Robust Mixed Control and LPV Control with Full Block Scalings

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Abstract

For systems affected by time-varying parametric uncertainties, we devise in this chapter a technique that allows to equivalently translate robust performance analysis specifications characterized through a single Lyapunov function into the corresponding analysis test with multipliers. Out of the multitude of possible applications of this so-called full block S-procedure, we will concentrate on a discussion of robust mixed control and of designing linear parametrically-varying (LPV) controllers.

1 Introduction

In the past years mixed control problems have attracted considerable interest, and the LMI approach to controller design has shown beneficial to tackle design specifications that have deemed untractable. For the design of output feedback controllers, it has been revealed quite recently how a suitable non-linear change of controller parameters allows to proceed in a straightforward fashion from an analysis specification for a controlled system formulated in terms of matrix inequalities to the corresponding synthesis inequalities for controller design [6, 12, 20].

One purpose of this chapter is to extend this general paradigm to robust performance specifications if the underlying system is affected by time-varying parametric uncertainties [13, 22].

We introduce a general technique that allows to equivalently translate robust performance objectives formulated in terms of a common Lyapunov function into the corresponding analysis test with multipliers. Similar approaches, as those in [13, 22], are usually based on the so-called S-procedure that introduces conservatism since the resulting multipliers have, in general, a block-diagonal structure. Instead, we propose to use full block multipliers that are only indirectly described by linear matrix inequalities such that the reformulation will not introduce conservatism. Hence we call the underlying abstract result for quadratic forms a full block S-procedure. For robust stability specifications with affine dependence on the uncertainties, such an equivalent reformulation has been provided before in [15].

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The main goal of this chapter is to generalize these ideas to robust performance specifications if the matrices describing the system can be rational functions of the parameters, and to base the transformation on one abstract result that can be applied with ease to several performance specifications. Moreover, we will provide a discussion of the (straightforward) consequence for robust controller design. Related results for the robust stabilization problem have appeared in the inspiring paper [10]. Finally, we will turn to the design of linear parametrically varying controllers that has been fully tackled for block-diagonal multipliers [14, 9, 21]. Our purpose is to show how to overcome this structural restriction to arrive at a solution even if the multipliers are full block matrices that are only indirectly described by LMIs. Contrary to previous approaches one has to modify the structure of the scheduling function: one cannot just employ a copy of the parameters to schedule the controller but one has to use a quadratic function thereof.

2 System Description

We concentrate on systems that are affected by time-varying parametric uncertainties. Since we deal with linear fractional system representations, we assume that all these parameters are collected in the matrix Δ. The admissible set of values that can be taken by the uncertainties is denoted as

$$\mathbf{\Delta} \subset \mathbf{R}^{k\times l} \text{ with } 0 \in \mathbf{\Delta}$$

and assumed to be compact. Note that $\mathbf{\Delta}$ captures both the *size* of the uncertainties as well as their *structure*. Typically, for a rational dependence of the system description on the parameters, the matrix can be taken block-diagonal, and the blocks take a real repeated structure; for the results in this chapter, such a structure is *not* required. Given the value set $\mathbf{\Delta}$, the actual time-varying parametric uncertainties consist of all continuous curves

$$\Delta : [0,\infty) \to \mathbf{\Delta}.$$

Let us now look at

(1)
$$\begin{pmatrix} \dot{x} \\ z_1 \\ z_2 \\ \vdots \\ z_m \\ y \end{pmatrix} = \left(\begin{array}{c|cccc|c} A & B_1 & B_2 & \cdots & B_m & B \\ \hline C_1 & D_{11} & D_{12} & \cdots & D_{1m} & E_1 \\ C_2 & D_{21} & D_{22} & \cdots & D_{2m} & E_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_m & D_{m1} & D_{m2} & \cdots & D_{mm} & E_m \\ \hline C & F_1 & F_2 & \cdots & F_m & 0 \end{array}\right) \begin{pmatrix} x \\ w_1 \\ w_2 \\ \vdots \\ w_m \\ u \end{pmatrix}, w_1 = \Delta(t) z_1,$$

a linear time-invariant system in which the uncertainties enter in a linear fractional fashion to define the time-varying uncertain system under investigation.

Obviously, $w_1 \to z_1$ constitutes the *uncertainty channel*, whereas $w_j \to z_j$, $j = 2,\ldots,m$, are the *performance channels* used to describe the desired performance specifications, and $u \to y$ is the *control channel*.

In fact, u is the control input and y is the measured output, and any linear time-invariant system that closes the loop as

$$\begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix} \tag{2}$$

is said to be a controller. The resulting controlled system admits the description

$$\begin{pmatrix} \dot{\xi} \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix} = \left(\begin{array}{c|cccc} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & \cdots & \mathcal{B}_m \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} & \cdots & \mathcal{D}_{1m} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} & \cdots & \mathcal{D}_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_m & \mathcal{D}_{m1} & \mathcal{D}_{m2} & \cdots & \mathcal{D}_{mm} \end{array}\right) \begin{pmatrix} \xi \\ w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}, w_1 = \Delta(t) z_1 \tag{3}$$

with the realization

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}_j \\ \mathcal{C}_i & \mathcal{D}_{ij} \end{pmatrix} = \left(\begin{array}{cc|c} A + BD_cC & BC_c & B_j + BD_cF_j \\ B_cC & A_c & B_cF_j \\ \hline C_i + E_iD_cC & E_iC_c & D_{ij} + E_iD_cF_j \end{array}\right). \tag{4}$$

The multi-objective robust controller design problem can be sketched as follows: Find a controller that exponentially stabilizes the closed-loop system, and that achieves a mixture of various performance specifications on the different performance channels of the controlled system for all possible uncertainties affecting the system.

In the next section we extend well-known analysis tests for nominal performance as described in [20] to the corresponding robust performance tests against time-varying uncertainties in terms of a constant quadratic Lyapunov function. The essential new ingredient will be the ease how one can derive the corresponding multiplier test by just referring to the abstract full block S-procedure that is presented in Lemma A.1 in the appendix.

3 Robust Performance Analysis with Constant Lyapunov Matrices

In this section, we provide analysis tests that are based on finding a constant quadratic Lyapunov function in order to guarantee the following properties:

- Well-posedness of the linear fractional transformation used to describe the uncertain system.
- Uniform (in the uncertainty) exponential stability.
- Robust performance, specified as quadratic performance (such as bounding the L_2-gain or dissipativity) or as bounding the H_2 norm, the generalized H_2 norm, or the peak-to-peak gain.

3.1 Well-Posedness and Robust Stability

We call the description (3) *well-posed* if

$$I - \Delta\mathcal{D}_{11} \text{ is non-singular for every } \Delta \in \boldsymbol{\Delta}. \tag{5}$$

Then the channel $w_j \to z_i$ of the uncertain closed-loop system admits the alternative representation

$$\begin{pmatrix} \dot{\xi} \\ z_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{A}}(\Delta(t)) & \boldsymbol{\mathcal{B}}_j(\Delta(t)) \\ \boldsymbol{\mathcal{C}}_i(\Delta(t)) & \boldsymbol{\mathcal{D}}_{ij}(\Delta(t)) \end{pmatrix} \begin{pmatrix} \xi \\ w_j \end{pmatrix} \tag{6}$$

with the function

$$\begin{pmatrix} \boldsymbol{\mathcal{A}}(\Delta) & \boldsymbol{\mathcal{B}}_j(\Delta) \\ \boldsymbol{\mathcal{C}}_i(\Delta) & \boldsymbol{\mathcal{D}}_{ij}(\Delta) \end{pmatrix} =$$

$$= \begin{pmatrix} \mathcal{A} + \mathcal{B}_1(I - \Delta\mathcal{D}_{11})^{-1}\Delta\mathcal{C}_1 & \mathcal{B}_j + \mathcal{B}_1(I - \Delta\mathcal{D}_{11})^{-1}\Delta\mathcal{D}_{1j} \\ \mathcal{C}_i + \mathcal{D}_{i1}(I - \Delta\mathcal{D}_{11})^{-1}\Delta\mathcal{C}_1 & \mathcal{D}_{ij} + \mathcal{D}_{i1}(I - \Delta\mathcal{D}_{11})^{-1}\Delta\mathcal{D}_{1j} \end{pmatrix}$$

that is, due to (5), continuous (and even smooth) on $\boldsymbol{\Delta}$.

If the interconnection is well-posed, (3) or (6) are *uniformly exponentially stable* if there exist constants K and $\alpha > 0$ such that, for every uncertainty $\Delta(.)$ and for every unforced ($w_2 = 0, \ldots, w_m = 0$) system trajectory $\xi(.)$,

$$\|\xi(t)\| \leq K e^{-\alpha(t-t_0)} \|\xi(t_0)\| \text{ for all } t \geq t_0 \geq 0.$$

Under the hypothesis (5), it is well-known that uniform exponential stability is guaranteed by the existence of an $\mathcal{X} > 0$ that satisfies the Lyapunov inequality $\boldsymbol{\mathcal{A}}(\Delta)^T\mathcal{X} + \mathcal{X}\boldsymbol{\mathcal{A}}(\Delta) < 0$ for all Δ in the admissible value set $\boldsymbol{\Delta}$. In the following result the Lyapunov inequality is rewritten into a form that facilitates the application of the full block S-procedure.

