Abstract

Gain scheduling is a popular approach for nonlinear control system design. A controller is obtained by designing a set of controllers at operating points and then linearly interpolating controller values between them. However, little guidance has been provided in the literature for the selection of operating points. We use interval mathematics and a classical synthesis design approach to determine a near minimal set of design points and assess the quality of a gain scheduled controller. A sufficient condition for the assignment of the system closed loop poles is developed, and an algorithm for selecting the operating points is provided. An example is given to demonstrate the approach.

Keywords Gain Scheduling, Interval Mathematics, Discrete-Time Control

1. Introduction

Gain scheduling is a method for designing controllers for slowly varying nonlinear systems based on linearized dynamics. Linearization transforms the nonlinear system into a linear parameter varying system whose parameters vary with some variable exogeneous to the system known as a scheduling variable. The system poles are placed near desired locations by designing controllers for a set of operating points, and then linearly interpolating parameter values between them. For further explanation of gain scheduling the reader is referred to references [9],[10],[11], and [12]. Controllers designed at the operating points place the system’s closed loop poles at desired locations, but only in the vicinity of the operating points. As the state of the system moves from one operating point to another, the poles will vary from their nominal design values. If the poles are allowed to migrate far from their nominal locations, the performance of the system will deteriorate. The selection of operating points is often heuristic and left to the control system designer. Thus there is a need for tools to guide the designer in selecting the operating points and assessing the quality of the gain scheduled controller. In this paper we apply interval
mathematics, and a classical synthesis design approach, to select an appropriate set of operating points and
ensure that the system’s closed loop poles are constrained to desired regions.

The use of a synthesis design approach and interval mathematics to localize the closed loop poles
of a system was introduced in [1]. In [2] the authors applied interval mathematics to a slowly varying
interval plant, and developed a test to assess the quality of the gain scheduled. Interval mathematics are
presented in detail in the texts [3], [4], and [5]. The classical synthesis design approach we use is covered
in [6]. In this paper, we restrict our study to systems with proper transfer functions whose coefficients
within known intervals vary slowly as a function of some external scheduling variable.

Interval mathematics, and the classical synthesis design approach of [6], can be used to assess the
quality of a gain scheduled controller. To this end, we relate the interval closed loop characteristic equation
of the system to the deviation of linearly interpolated controller parameter values from those which place
the poles at their nominal design locations. The results of [2] are used to develop a sufficient condition
under which the linearly interpolated controller keeps the system poles within desired regions. By applying
this condition, we develop an algorithm for selecting a small set of operating points for the gain scheduled
control system. Our algorithm requires that the behavior of the scheduling variable be known, but an
interpolated approximation of the system plant parameter functions may be used since these functions are
rarely completely known. In the latter case the results of applying the algorithm will be subject to the
accuracy of the interpolated approximation. Because a gain scheduled controller will most likely be
implemented by a digital computer our results are presented in discrete time.

The paper begins by reviewing the interval mathematics underlying the method in Section 2.
Sections 3 extends the gain scheduling process of [2] to discrete time plants with no uncertainty. In Section
4, we develop sufficient conditions to ensure the placement of closed loop poles of a gain scheduled
controller are developed. The method for selecting operating points is discussed in Section 5, and an
example of the process is given in Section 6. Conclusions and directions for future work are given in
Section 7.
2. Review of Interval Mathematics

In this section, the basics of interval mathematics are reviewed as necessary to understand the remainder of the paper. The material here is based on the extensive treatment of [3], [4], and [5].

A real interval with real bounds \(a^-, a^+\) is defined as

\[
A = [a^-, a^+] = [\min(a), \max(a)]
\] (2.1)

The interval is said to be thin if \(a^- = a^+\).

For real intervals \(A_i = [a^-_i, a^+_i], i = 1, 2\), the basic interval arithmetic operations are defined by

**Interval Addition**  
\[
A_1 + A_2 = [a^-_1 + a^-_2, a^+_1 + a^+_2]
\] (2.2)

**Interval Subtraction**  
\[
A_1 - A_2 = [a^-_1 - a^+_2, a^+_1 - a^-_2]
\] (2.3)

**Interval Multiplication**  
\[
A_1 \cdot A_2 = [\min\{a^-_1 \cdot a^-_2, a^-_1 a^+_2, a^+_1 a^-_2, a^+_1 a^+_2\}, \max\{a^-_1 a^-_2, a^-_1 a^+_2, a^+_1 a^-_2, a^+_1 a^+_2\}] 
\] (2.4)

**Interval Division**  
\[
A_1 / A_2 = [a^-_1 / a^-_2, a^-_1 / a^+_2].[1/a^+_1,1/a^-_2]
\] (2.5)

**Absolute Value**  
\[
|A| = \max\{|a^-|, |a^+|\}
\] (2.6)

**Midpoint**  
\[
\text{mid}(A) = (a^- + a^+)/2
\] (2.7)

**Radius**  
\[
\text{rad}(A) = (a^+ - a^-)/2
\] (2.8)

If \(f(x)\) is a continuous unary operation \(f(A) = [\min(f(x)), \max(f(x))]\), \(x \in A\) (2.9)

For the interval matrices \(A = [A_{ij}], B = [B_{ij}]\), we define matrix operations based on scalar operations as follows

**Matrix Addition/Subtraction**  
\[
A \pm B = [A_{ij} \pm B_{ij}], A \text{ and } B \text{ m} \times \text{n}
\] (2.10)

**Matrix Multiplication**  
\[
A.B = \left[ \sum_{k=1}^n A_{ik}B_{kj} \right], A \text{ m} \times \text{n} \text{ and } B \text{ n} \times \text{p}
\] (2.11)

