \quad (7.12)$$

Now let us note that after the substitution of (7.12) into the right-hand side of (7.10), we obtain

$$\begin{aligned} \omega &= \omega_0 + i\hbar\bar{\partial}\partial \ln \theta\left(\frac{i\pi(\bar{z} + z - \hbar)}{\hbar}, e^{-\pi^2/\hbar}\right) \\ &= \omega_0 + O(e^{-\pi^2/\hbar}) \quad \text{as } \hbar \rightarrow 0. \end{aligned} \quad (7.13)$$

Here ω_0 denotes the classical symplectic form on the leaf \mathfrak{X} (7.1):

$$\omega_0 = \frac{i}{2} d\bar{z} \wedge dz.$$

And also from (7.11) we obtain the asymptotics

$$dm = dm^{\omega_0} + O(e^{-\pi^2/\hbar}), \quad \text{where} \quad dm^{\omega_0} = \frac{1}{2} d\bar{z} dz. \quad (7.14)$$

Let us stress that the exponentially small remainders $O(e^{-\pi^2/\hbar})$ in (7.13), (7.14) are not zero and *explicitly given by the theta-series*. This means that the corresponding Wick product over the leaf \mathfrak{X} has the asymptotic expansion

$$\psi * \chi \simeq \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \partial^r \psi \bar{\partial}^r \chi + O(e^{-\pi^2/\hbar}). \quad (7.15)$$

The formal \hbar -series on the right-hand side of (7.15) is the well-known expansion of the Wick product over \mathbb{R}^2 corresponding to the symplectic form ω_0 and the measure dm^{ω_0} . This series does not know anything about the periodic condition on functions transforming the plane \mathbb{R}^2 to the cylinder $\mathbb{R} \times \mathbb{S}$ (in our case: $2\pi i$ -periodicity by the z variable). Only the exponentially small remainder in (7.15) “knows” about this additional periodic condition, and so, about the topology of \mathfrak{X} . This remainder is invisible within the frames of formal constructions of $*$ -products by power series in \hbar . In order to take into account the nontrivial 1-cycle on \mathfrak{X} , we have to exploit formulas for the Wick $*$ -product including the exponentially small quantities $\exp\{-j\pi^2/\hbar\}$ for all $j = 1, 2, \dots$. We consider this fact as a *tunneling* along the noncontractible cycle on the quantum cylinder.

The same results can be obtained in the case where \mathfrak{X} is homeomorphic to the torus $\mathbb{T}^2 = \mathbb{S} \times \mathbb{S}$ [29], and obviously, this is a general property of the quantum geometry over not simply connected Kähler manifolds.

Example 7.1 The simplest cylindric symplectic leaves are one-sheet hyperboloids in $\mathbb{R}^3 = \mathfrak{su}(1, 1)^*$. So, let us return to the Lie algebra $\mathfrak{g} = \mathfrak{su}(1, 1)$ already considered in Example 6.1. Now we take the following family of symplectic leaves:

$$\mathfrak{X} = \{BC - A^2 = \lambda^2\}, \quad \lambda > 0. \quad (7.16)$$

In this case $K = -\lambda^2$ and $\rho(A) = A^2$. We choose $g(A) \equiv -\lambda^2$, $a_0 = 0$ and so $\rho(A) - g(A) = A^2 + \lambda^2 = (A + i\lambda)(A - i\lambda)$. Thus one can take $M(A) = A - i\lambda$ in (7.2) and introduce the complex structure to the hyperboloid (7.16):

$$e^z = C \frac{e^A}{A + i\lambda} \quad \text{or} \quad A = \operatorname{Re} z, \quad C = (\operatorname{Re} z + i\lambda)e^{i \operatorname{Im} z}. \quad (7.17)$$

The Kählerian form and the reproducing measure on the hyperboloid (7.16) are defined by the general formulas (7.10), (7.11). The irreducible Hermitian representation of the Lie algebra $\mathfrak{su}(1, 1)$ (the “prime series”) is given by formulas (7.9) which read:

$$\begin{aligned} \widehat{A} &= \hbar + \hbar \bar{\partial}, & \widehat{B} &= e^{\bar{z} - \hbar \bar{\partial} - \hbar/2} (\hbar - i\lambda + \hbar \bar{\partial}), \\ \widehat{C} &= (\hbar + i\lambda + \hbar \bar{\partial}) e^{\hbar \bar{\partial} - \bar{z} + \hbar/2}. \end{aligned} \quad (7.18)$$

In our case, the representation contains pseudodifferential operators since it corresponds to the complex structure (7.17) which is not $\mathfrak{su}(1, 1)$ -invariant. The Casimir element in this representation takes the value $\widehat{A}^2 - \widehat{C}\widehat{B} = -\lambda^2 \cdot I$. Usually, the prime series is realized by differential operators on the circle what corresponds to the choice of the real polarization over the hyperboloid \mathfrak{X} (7.16).

Note that the use of a real polarization over not simply connected symplectic leaves generates associative algebras of functions which are not central (that is, they have a nontrivial center). To make them central, one has to restrict all functions to a lattice, and so, to disconnect the leaf. After that we lose the effect of tunneling.

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