THEOREM 3.1. *Suppose the interconnection (3) is well-posed, and suppose there exists an $\mathcal{X} > 0$ satisfying*

$$\begin{pmatrix} I \\ \boldsymbol{\mathcal{A}}(\Delta) \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{X} \\ \mathcal{X} & 0 \end{pmatrix} \begin{pmatrix} I \\ \boldsymbol{\mathcal{A}}(\Delta) \end{pmatrix} < 0 \textit{ for all } \Delta \in \boldsymbol{\Delta}. \tag{7}$$

Then the uncertain system (3) is uniformly exponentially stable.

Proof. The very standard proof is only included for the convenience of the reader. By compactness of $\boldsymbol{\Delta}$, there exists an $\epsilon > 0$ such that

$$\begin{pmatrix} I \\ \boldsymbol{\mathcal{A}}(\Delta) \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{X} \\ \mathcal{X} & 0 \end{pmatrix} \begin{pmatrix} I \\ \boldsymbol{\mathcal{A}}(\Delta) \end{pmatrix} + \epsilon\mathcal{X} < 0$$

for all $\Delta \in \boldsymbol{\Delta}$. Now suppose $\Delta(.)$ is any admissible uncertainty curve, and let $\xi(.)$ be an arbitrary unforced system trajectory. With $v(t) := \xi(t)^T\mathcal{X}\xi(t)$, we obtain

$$\dot{v}(t) + \epsilon v(t) \leq 0 \text{ for all } t \geq 0$$

by left-multiplying $\xi(t)^T$ and right-multiplying $\xi(t)$. With the integrating factor $e^{\epsilon t}$, this implies

$$v(t) \leq v(t_0)e^{-\epsilon(t-t_0)} \text{ for all } t \geq t_0 \geq 0.$$

Due to $\lambda_{\min}(\mathcal{X})\|\xi(t)\|^2 \leq v(t) \leq \lambda_{\max}(\mathcal{X})\|\xi(t)\|^2$, we finally obtain

$$\|\xi(t)\| \leq \sqrt{\frac{\lambda_{\max}(\mathcal{X})}{\lambda_{\min}(\mathcal{X})}} e^{-\frac{\epsilon}{2}(t-t_0)}\|\xi(t_0)\| \text{ for all } t \geq t_0 \geq 0$$

what finishes the proof with explicit formulas for the constants α and K. ◻

The condition (7) is formulated in terms of the rational function $\boldsymbol{\mathcal{A}}(\Delta)$ on the whole set $\boldsymbol{\Delta}$ such that it is pretty hard to verify directly. The main purpose of the full block S-procedure proposed in this chapter is to *equivalently* reformulate this test into a more explicit condition that makes use of multipliers. Here, the relevant set of multipliers is defined as

(8)
$$\mathcal{P} := \left\{ P \in \mathbf{R}^{(k+l)\times(k+l)} : \ P = P^T, \begin{pmatrix} \Delta \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta \\ I \end{pmatrix} > 0 \text{ for all } \Delta \in \boldsymbol{\Delta} \right\}.$$

Whenever required we will tacitly assume that any such multiplier is partitioned as

$$P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \text{ conformable to } \begin{pmatrix} \Delta \\ I \end{pmatrix}.$$

For robust stability, we will reveal in detail how to apply Lemma A.1 such that we can be very short in extending the technique to robust performance tests.

THEOREM 3.2. *The interconnection (3) is well-posed and $\mathcal{X}$ satisfies (7) if and only if there exists a multiplier $P \in \mathcal{P}$ with*

(9)
$$\begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B}_1 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B}_1 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \end{pmatrix} < 0.$$

Proof. To apply Lemma A.1, introduce

$$N = \left(\begin{array}{cc|cc} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \ \mathcal{S} = \operatorname{im} \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B}_1 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \end{pmatrix}, \ \mathcal{S}_0 = \operatorname{im} \begin{pmatrix} 0 \\ \mathcal{B}_1 \\ \hline I \\ \mathcal{D}_{11} \end{pmatrix}$$

and

$$U = \begin{pmatrix} I & -\Delta \end{pmatrix}, \ T = \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right).$$

Hence $\mathcal{S}_U = \ker(UT) \cap \mathcal{S}$ is described as

$$\left\{ \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B}_1 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} : \; -\Delta \mathcal{C}_1 \xi_1 + (I - \Delta \mathcal{D}_{11}) \xi_2 = 0 \right\}.$$

Therefore, we conclude that $\mathcal{S}_U$ is complementary to $\mathcal{S}_0$ iff $I - \Delta\mathcal{D}_{11}$ is non-singular. Moreover, if $I - \Delta\mathcal{D}_{11}$ is non-singular, we arrive at the alternative description of $\mathcal{S}_U$ as

$$\operatorname{im} \begin{pmatrix} I \\ \boldsymbol{\mathcal{A}}(\Delta) \\ (I - \Delta\mathcal{D}_{11})^{-1}\Delta\mathcal{C}_1 \\ \mathcal{C}_1 + \mathcal{D}_{11}(I - \Delta\mathcal{D}_{11})^{-1}\Delta\mathcal{C}_1 \end{pmatrix}.$$

Then it is obvious that the following conditions are equivalent: The inequality (7) and N being negative definite on $\mathcal{S}_U$, the inequality (9) and $N + T^T PT$ being negative on $\mathcal{S}$, and, finally, the condition $P \in \mathcal{P}$ and P being positive on $\ker(U)$. The latter just follows from the observation that the kernel of U is nothing but the image of $\begin{pmatrix} \Delta \\ I \end{pmatrix}$. All this shows that Theorem 9 is a reformulation of Lemma A.1 what finishes the proof. □

3.2 Robust Quadratic Performance

Suppose P_{pi} is a symmetric matrix that defines a performance index. Then the *robust quadratic performance* (QP) specification on the channel i is formulated as follows: There exists an $\epsilon > 0$ such that

$$\int_0^\infty \begin{pmatrix} w_i(t) \\ z_i(t) \end{pmatrix}^T P_{pi} \begin{pmatrix} w_i(t) \\ z_i(t) \end{pmatrix} dt \le -\epsilon \int_0^\infty w_i(t)^T w_i(t)\, dt \tag{10}$$

holds for any trajectory of the uncertain system (3) with $\xi(0) = 0$.

The following technical hypothesis on the performance index is both indispensable to apply the full block S-procedure, and to arrive at nominal controller synthesis procedures that can be based on solving LMIs:

$$P_{pi} = \begin{pmatrix} Q_{pi} & S_{pi} \\ S_{pi}^T & R_{pi} \end{pmatrix} \text{ satisfies } R_{pi} \ge 0. \tag{11}$$

Among others, this hypothesis holds true for the following very important special cases: Robust quadratic performance with

- $P_{pi} = \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix}$ guarantees that the L_2-gain of the channel $w_i \to z_i$ is robustly smaller than γ.

- $P_{pi} = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$ guarantees strict robust dissipativity of the channel $w_i \to z_i$, generalizing positive realness.

Based on Lyapunov arguments, one can easily obtain a sufficient LMI condition for robust quadratic performance.

THEOREM 3.3. *Suppose the interconnection (3) is well-posed, and suppose there exists an* $\mathcal{X}$ *satisfying*

(12)
$$\mathcal{X} > 0, \quad \begin{pmatrix} * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q_{pi} & S_{pi} \\ 0 & 0 & S_{pi}^T & R_{pi} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \boldsymbol{\mathcal{A}}(\Delta) & \boldsymbol{\mathcal{B}}_i(\Delta) \\ \hline 0 & I \\ \boldsymbol{\mathcal{C}}_i(\Delta) & \boldsymbol{\mathcal{D}}_{ii}(\Delta) \end{array}\right) < 0$$

for all $\Delta \in \boldsymbol{\Delta}$. *Then the uncertain system (3) is uniformly exponentially stable and satisfies the robust quadratic performance specification for the channel* $w_i \to z_i$.

Proof. The left-upper block of (12) reads as

$$\begin{pmatrix} I \\ \boldsymbol{\mathcal{A}}(\Delta) \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{X} \\ \mathcal{X} & 0 \end{pmatrix} \begin{pmatrix} I \\ \boldsymbol{\mathcal{A}}(\Delta) \end{pmatrix} + \boldsymbol{\mathcal{C}}_i(\Delta)^T R_{pi} \boldsymbol{\mathcal{C}}_i(\Delta) < 0.$$

At this point we exploit (11) to infer (7) such that we can conclude uniform robust exponential stability.

The proof of robust performance is, again, straightforward. By continuity and compactness of $\boldsymbol{\Delta}$, we can add $\begin{pmatrix} 0 & 0 \\ 0 & \epsilon I \end{pmatrix}$ on the left-hand side of (12) for some small $\epsilon > 0$ without violating the inequality. For any $w_i \in L_2$, let $\xi(.)$ and $z_i(.)$ be the corresponding state- and output trajectory of the uncertain system (3) with $\xi(0) = 0$. By right- and left-multiplication with $\begin{pmatrix} \xi(t) \\ w_i(t) \end{pmatrix}$ and its transpose, we infer

$$\frac{d}{dt}\xi(t)^T \mathcal{X} \xi(t) + \begin{pmatrix} w_i(t) \\ z_i(t) \end{pmatrix}^T P_{pi} \begin{pmatrix} w_i(t) \\ z_i(t) \end{pmatrix} + \epsilon w_i(t)^T w_i(t) \leq 0.$$

Integrating over $[0, T]$ and letting T go to infinity implies the desired inequality (10) if we recall $\xi(0) = 0$ and $\lim_{T\to\infty} \xi(T) = 0$. □

On the basis of Lemma A.1, we can obtain with ease the *equivalent* multiplier test.

THEOREM 3.4. *The interconnection (3) is well-posed and* $\mathcal{X}$ *satisfies (12)* *if and only if there exists a* $P \in \mathcal{P}$ *such that*

(13)

$$\mathcal{X} > 0, \quad \begin{pmatrix} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & \mathcal{X} & 0 & 0 & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_{pi} & S_{pi} \\ 0 & 0 & 0 & 0 & S_{pi}^T & R_{pi} \end{array}\right) \begin{pmatrix} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_i \\ \hline 0 & I & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{1i} \\ \hline 0 & 0 & I \\ \mathcal{C}_i & \mathcal{D}_{i1} & \mathcal{D}_{ii} \end{pmatrix} < 0.$$

Proof. As in the proof of Theorem 3.2, apply Lemma A.1 to

$$N = \left(\begin{array}{cc|cc|cc} 0 & \mathcal{X} & 0 & 0 & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_{pi} & S_{pi} \\ 0 & 0 & 0 & 0 & S_{pi}^T & R_{pi} \end{array}\right), \quad \mathcal{S}_0 = \operatorname{im} \begin{pmatrix} 0 \\ \mathcal{B}_1 \\ \hline I \\ \mathcal{D}_{11} \\ \hline 0 \\ \mathcal{D}_{i1} \end{pmatrix},$$

(14)

$$\mathcal{S} = \operatorname{im} \begin{pmatrix} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_i \\ \hline 0 & I & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{1i} \\ \hline 0 & 0 & I \\ \mathcal{C}_i & \mathcal{D}_{i1} & \mathcal{D}_{ii} \end{pmatrix}$$

and

$$U = \begin{pmatrix} I & -\Delta \end{pmatrix}, \quad T = \left(\begin{array}{cc|cc|cc} 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{array}\right).$$

3.3 Robust H_2 Performance

If the disturbance w_i that affects the system is stochastic in nature, we can also consider the H_2 criterion as a performance measure for the perturbed system. Let us assume that $\xi(0) = 0$, that w_i is white noise, and that we are interested in bounding the variance of the output z_i by a given number γ.

We need to assume that the uncertainty model and the controller are such that

(15)
$$\forall \Delta \in \boldsymbol{\Delta} : \; \boldsymbol{\mathcal{D}}_{ii}(\Delta) = 0.$$

Then it is easy to arrive at the following analysis result.

THEOREM 3.5. *Suppose the interconnection (3) is well-posed, and suppose there exist $\mathcal{X} > 0$, $Z > 0$ satisfying, for all $\Delta \in \mathbf{\Delta}$,*

(16)
$$\left(\begin{array}{cc} I & 0 \\ \mathcal{A}(\Delta) & \mathcal{B}_i(\Delta) \\ \hline 0 & I \\ \mathcal{C}_i(\Delta) & \mathcal{D}_{ii}(\Delta) \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \mathcal{A}(\Delta) & \mathcal{B}_i(\Delta) \\ \hline 0 & I \\ \mathcal{C}_i(\Delta) & \mathcal{D}_{ii}(\Delta) \end{array}\right) < 0$$

(17)
$$\left(\begin{array}{c} I \\ \mathcal{C}_i(\Delta) \end{array}\right)^T \left(\begin{array}{cc} -\mathcal{X} & 0 \\ 0 & Z^{-1} \end{array}\right) \left(\begin{array}{c} I \\ \mathcal{C}_i(\Delta) \end{array}\right) < 0, \;\; \mathbf{Tr}(Z) < 1.$$

Then the uncertain system (3) is uniformly exponentially stable, and the variance of the output z_i on the time-interval $[0, \infty)$ is smaller than γ.

Due to the zero row and column, the inequality in (16) could be simplified. We kept it in this form to display the similarity to the robust quadratic performance test.

Proof. Uniform exponential stability follows as in the proof of Theorem 3.3. For verifying robust performance, one can easily rewrite (16) with $\mathcal{Y} = \mathcal{X}^{-1}$ to

$$\mathcal{A}(\Delta)\mathcal{Y} + \mathcal{Y}\mathcal{A}(\Delta)^T + \frac{1}{\gamma}\mathcal{B}_i(\Delta)\mathcal{B}_i(\Delta)^T < 0.$$

Using Lemma A.2, (17) is seen to be equivalent to

$$\mathcal{C}_i(\Delta)\mathcal{Y}\mathcal{C}_i(\Delta)^T < Z, \;\; \mathbf{Tr}(Z) < 1.$$

Let us now recall that the state covariance matrix of the uncertain system is given as the solution of the initial value problem

$$\dot{\mathcal{K}} = \mathcal{A}(\Delta(t))\mathcal{K} + \mathcal{K}\mathcal{A}(\Delta(t))^T + \mathcal{B}_i(\Delta(t))\mathcal{B}_i(\Delta(t))^T, \;\; \mathcal{K}(0) = 0.$$

Standard comparison results for linear matrix differential equations imply $\mathcal{K}(t) < \gamma\mathcal{Y}$ for $t \geq 0$ and hence we infer

$$\mathbf{Tr}\left[\mathcal{C}_i(\Delta(t))\mathcal{K}(t)\mathcal{C}_i(\Delta(t))^T\right] \leq \gamma\, \mathbf{Tr}\left[\mathcal{C}_i(\Delta(t))\mathcal{Y}\mathcal{C}_i(\Delta(t))^T\right] < \gamma \text{ for all } t \geq 0$$

what leads to the desired bound on the output variance. ∎

We end up with two inequalities in the parameter Δ. Therefore, we have to apply Lemma A.1 to each of these inequalities individually. This leads to two independent multipliers to equivalently reformulate the robust H_2 analysis condition to the multiplier version.

THEOREM 3.6. *The interconnection (3) is well-posed and $\mathcal{X}$, Z satisfy (16)-(17) if and only if there exist $P_1, P_2 \in \mathcal{P}$ such that*

(18)
$$\left(\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_i \\ \hline 0 & I & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{1i} \\ \hline 0 & 0 & I \\ \mathcal{C}_i & \mathcal{D}_{i1} & \mathcal{D}_{ii} \end{array}\right)^T \left(\begin{array}{cc|cc|cc} 0 & \mathcal{X} & 0 & 0 & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q_1 & S_1 & 0 & 0 \\ 0 & 0 & S_1^T & R_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_i \\ \hline 0 & I & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{1i} \\ \hline 0 & 0 & I \\ \mathcal{C}_i & \mathcal{D}_{i1} & \mathcal{D}_{ii} \end{array}\right) < 0$$

(19)
$$\left(\begin{array}{cc} I & 0 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \\ \hline \mathcal{C}_i & \mathcal{D}_{i1} \end{array}\right)^T \left(\begin{array}{c|cc|c} -\mathcal{X} & 0 & 0 & 0 \\ \hline 0 & Q_2 & S_2 & 0 \\ 0 & S_2^T & R_2 & 0 \\ \hline 0 & 0 & 0 & Z^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \\ \hline \mathcal{C}_i & \mathcal{D}_{i1} \end{array}\right) < 0, \mathbf{Tr}(Z) < 1.$$

Remark. Instead of the stochastic interpretation, the criterion derived here admits as well a deterministic interpretation; it guarantees a bound on the sum of the energy of the output responses to initial conditions taken as the columns of $\boldsymbol{\mathcal{B}}_i(\Delta(0))$.

3.4 Robust Generalized H_2 Performance

The generalized H_2 norm is the gain of the system mapping $w_i \in L_2$ into $z_i \in L_\infty$, i.e., the energy to peak gain [16]. The corresponding performance specification is to robustly guarantee the bound

$$\sup_{0<\|w_i\|_2<\infty} \frac{\|z_i\|_\infty}{\|w_i\|_2} < \gamma. \tag{20}$$

Note that the gain can only be finite if (15) holds what is assumed throughout this section.

THEOREM 3.7. *Suppose the interconnection (3) is well-posed, and that there exists an $\mathcal{X} > 0$ such that, for all $\Delta \in \mathbf{\Delta}$, the inequality (16) and*

$$\left(\begin{array}{c} I \\ \boldsymbol{\mathcal{C}}_i(\Delta) \end{array}\right)^T \left(\begin{array}{cc} -\mathcal{X} & 0 \\ 0 & \frac{1}{\gamma}I \end{array}\right) \left(\begin{array}{c} I \\ \boldsymbol{\mathcal{C}}_i(\Delta) \end{array}\right) < 0 \tag{21}$$

hold true. Then (3) is robustly exponentially stable, and the gain of $L_2 \ni w_i \to z_i \in L_\infty$ is smaller than γ.

Proof. As for quadratic performance, the proof of stability is obvious, and one can infer from (16) that there exists an $\epsilon > 0$ such that

$$\frac{d}{dt}\xi(t)^T \mathcal{X} \xi(t) \le (\gamma - \epsilon) w_i(t)^T w_i(t)$$

for any $w_i \in L_2$. Integrating over $[0,T]$ leads to $\xi(T)^T \mathcal{X} \xi(T) \leq (\gamma - \epsilon)\int_0^T w_i(t)^T w_i(t)\,dt$ and hence

$$\xi(T)^T \mathcal{X} \xi(T) \leq (\gamma - \epsilon)\|w_i\|_2^2 \text{ for all } T \geq 0.$$

The second inequality (21) implies $\boldsymbol{\mathcal{C}}_i(\Delta)^T \boldsymbol{\mathcal{C}}_i(\Delta) < \gamma \mathcal{X}$ and therefore, due to $\boldsymbol{\mathcal{D}}_{ii}(\Delta) = 0$,

$$z_i(T)^T z_i(T) \leq \gamma \xi(T)^T \mathcal{X} \xi(T) \text{ for all } T \geq 0.$$

Both relations taken together lead to (20). □

Similarly as for the H_2 performance specification, we have to introduce two independent multipliers to equivalently reformulate these conditions to the corresponding multiplier versions.

THEOREM 3.8. *The interconnection (3) is well-posed and $\mathcal{X}$ satisfies (16) and (21) if and only if there exist multipliers $P_1, P_2 \in \mathcal{P}$ with (18) and*

$$\text{(22)} \quad \left(\begin{array}{cc} I & 0 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \\ \hline \mathcal{C}_i & \mathcal{D}_{i1} \end{array}\right)^T \left(\begin{array}{c|cc|c} -\mathcal{X} & 0 & 0 & 0 \\ \hline 0 & Q_2 & S_2 & 0 \\ 0 & S_2^T & R_2 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\gamma} I \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \hline 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{11} \\ \hline \mathcal{C}_i & \mathcal{D}_{i1} \end{array}\right) < 0.$$

3.5 Robust Bound on Peak-to-Peak Gain

Let us finally include in our list the important time-domain specification to keep the peak-to-peak gain $L_\infty \ni w_i \to z_i \in L_\infty$ smaller than γ. To be precise, we intend to robustly guarantee

$$\text{(23)} \qquad \sup_{0 < \|w_i\|_\infty < \infty} \frac{\|z_i\|_\infty}{\|w_i\|_\infty} < \gamma.$$

Again, simple Lyapunov arguments provide sufficient conditions for this inequality to hold.

THEOREM 3.9. *Suppose the interconnection (3) is well-posed. If there exist $\mathcal{X}$, $\lambda > 0$, μ such that, for all $\Delta \in \boldsymbol{\Delta}$,*

(24)

$$\left(\begin{array}{cc} I & 0 \\ \boldsymbol{\mathcal{A}}(\Delta) & \boldsymbol{\mathcal{B}}_i(\Delta) \\ \hline 0 & I \\ \boldsymbol{\mathcal{C}}_i(\Delta) & \boldsymbol{\mathcal{D}}_{ii}(\Delta) \end{array}\right)^T \left(\begin{array}{cc|cc} \lambda\mathcal{X} & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & -\mu I & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \boldsymbol{\mathcal{A}}(\Delta) & \boldsymbol{\mathcal{B}}_i(\Delta) \\ \hline 0 & I \\ \boldsymbol{\mathcal{C}}_i(\Delta) & \boldsymbol{\mathcal{D}}_{ii}(\Delta) \end{array}\right) < 0,$$

(25)

$$\left(\begin{array}{cc} I & 0 \\ \hline 0 & I \\ \boldsymbol{\mathcal{C}}_i(\Delta) & \boldsymbol{\mathcal{D}}_{ii}(\Delta) \end{array}\right)^T \left(\begin{array}{c|cc} -\lambda\mathcal{X} & 0 & 0 \\ \hline 0 & (\mu - \gamma) I & 0 \\ 0 & 0 & \frac{1}{\gamma} I \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \hline 0 & I \\ \boldsymbol{\mathcal{C}}_i(\Delta) & \boldsymbol{\mathcal{D}}_{ii}(\Delta) \end{array}\right) < 0,$$

then (3) is robustly exponentially stable, and the gain of $L_\infty \ni w_i \to z_i \in L_\infty$ is robustly smaller than γ.

Proof. As before we observe that (24) implies

$$\frac{d}{dt}\xi(t)^T\mathcal{X}\xi(t) + \lambda\xi(t)^T\mathcal{X}\xi(t) - \mu w_i(t)^T w_i(t) \leq 0.$$

We infer $\xi(t)^T\mathcal{X}\xi(t) \leq \mu\int_0^t e^{-\lambda(t-\tau)}\|w_i(\tau)\|^2\, d\tau$ and hence

$$\xi(t)^T\mathcal{X}\xi(t) \leq \frac{\mu}{\lambda}\|w_i\|_\infty^2 \text{ for all } t \geq 0.$$

The inequality (25) leads (due to $\gamma - \mu > 0$) to

$$\frac{1}{\gamma-\epsilon}\|z_i(t)\|^2 \leq \lambda\xi(t)^T\mathcal{X}\xi(t) + (\gamma-\mu)\|w_i(t)\|^2 \leq \lambda\xi(t)^T\mathcal{X}\xi(t) + (\gamma-\mu)\|w_i\|_\infty^2$$

for some small $\epsilon > 0$. If we combine we infer $\|z_i\|_\infty^2 \leq (\gamma-\epsilon)\gamma\|w_i\|_\infty^2$ and hence (23). □

The reformulation into the corresponding multiplier version is, again, straightforward.

THEOREM 3.10. *The interconnection (3) is well-posed and (24)-(25) hold if and only if there exist multipliers $P_1, P_2 \in \mathcal{P}$ that satisfy*

(26)

$$\begin{pmatrix} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} \lambda\mathcal{X} & \mathcal{X} & 0 & 0 & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q_1 & S_1 & 0 & 0 \\ 0 & 0 & S_1^T & R_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\mu I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_i \\ \hline 0 & I & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{1i} \\ \hline 0 & 0 & I \\ \mathcal{C}_i & \mathcal{D}_{i1} & \mathcal{D}_{ii} \end{array}\right) < 0$$

(27)

$$\begin{pmatrix} * \\ \hline * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{c|cc|cc} -\lambda\mathcal{X} & 0 & 0 & 0 & 0 \\ \hline 0 & Q_2 & S_2 & 0 & 0 \\ 0 & S_2^T & R_2 & 0 & 0 \\ \hline 0 & 0 & 0 & (\mu-\gamma)I & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma}I \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ \hline 0 & I & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{1i} \\ \hline 0 & 0 & I \\ \mathcal{C}_i & \mathcal{D}_{i1} & \mathcal{D}_{ii} \end{array}\right) < 0.$$

Note that $\mathcal{X}$ and λ enter these inequalities non-linearly. In an analysis problem, it is advisable to search for the best upper bound on the peak-to-peak gain by fixing $\lambda > 0$, and minimizing the bound γ over the LMIs (26)-(27) (what is a convex optimization problem) to get the optimal value $\gamma_*(\lambda)$. Then one can perform an additional line-search over λ to further minimize $\gamma_*(\lambda)$ and to get to the smallest achievable bound.

4 Dualization

On the basis of Lemma A.2, one can dualize all robust performance tests we have provided in the previous section. Let us concentrate on quadratic performance only with an

$$\text{index matrix} \begin{pmatrix} Q_{pi} & S_{pi} \\ S_{pi}^T & R_{pi} \end{pmatrix} \text{ that is non-singular}$$

and whose inverse is denoted as

$$\begin{pmatrix} \tilde{Q}_{pi} & \tilde{S}_{pi} \\ \tilde{S}_{pi}^T & \tilde{R}_{pi} \end{pmatrix} = \begin{pmatrix} Q_{pi} & S_{pi} \\ S_{pi}^T & R_{pi} \end{pmatrix}^{-1}.$$

Note that this property can be enforced by a slight perturbation that neither changes the problem formulation nor its solution.

For an arbitrary matrix M, we observe that

$$\text{im}\begin{pmatrix} M \\ I \end{pmatrix}^{-} = \text{im}\begin{pmatrix} I \\ -M^T \end{pmatrix}. \tag{28}$$

Hence the set of dual scalings has to be defined as

$$\tilde{\mathcal{P}} := \left\{ \tilde{P} \in \mathbf{R}^{(k+l)\times(k+l)} :\ \tilde{P} = \tilde{P}^T,\ \forall \Delta \in \boldsymbol{\Delta} : \begin{pmatrix} I \\ -\Delta^T \end{pmatrix}^T \tilde{P} \begin{pmatrix} I \\ -\Delta^T \end{pmatrix} < 0 \right\}$$

and each $\tilde{P} \in \tilde{\mathcal{P}}$ is partitioned in the same fashion as P.

COROLLARY 4.1. *There exists an $\mathcal{X}$ and a multiplier $P \in \mathcal{P}$ with (12) if and only if there is a dual Lyapunov matrix $\mathcal{Y} > 0$ and a dual multiplier $\tilde{P} \in \tilde{\mathcal{P}}$ with*

$$\begin{pmatrix} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & \mathcal{Y} & 0 & 0 & 0 & 0 \\ \mathcal{Y} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q}_{pi} & \tilde{S}_{pi} \\ 0 & 0 & 0 & 0 & \tilde{S}_{pi}^T & \tilde{R}_{pi} \end{array}\right) \left(\begin{array}{ccc} -\mathcal{A}^T & \mathcal{C}_1^T & \mathcal{C}_i^T \\ I & 0 & 0 \\ \hline -\mathcal{B}_1^T & -\mathcal{D}_{11}^T & -\mathcal{D}_{j1}^T \\ 0 & I & 0 \\ \hline -\mathcal{B}_i^T & -\mathcal{D}_{1j}^T & -\mathcal{D}_i^T \\ 0 & 0 & I \end{array}\right) > 0$$

The Lyapunov matrices and multipliers are related as

$$\mathcal{X} = \mathcal{Y}^{-1} \text{ and } P = \tilde{P}^{-1}.$$

Proof. Suppose $\mathcal{X}$ and P satisfy (12). Since $\Delta = 0$ is in the value set $\boldsymbol{\Delta}$, we conclude that $R > 0$. Hence we infer

$$\left(\begin{array}{ccc} 0 & 0 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & I \end{array}\right)^T \underbrace{\left(\begin{array}{cc|cc|cc} 0 & \mathcal{X} & 0 & 0 & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_{pi} & S_{pi} \\ 0 & 0 & 0 & 0 & S_{pi}^T & R_{pi} \end{array}\right)}_{N} \left(\begin{array}{ccc} 0 & 0 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & I \end{array}\right) \geq 0.$$

This implies that $\mathcal{S}$ as defined in (14) is, in fact, a negative subspace of N of maximal dimension. Moreover, since (13) implies $\begin{pmatrix} I \\ \mathcal{D}_{11} \end{pmatrix}^T P \begin{pmatrix} I \\ \mathcal{D}_{11} \end{pmatrix} < 0$, P is non-singular. Hence the same is true of N. Due to (28) (after a row permutation), the proof is finished by applying Lemma A.2. □

5 How to Verify the Robust Performance Tests

The set of multipliers $\mathcal{P}$ is obviously convex. However, since this set is described by infinitely many inequalities, and since it is not obvious, in general, how to reduce to finitely many conditions, it is hard to even decide whether a matrix P is indeed contained in $\mathcal{P}$ or not. This precludes the verification of the robust performance tests in Section 3 by standard algorithms.

This is the motivation to confine, at the expense of conservatism, the search to a *smaller* set of multipliers that admits a simple description, preferably in terms of finitely many LMIs.

For that purpose we assume that the value set $\mathbf{\Delta}$ is the convex hull of finitely many prespecified matrices:

$$\mathbf{\Delta} = \mathbf{Co}\{\Delta_1, ..., \Delta_N\}. \tag{29}$$

Let us then introduce the multiplier set

$$\mathcal{P}_1 := \{P \in \mathcal{P} : \; Q < 0\} \text{ which satisfies } \mathcal{P}_1 \subset \mathcal{P}.$$

Note that, in general, this is a *strict* inclusion such that the restriction of the search to $\mathcal{P}_1$ introduces conservatism. However, a straightforward convexity argument (based on $Q < 0$) reveals that

$$P \in \mathcal{P}_1 \text{ if and only if } Q < 0, \; \begin{pmatrix} \Delta_\nu \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta_\nu \\ I \end{pmatrix} > 0 \text{ for all } \nu = 1, \ldots, N.$$

Hence, the set $\mathcal{P}_1$ can indeed be fully described by finitely many linear matrix inequalities, and the search for $P \in \mathcal{P}_1$ renders our robust performance test verifiable by solving standard LMI problems.

Remark. If $\mathcal{D}_{11} = 0$ such that the parameters enter (3) affinely, we observe that all the robust performance inequalities already *imply* that any multiplier in $\mathcal{P}$ satisfies $Q < 0$ and is, hence, contained in $\mathcal{P}_1$. Hence, in this case, $\mathcal{P}$ and $\mathcal{P}_1$ conincide and no conservatism has been introduced.

It is possible to reduce the conservatism for block-diagonal uncertainties. For that purpose let us assume that

$$\mathbf{\Delta} = \left\{ \Delta = \begin{pmatrix} \delta_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \delta_m I_m \end{pmatrix} : \; \delta_i \in [-1, 1] \right\}.$$

Then we obtain (29) where the $N = 2^m$ generators Δ_ν are defined by letting each δ_i vary in $\{-1, 1\}$. Let us now partition the left-upper block Q of the multiplier P as Δ what defines m diagonal blocks $Q_1, \ldots, Q_m$. If we introduce

$$\mathcal{P}_2 := \{P \in \mathcal{P} : \; Q_\mu < 0, \; \mu = 1, \ldots, m\} \text{ satisfying } \mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P},$$

we infer

$$P \in \mathcal{P}_2 \;\text{ iff }\; Q_\mu < 0, \; \mu = 1, \ldots, m, \; \begin{pmatrix} \Delta_\nu \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta_\nu \\ I \end{pmatrix} > 0, \; \nu = 1, \ldots, N$$

on the basis of a simple multi-convexity argument. Again, the set $\mathcal{P}_2$ is described in terms of finitely many LMIs. As expected, one can demonstrate by simple examples that the set $\mathcal{P}_2$ is in general larger than $\mathcal{P}_1$, and that $\mathcal{P}_2$ can lead to less conservative robust performance tests.

All these techniques apply in the same fashion to the corresponding dual inequalities as provided in Section A.2 for the corresponding sets of dual multipliers $\tilde{\mathcal{P}}$, $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_2$.

6 Mixed Robust Controller Design

We have listed several important analysis tests for robust performance. In a typical multi-objective robust controller design problem one tries to find a controller that meets a selection of all these specifications on various channels of the controlled system.

In order to render the underlying analysis problems algorithmically tractable, we constrain ourselves to the multiplier set $\mathcal{P}_1$ or, for block-diagonal uncertainties, to $\mathcal{P}_2$ respectively.

Even for the nominal performance specifications, the corresponding multi-objective control problems are still hard and mostly open. It is well-known that the main difficulty arises due to the fact that each of the performance specification requires, in general, a different Lyapunov matrix to render it satisfied. The presently known design techniques do not allow to easily overcome this obstacle. This has been the motivation to consider, instead, the so-called mixed control problems in which the goal is to render the specifications satisfied with a common Lyapunov function $\mathcal{X}$ for all specifications under investigation.

To be specific, we consider the robust mixed QP/H_2 problem: Try to robustly achieve quadratic performance with index P_{pi} on channel $w_i \to z_i$ and an H_2 bound γ on channel $w_j \to z_j$. Note that the generalization to any other combination of (possibly repeated) performance specifications on other channels is obvious.

The robust mixed QP/H_2 control problem hence aims at finding a controller, multipliers $P, P_1, P_2 \in \mathcal{P}$ and a Lyapunov matrix $\mathcal{X}$ as well as an auxiliary variable $Z > 0$ that render the LMIs (13) and (18)-(19) with i replaced by j satisfied.

6.1 Synthesis Inequalities for Output Feedback Control

Quite recently it has been observed [12, 6, 20, 18] how to step in a formal manner from the analysis to the corresponding synthesis inequalities. This procedure is based on transforming the Lyapunov matrix $\mathcal{X}$ and the controller parameters as

$$(30) \qquad \begin{pmatrix} \mathcal{X} & A_c & B_c & C_c & D_c \end{pmatrix} \to \begin{pmatrix} X & Y & K & L & M & N \end{pmatrix} =: v$$

into the new symmetric blocks X, Y and the transformed controller parameters K, L, M, N that are collected in the variable v. Let us now introduce the functions

$$\boldsymbol{X}(v) = \begin{pmatrix} Y & I \\ I & X \end{pmatrix}$$

and

$$\left(\begin{array}{c|c} \boldsymbol{A}(v) & \boldsymbol{B}_j(v) \\ \hline \boldsymbol{C}_i(v) & \boldsymbol{D}_{ij}(v) \end{array}\right) =$$
$$= \left(\begin{array}{cc|c} AY & A & B_j \\ 0 & XA & XB_j \\ \hline C_iY & C_i & D_{ij} \end{array}\right) + \left(\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & E_i \end{array}\right) \begin{pmatrix} K & L \\ M & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F_j \end{array}\right)$$

that are *affine* in v.

In order to transform the synthesis inequalities, one needs to find a formal congruence transformation involving $\mathcal{Z}$ such that the blocks in the analysis inequalities transform as

$$\mathcal{X} \to \mathcal{Z}^T \mathcal{X} \mathcal{Z}, \quad \begin{pmatrix} \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_j \\ \mathcal{C}_i & \mathcal{D}_{ij} \end{pmatrix} \to \begin{pmatrix} \mathcal{Z}^T[\mathcal{X}\mathcal{A}]\mathcal{Z} & \mathcal{Z}^T[\mathcal{X}\mathcal{B}_j] \\ \mathcal{C}_i\mathcal{Z} & \mathcal{D}_{ij} \end{pmatrix}.$$

Then it suffices to perform the substitution

$$\mathcal{Z}^T \mathcal{X} \mathcal{Z} \to \boldsymbol{X}(v), \quad \begin{pmatrix} \mathcal{Z}^T[\mathcal{X}\mathcal{A}]\mathcal{Z} & \mathcal{Z}^T[\mathcal{X}\mathcal{B}_j] \\ \mathcal{C}_i\mathcal{Z} & \mathcal{D}_{ij} \end{pmatrix} \to \begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}_j(v) \\ \boldsymbol{C}_i(v) & \boldsymbol{D}_{ij}(v) \end{pmatrix}$$

to arrive at the synthesis inequalities in the new variables v and all other variables that appear in the analysis test (such as multipliers and auxiliary parameters).

Once having solved the synthesis inequalities, the inversion of (30) leads back to the Lyapunov matrix and to the desired controller parameters. This inverse is easily calculated by finding non-singular matrices U, V with $I - XY = UV^T$ and solving the equations

$$\begin{pmatrix} Y & V \\ I & 0 \end{pmatrix} \mathcal{X} = \begin{pmatrix} I & 0 \\ X & U \end{pmatrix},$$

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} XAY & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix} \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} V^T & 0 \\ CY & I \end{pmatrix}$$

for $\mathcal{X}$ and A_c, B_c, C_c, D_c.

If performing these two steps for the robust mixed QP/H_2 problem, one arrives at the synthesis inequalities

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0,$$

$$\begin{pmatrix} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_{pi} & S_{pi} \\ 0 & 0 & 0 & 0 & S_{pi}^T & R_{pi} \end{array}\right) \begin{pmatrix} I & 0 & 0 \\ \boldsymbol{A} & \boldsymbol{B}_1 & \boldsymbol{B}_i \\ \hline 0 & I & 0 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{1i} \\ \hline 0 & 0 & I \\ \boldsymbol{C}_i & \boldsymbol{D}_{i1} & \boldsymbol{D}_{ii} \end{pmatrix} < 0,$$

$$\begin{pmatrix} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q_1 & S_1 & 0 & 0 \\ 0 & 0 & S_1^T & R_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} I & 0 & 0 \\ \boldsymbol{A} & \boldsymbol{B}_1 & \boldsymbol{B}_j \\ \hline 0 & I & 0 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{1j} \\ \hline 0 & 0 & I \\ \boldsymbol{C}_j & \boldsymbol{D}_{j1} & \boldsymbol{D}_{jj} \end{pmatrix} < 0,$$

$$\begin{pmatrix} I & 0 \\ \hline 0 & I \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} \\ \hline \boldsymbol{C}_j & \boldsymbol{D}_{j1} \end{pmatrix}^T \left(\begin{array}{c|cc|c} -\boldsymbol{X} & 0 & 0 & 0 \\ \hline 0 & Q_2 & S_2 & 0 \\ 0 & S_2^T & R_2 & 0 \\ \hline 0 & 0 & 0 & Z^{-1} \end{array}\right) \begin{pmatrix} I & 0 \\ \hline 0 & I \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} \\ \hline \boldsymbol{C}_j & \boldsymbol{D}_{j1} \end{pmatrix} < 0, \ \mathbf{Tr}(Z) < 1,$$

where we suppress the variable v for reasons of space.

Unfortunately, testing the feasibility of these inequalities does, in general, not amount to solving a convex optimization or LMI problem. Hence one has to resort to controller scalings iteration as they are known from μ-theory [13].

One might suggest the following procedure: Consider the scaled uncertainty set $r\boldsymbol{\Delta}$ and try to maximize r such that the synthesis inequalities are satisfied for the value set $r\boldsymbol{\Delta}$. Start with a nominal design for $r = 0$. Then the iteration proceeds as follows: In the first step, fix K, L, M, N and find X, Y and multipliers P, P_1, P_2 which correspond to $r\boldsymbol{\Delta}$ such that the synthesis inequalities hold and r is maximal. If parametrizing the multipliers by finitely many LMIs, testing the feasibility for fixed r corresponds to an analysis problem and hence reduces to a standard LMI feasibility test; the maximization of r can be performed by bisection. In the second step, one fixes P, P_1, P_2, and maximizes r by varying the whole variable v. For fixed r, this corresponds to a nominal performance design problem (such that it reduces to an LMI feasibility test), and the maximization of r is done by bisection.

One can immediately advise many variations of this principal procedure. In particular, one might choose other combinations of fixed and varying parameters in the optimization steps to increase r [18].

6.2 Synthesis Inequalities for State-Feedback Control

The procedure described in the previous section applies literally to state-feedback design. One just needs to employ the functions

$$\boldsymbol{X}(v) = Y,\ \left(\begin{array}{c|c} \boldsymbol{A}(v) & \boldsymbol{B}_j(v) \\ \hline \boldsymbol{C}_i(v) & \boldsymbol{D}_{ij}(v) \end{array}\right) = \left(\begin{array}{c|c} AY + BM & B_j \\ \hline C_iY + E_iM & D_{ij} \end{array}\right),$$

and the equations to reconstruct the Lyapunov matrix and the (static) controller read as

$$\mathcal{X} = Y^{-1},\ D_c = MY^{-1}.$$

As such, the synthesis inequalities are not affine in all variables. However, as demonstrated for the analysis inequalities, one can straightforwardly dualize the synthesis inequalities that we have derived above. Due to the mere fact that $\boldsymbol{B}_j(v)$ and $\boldsymbol{D}_{ij}(v)$ do actually *not depend on* v, the dual inequalities define convex constraints even if letting both the multipliers, the auxiliary parameter Z, and the whole parameter v vary; in fact, they can be easily rearranged to become affine in all unknowns. Hence, what is known from single-objective control problems or for block-diagonal multipliers completely generalizes to robust mixed control problems with full block scalings.

6.3 Elimination of Controller Parameters

It is well-known how to eliminate the controller parameters in single-objective nominal design problems. This is also possible in single-objective robust control problems. Let us consider, as an example, the robust quadratic performance problem with index P_p on the channel $w_2 \to z_2$. We will provide a variation of the standard procedure based on the projection lemma that leads, even for a general quadratic performance index, to particularly simple formulas.

We just apply Lemma A.3 and observe we can work with basis matrices K_1 and K_2 of the kernels of

$$\ker \begin{pmatrix} B^T & E_1^T & E_2^T \end{pmatrix} \text{ and } \ker \begin{pmatrix} C & F_1 & F_2 \end{pmatrix}$$

respectively. Then we arrive at the following equivalent synthesis test: Find X, Y and multipliers $P \in \mathcal{P}$, $\tilde{P} \in \tilde{\mathcal{P}}$ that satisfy

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0, \tag{31}$$

(32)

$$K_2^T \left(\begin{array}{c} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{array}\right)^T \left(\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ XA & XB_1 & XB_2 \\ \hline 0 & I & 0 \\ C_1 & D_{11} & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_{22} \end{array}\right) K_2 < 0,$$

(33)

$$K_1^T \left(\begin{array}{c} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{array}\right)^T \left(\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q}_p & \tilde{S}_p \\ 0 & 0 & 0 & 0 & \tilde{S}_p^T & \tilde{R}_p \end{array}\right) \left(\begin{array}{ccc} -YA^T & -YC_1^T & -YC_2^T \\ I & 0 & 0 \\ \hline -B_1^T & -D_{11}^T & -D_{21}^T \\ 0 & I & 0 \\ \hline -B_2^T & -D_{12}^T & -D_{22}^T \\ 0 & 0 & I \end{array}\right) K_1 > 0$$

and the duality coupling condition

$$\left(\begin{array}{cc} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{array}\right) = \left(\begin{array}{cc} Q & S \\ S^T & R \end{array}\right)^{-1}. \tag{34}$$

Non-convexity enters via the generally non-convex constraint (34). Although this result has not appeared in the literature, it is a straightforward (and notationally simple) extension of those in [15, 10] to robust quadratic performance. The main purpose for being so detailed is to point out the relation to LPV control in the next section.

7 LPV Design with Full Scalings

In LPV control, it is assumed that the parameters $\Delta(t)$ that enter the system are not unknown but that they can be measured on-line. This allows, among other applications, to approach (robust) gain-scheduling synthesis problems for non-linear control systems. So far, the LPV problem has been solved for block-diagonal parameter matrices with the corresponding block-diagonal scalings [1, 9, 14, 21]. In [19] we have sketched how to extend these techniques to full block scalings, and in this chapter we give for the first time the full problem solution.

Adjusted to the structure of (1), we assume that the measured parameter curve enters the controller also in a linear fractional fashion. Hence an LPV controller is defined by scheduling the LTI system

$$\dot{x}_c = A_c x_c + B_c \left(\begin{array}{c} y \\ w_c \end{array}\right), \quad \left(\begin{array}{c} u \\ z_c \end{array}\right) = C_c x_c + D_c \left(\begin{array}{c} y \\ w_c \end{array}\right) \tag{35}$$

with the actual parameter curve as

$$w_c = \Delta_c(\Delta(t)) z_c.$$

The controller is hence parameterized through the matrices A_c, B_c, C_c, D_c, and through a possibly non-linear scheduling function $\Delta_c(\Delta)$.

Note that previous approaches were based on $\Delta_c(\Delta) = \Delta$ such that the controller is scheduled with an identical copy of the parameters. However, full block scalings require the extension to more general scheduling function that will - a posteriori - turn out to be quadratic functions.

The goal is to construct an LPV controller that renders the quadratic performance specification with index P_p for the channel $w_2 \to z_2$ for all possible parameter curves satisfied.

As standard, the solution of this problem is obtained with a simple trick. In fact, the controlled system can, alternatively, be obtained by scheduling the LTI system

$$
\begin{pmatrix} \dot{x} \\ \hline z_1 \\ z_c \\ \hline z_2 \\ \hline y \\ w_c \end{pmatrix} = \left(\begin{array}{c|cc|c|cc} A & B_1 & 0 & B_2 & B & 0 \\ \hline C_1 & D_{11} & 0 & D_{12} & E_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \hline C_2 & D_{21} & 0 & D_{22} & E_2 & 0 \\ \hline C & F_1 & 0 & F_2 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w_1 \\ w_c \\ \hline w_2 \\ \hline u \\ z_c \end{pmatrix} \tag{36}
$$

with the parameters as

$$
\begin{pmatrix} w_1 \\ w_c \end{pmatrix} = \begin{pmatrix} \Delta(t) & 0 \\ 0 & \Delta_c(\Delta(t)) \end{pmatrix} \begin{pmatrix} z_1 \\ z_c \end{pmatrix}, \tag{37}
$$

and then controlling this parameter dependent system with the LTI controller (35).

Hence, with the chosen controller structure, the LPV problem we started out with is *equivalently* reformulated to a robust performance design problem as discussed previously: Find an LTI controller (35) that renders the system (36)-(37) uniformly exponentially stable such that the robust quadratic performance specification for $w_2 \to z_2$ with non-singular index P_p is satisfied.

Due to the fact that the parameters are measured on-line, the uncertainties enter the system in a very specific form what is reflected in the particular structure of the describing matrices in (36) and of the parameters in (37). This particular structure implies that the synthesis inequalities related to this robust performance problem are standard LMI problems. Hence they can be solved (without conservatism) using existing algorithms.

For guaranteeing robust stability and performance of the closed-loop system, we employ *extended* multipliers adjusted to the extended uncertainty structure that are given as

(38)

$$
P_e = \left(\begin{array}{c|c} Q_e & S_e \\ \hline S_e^T & R_e \end{array}\right) = \left(\begin{array}{cc|cc} Q & Q_{12} & S & S_{12} \\ Q_{21} & Q_{22} & S_{21} & S_{22} \\ \hline * & * & R & R_{12} \\ * & * & R_{21} & R_{22} \end{array}\right) \text{ with } Q_e < 0,\ R_e > 0
$$

and that satisfy

$$
\begin{pmatrix} \Delta & 0 \\ 0 & \Delta_c(\Delta) \\ \hline I & 0 \\ 0 & I \end{pmatrix}^* P_e \begin{pmatrix} \Delta & 0 \\ 0 & \Delta_c(\Delta) \\ \hline I & 0 \\ 0 & I \end{pmatrix} > 0 \text{ for all } \Delta \in \boldsymbol{\Delta}. \tag{39}
$$

The corresponding dual multipliers $\tilde{P}_e = P_e^{-1}$ are partitioned similarly as

(40)
$$\tilde{P}_e = \left(\begin{array}{c|c} \tilde{Q}_e & \tilde{S}_e \\ \hline \tilde{S}_e^T & \tilde{R}_e \end{array}\right) = \left(\begin{array}{cc|cc} \tilde{Q} & \tilde{Q}_{12} & \tilde{S} & \tilde{S}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{S}_{21} & \tilde{S}_{22} \\ \hline * & * & \tilde{R} & \tilde{R}_{12} \\ * & * & \tilde{R}_{21} & \tilde{R}_{12} \end{array}\right) \text{ with } \tilde{Q}_e < 0,\ \tilde{R}_e > 0.$$

As indicated by this notation, it will turn out that the LPV synthesis inequalities will be only influenced by the multipliers blocks in P_e and $\tilde{P}_e$ without indices, and they will be actually *identical* to those of the robust control problem apart from the coupling condition (34). Therefore, testing these synthesis conditions indeed amounts to solving a standard LMI problem.

THEOREM 7.1. *There exists a controller (35) and a scheduling function such that the system (36)-(37) controlled with (35) satisfies the analysis conditions for robust quadratic performance with multipliers (38)-(39) if and only if there exist X, Y and scalings $P \in \mathcal{P}_1$, $\tilde{P} \in \tilde{\mathcal{P}}_1$ that satisfy the linear matrix inequalities (31)-(33).*

Proof. The proof of 'only if' is straightforward: Eliminate the controller parameters in the analysis inequalities. Due to the specific structure of the describing matrices in (36), the resulting synthesis inequalities simplify to (31)-(33) such that only the multiplier parts

$$P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \text{ and } \tilde{P} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix}$$

appear.

The constructive proof of 'if' is more involved. Let us assume that we have found a solution to (31)-(33).

First step: Extension of Scalings. Let us define the matrices

$$Z = \begin{pmatrix} I \\ 0 \end{pmatrix} \text{ and } \tilde{Z} = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

with the row partition of P. Note that $\mathrm{im}(\tilde{Z})$ is the orthogonal complement of $\mathrm{im}(Z)$, and recall

(41) $$Z^T P Z < 0,\ \tilde{Z}^T P \tilde{Z} > 0 \text{ as well as } Z^T \tilde{P} Z < 0,\ \tilde{Z}^T \tilde{P} \tilde{Z} > 0.$$

For the given P and $\tilde{P}$, we try to find an extension P_e with (38) such that the dual multiplier $\tilde{P}_e = P_e^{-1}$ is related to the given $\tilde{P}$ as in (40). After a suitable permutation, this amounts to finding an extension

(42) $$\begin{pmatrix} P & T \\ T^T & T^T N T \end{pmatrix} \text{ with } \begin{pmatrix} \tilde{P} & * \\ * & * \end{pmatrix} = \begin{pmatrix} P & T \\ T^T & T^T N T \end{pmatrix}^{-1},$$

where the specific parametrization of the new blocks in terms of a non-singular matrix T and some symmetric N will turn out convenient. Such an extension is very simple to obtain. However, we also need to obey the positivity/negativity constraints in (38) that amount to

$$\left(\begin{array}{cc} Z & 0 \\ 0 & Z \end{array}\right)^T \left(\begin{array}{cc} P & T \\ T^T & T^TNT \end{array}\right)\left(\begin{array}{cc} Z & 0 \\ 0 & Z \end{array}\right) < 0 \tag{43}$$

and

$$\left(\begin{array}{cc} \tilde{Z} & 0 \\ 0 & \tilde{Z} \end{array}\right)^T \left(\begin{array}{cc} P & T \\ T^T & T^TNT \end{array}\right)\left(\begin{array}{cc} \tilde{Z} & 0 \\ 0 & \tilde{Z} \end{array}\right) > 0. \tag{44}$$

We can assume w.l.o.g. (perturb, if necessary) that $P - \tilde{P}^{-1}$ is non-singular. Then we set

$$N = (P - \tilde{P}^{-1})^{-1}$$

and observe that (42) holds for any non-singular T.

The main goal is to adjust T to render (43)-(44) satisfied. We will in fact construct the subblocks $T_1 = TZ$ and $T_2 = T\tilde{Z}$ of $T = (T_1\ T_2)$. Due to (41), the conditions (43)-(44) read in terms of these blocks as

$$T_1^T\left[N - Z(Z^TPZ)^{-1}Z^T\right]T_1 < 0 \text{ and } T_2^T\left[N - \tilde{Z}(\tilde{Z}^TP\tilde{Z})^{-1}\tilde{Z}^T\right]T_2 > 0. \tag{45}$$

If we denote by $n_+(S)$, $n_-(S)$ the number of positive, negative eigenvalues of the symmetric matrix S, we hence have to calculate $n_-(N - Z(Z^TPZ)^{-1}Z^T)$ and $n_+(N - \tilde{Z}(\tilde{Z}^TP\tilde{Z})^{-1}\tilde{Z}^T)$. Simple Schur complement arguments reveal that

$$n_-\left(\begin{array}{cc} Z^TPZ & Z^T \\ Z & N \end{array}\right)$$

equals

$$n_-(N - Z(Z^TPZ)^{-1}Z^T) + n_-(Z^TPZ) \text{ and } n_-(Z^T\tilde{P}^{-1}Z) + n_-(N).$$

By Lemma A.2 and $\tilde{Z}^TP\tilde{Z} > 0$, we infer $Z^T\tilde{P}^{-1}Z < 0$, and hence $n_-(Z^TPZ) = n_-(Z^T\tilde{P}^{-1}Z)$. This leads to

$$n_-(N - Z(Z^TPZ)^{-1}Z^T) = n_-(N) \text{ and } n_+(N - \tilde{Z}(\tilde{Z}^TP\tilde{Z})^{-1}\tilde{Z}^T) = n_+(N).$$

This implies that there exist T_1, T_2 with $n_-(N)$, $n_+(N)$ columns that satisfy (45). Since $n_+(N) + n_-(N)$ equals the number of rows of T, these two blocks indeed define a *square* T that can be assumed (after perturbation, if necessary) to be non-singular.

This finishes the construction of the extended multiplier (38) where we observe that the dimensions of Q_{22}/R_{22} equal the number of columns of T_1/T_2 which are, in turn, identical to $n_-(N)/n_+(N)$.

Second Step: Construction of the scheduling function. Let us recall that

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta \\ I \end{pmatrix} > 0 \text{ and } \begin{pmatrix} I \\ -\Delta^T \end{pmatrix}^T \tilde{P} \begin{pmatrix} I \\ -\Delta^T \end{pmatrix} < 0$$

on $\mathbf{\Delta}$. By Lemma A.3, we can hence infer that, for each $\Delta \in \mathbf{\Delta}$, there indeed exists a $\Delta_c(\Delta)$ that satisfies (39). Due to the structural simplicity, we can even provide an explicit formula which shows that $\Delta_c(\Delta)$ can be selected to depend smoothly on Δ. Indeed, by a straightforward Schur-complement argument, (39) is equivalent to

$$\left(\begin{array}{cc|cc} U_{11} & U_{12} & (W_{11}+\Delta)^T & W_{21}^T \\ U_{21} & U_{22} & W_{12}^T & (W_{22}+\Delta_c(\Delta))^T \\ \hline W_{11}+\Delta & W_{12} & V_{11} & V_{12} \\ W_{21} & W_{22}+\Delta_c(\Delta) & V_{21} & V_{22} \end{array}\right) > 0$$

for $U = R_e - S_e^T Q_e^{-1} S_e > 0$, $V = -Q_e^{-1} > 0$, $W = Q_e^{-1} S_e$. Since there exists a solution, the inequality can be rearranged to

$$\begin{pmatrix} U_{22} & * \\ W_{22}+\Delta_c(\Delta) & V_{22} \end{pmatrix} -$$
$$- \begin{pmatrix} U_{21} & W_{12}^T \\ W_{21} & V_{21} \end{pmatrix} \begin{pmatrix} U_{11} & * \\ W_{11}+\Delta & V_{11} \end{pmatrix}^{-1} \begin{pmatrix} U_{12} & W_{21}^T \\ W_{12} & V_{12} \end{pmatrix} > 0,$$

and it is clear that one solution is obtained by rendering the $(2,1)$-block to vanish:

$$\Delta_c(\Delta) = -W_{22} + \begin{pmatrix} W_{21} & V_{21} \end{pmatrix} \begin{pmatrix} U_{11} & * \\ W_{11}+\Delta & V_{11} \end{pmatrix}^{-1} \begin{pmatrix} U_{12} \\ W_{12} \end{pmatrix}.$$

Note that $\Delta_c(\Delta)$ has dimension $n_-(N) \times n_+(N)$.

Third Step: LTI controller construction. After having reconstructed the scalings, the design of the LTI part of the controller amounts to solving a nominal quadratic performance problem what can be done along standard lines, as e.g. shown in [20]. □

Remark. The proof reveals that the scheduling function $\Delta_c(\Delta)$ has a many rows/colums as there are negative/positive eigenvalues of $P - \tilde{P}^{-1}$ (if assuming w.l.o.g. that the latter is non-singular.) If it happens that $P - \tilde{P}^{-1}$ is posititve or negative definite, there is no need to schedule the controller at all; then we obtain a controller that solves the robust quadratic performance problem.

The search for X and Y and for the multipliers $P \in \mathcal{P}_1$ and $\tilde{P} \in \tilde{\mathcal{P}}_1$ to satisfy (31)-(33) amounts to testing the feasibility of a standard LMI. Moreover, the controller construction in the proof of Theorem 7.1 is constructive. Hence we conclude that we have found a full solution to the quadratic performance LPV control problem (including L_2-gain and dissipativity specifications) for full block scalings P_e that satisfy $Q_e < 0$. The more interesting general case without this still restrictive negativity hypotheses is dealt with in a forthcoming paper.

8 Conclusions

In this chapter we have discussed how to handle multi-objective robust performance analysis and synthesis problems for systems that are affected by time-varying parametric uncertainties. We introduced a general technique, the so-called full block S-procedure, that allows to formally introduce full block multipliers in order to reduce the conservatism that could result from the restriction to standard block-diagonal multipliers. Finally, we provided a solution to the single-objective LPV control problem if employing a subclass of full block scalings; the fully general case is left to a forthcoming paper.

A Auxiliary Results on Quadratic Forms

A.1 A Full Block S-Procedure

Suppose $\mathcal{S}$ is a subspace of $\mathbf{R}^n$, $T \in \mathbf{R}^{l\times n}$ is a full row rank matrix, and $\boldsymbol{U} \subset \mathbf{R}^{k\times l}$ is a compact set of matrices of full row rank. Define the family of subspaces

$$\mathcal{S}_U := \mathcal{S} \cap \ker(UT) = \{x \in \mathcal{S} :\ UTx = 0\} = \{x \in \mathcal{S} :\ Tx \in \ker(U)\}$$

indexed by $U \in \boldsymbol{U}$.

Suppose $N \in \mathbf{R}^{n\times n}$ is a fixed symmetric matrix. The goal is to render the implicit negativity condition

$$\forall U \in \boldsymbol{U} :\ N < 0 \text{ on } \mathcal{S}_U$$

explicit. We want to relate this property, under certain technical hypotheses, to the existence of a symmetric multiplier P that satisfies

$$N + T^T P T < 0 \text{ on } \mathcal{S} \text{ and } \forall U \in \boldsymbol{U} :\ P > 0 \text{ on } \ker(U).$$

As a technical hypothesis, we require that all subspaces $\mathcal{S}_U$ are complementary to a fixed subspace $\mathcal{S}_0 \subset \mathcal{S}$ that has the two properties

$$\dim(\mathcal{S}_0) \geq k \text{ and } N \geq 0 \text{ on } \mathcal{S}_0.$$

In the intended applications, $\mathcal{S}$ is an unperturbed system, T picks the interconnection variables that are constrained by the uncertainties, the elements of $U \in \boldsymbol{U}$ define kernel representations of the uncertainties, and $\mathcal{S}_U$ is the uncertain system. The complementarity condition amounts to a well-posedness property of the uncertain system description, and the non-negativity of N is a condition on the performance index of interest.

LEMMA A.1. *The two conditions*

$$\forall U \in \boldsymbol{U} :\ \mathcal{S}_U \cap \mathcal{S}_0 = \{0\} \textit{ and } N < 0 \textit{ on } \mathcal{S}_U \tag{46}$$

hold iff there exists a matrix P that satisfies

$$\forall U \in \boldsymbol{U}: \ N + T^T P T < 0 \ \text{on } \mathcal{S} \ \text{ and } \ P > 0 \ \text{on } \ker(U). \tag{47}$$

A.2 Dualization

The dualization of robust performance tests is most easily achieved with the following well-known auxiliary result that is the abstract version of a dualization argument in [10] which is based on manipulating block matrices.

LEMMA A.2. *Suppose that N is symmetric and non-singular, and that $\mathcal{S}$ is a negative subspace of N of maximal dimension. (N is negative definite on $\mathcal{S}$, and the number of negative eigenvalues of N coincides with the dimension of $\mathcal{S}$.) Then*

$$N^{-1} \text{ is positive definite on } \mathcal{S}^{-}.$$

A.3 Solvabiltiy Test for a Quadratic Inequality

The following explicit solvability characterization for a quadratic matrix inequality serves to conveniently eliminate controller parameters in synthesis tests.

Consider the quadratic inequality

$$\begin{pmatrix} I \\ A^T X B + C \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ A^T X B + C \end{pmatrix} < 0 \tag{48}$$

in the unstructured unknown X. We assume that

$$R \geq 0 \text{ and that } \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \text{ is non-singular.}$$

By Lemma A.2, we can dualize to equivalently reformulate the inequality as

$$\begin{pmatrix} -B^T X^T A - C^T \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}^{-1} \begin{pmatrix} -B^T X^T A - C^T \\ I \end{pmatrix} > 0.$$

The solvability test makes use of matrices A_- and B_- whose columns form a basis of the kernels of A and B respectively.

LEMMA A.3. *The quadratic inequality (48) has a solution X if and only if*

$$\begin{pmatrix} B_- \\ C B_- \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} B_- \\ C B_- \end{pmatrix} < 0$$

and

$$\begin{pmatrix} -C^T A_- \\ A_- \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}^{-1} \begin{pmatrix} -C^T A_- \\ A_- \end{pmatrix} > 0.$$

The proof of 'only if' is obvious. The proof of the converse can be based on the projection lemma and is constructive; if a solution is known to exist, one can explicitly calculate it.

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