**Absolute Value Matrix**  
\[
|A| = [|A_{ij}|]
\] (2.12)
The midpoint matrix and the radius matrix have the following properties [5, p. 84]

\[
\text{mid}(A \pm B) = \text{mid}(A) \pm \text{mid}(B) \tag{2.15}
\]

\[
\text{mid}(AB) = \text{mid}(A)\text{mid}(B) \text{ if } A \text{ or } B \text{ is thin} \tag{2.16}
\]

\[
\text{rad}(A) |B| \leq \text{rad}(AB) \leq \text{rad}(A) |B| + \text{mid}(A) \text{ rad}(B) \tag{2.17}
\]

\[
\text{rad}(AB) = \text{rad}(A) |B| \text{ if } B \text{ is thin} \tag{2.18}
\]

\[
\text{rad}(A \pm B) = \text{rad}(A) + \text{rad}(B) \tag{2.19}
\]

### 3. Synthesis Design of a Discrete Time Gain Scheduled Controller

To design a discrete-time gain scheduled controller, we apply the procedure used in [2] to develop continuous-time controllers. The system is first linearized to create a linear parameter varying transfer function

\[
G(w, s) = \frac{n_0(w) + n_1(w)s + n_2(w)s^2 + \ldots + n_l(w)s^l}{d_0(w) + d_1(w)s + d_2(w)s^2 + \ldots + d_l(w)s^l} \tag{3.1}
\]

where \( l \) is the order of the system, \( w \) is a slowly varying exogenous scheduling variable, and \( n_i(w), d_i(w) \) for \( i = 0 \ldots l \) are coefficients that depend on \( w \).

Each discrete time transfer function is written in the form [6]

\[
G_i(w, z) = \frac{N_i(w, z)}{D_i(w, z)}, i \in p, c, f \tag{3.2}
\]

where \( p \) denotes the discrete time plant transfer function, \( c \) denotes the controller, and \( f \) denotes the desired closed loop transfer function. The discrete-time plant transfer function can be easily obtained from its continuous-time counterpart using hold equivalence [7], [8]. The transfer function for an \( n^{th} \) order system is given by

\[
G_i(w, z) = \frac{n_{i0}(w) + n_{i1}(w)z + n_{i2}(w)z^2 + \ldots + n_{in}(w)z^n}{d_{i0}(w) + d_{i1}(w)z + d_{i2}(w)z^2 + \ldots + d_{in}(w)z^n} \tag{3.3}
\]
Using simple block diagram manipulation, we obtain the closed loop transfer function in terms of the plant and compensator transfer functions

\[
G_f(z) = \frac{N_c(w,z)N_p(w,z)}{D_c(w,z)D_p(w,z) + N_c(w,z)N_p(w,z)}
\]  

(3.4)

For \( \text{deg}\{G_p\} = n \), \( \text{deg}\{G_c\} = m \), \( \text{deg}\{G_f\} = m + n \), where \( \text{deg}\{\} \) denotes the order of the polynomial, we rewrite the desired transfer function as

\[
G_f(z) = \frac{n_f_0 + n_f_1 z + n_f_2 z^2 + \ldots + n_{f,m+n} z^{m+n}}{d_f_0 + d_f_1 z + d_f_2 z^2 + \ldots + d_{f,m+n} z^{m+n}}
\]  

(3.5)

By equating the denominators of (3.4) and (3.5) we obtain the Diophantine equation [6]

\[
D_f(w,z) = D_c(w,z)D_p(w,z) + N_c(w,z)N_p(w,z)
\]  

(3.6)

Equating coefficients and rewriting the result gives the linear system

\[
\begin{bmatrix}
    d_{c0}(w) \\
    n_{c0}(w) \\
    d_{c1}(w) \\
    n_{c1}(w) \\
    \vdots \\
    d_{cm}(w) \\
    n_{cm}(w)
\end{bmatrix}^T
= 
\begin{bmatrix}
    f_0 \\
    f_1 \\
    f_2 \\
    \vdots \\
    f_{m+n}
\end{bmatrix}
\]  

(3.7)

where \( f_i, i=1 \ldots m+n \) are the coefficients of the desired closed loop transfer function denominator,

\[
S = 
\begin{bmatrix}
    d_{p0}(w) & d_{p1}(w) & \ldots & d_{p,m-1}(w) & d_{p,m}(w) & 0 & \ldots & 0 \\
    n_{p0}(w) & n_{p1}(w) & \ldots & n_{p,m-1}(w) & n_{p,m}(w) & 0 & \ldots & 0 \\
    0 & d_{p0}(w) & \ldots & d_{p,m-1}(w) & d_{p,m}(w) & \ldots & 0 \\
    0 & n_{p0}(w) & \ldots & n_{p,m-1}(w) & n_{p,m}(w) & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0 & d_{p0}(w) & d_{p1}(w) & \ldots & d_{p,m}(w) \\
    0 & 0 & \ldots & 0 & n_{p0}(w) & n_{p1}(w) & \ldots & n_{p,m}(w)
\end{bmatrix}
\]  

(3.8)

and \( S \) is a \( 2(m+1) \times (m+n+1) \) real matrix. In the remainder of this paper the vector of controller coefficients is denoted by \( x \), and the vector of closed loop system coefficients is denoted by \( f \).
The design goal is now to solve (3.8) for $x$ in terms of the scheduling variable $w$. If $m = n - 1$ the matrix $S$ is square, and therefore

$$x = (S^T)^{-1} f$$ (3.9)

In some cases where it is not feasible to restrict the degree of the controller so that $m = n - 1$ [6], we use the left inverse of $S^T$ [13].

$$(S^T)^{-} = (SS^T)^{-1} S$$ (3.10)

and then

$$x = (S^T)^{-} f$$ (3.11)

If $S$ is square, its inverse and left inverse are identical. In the remainder of the paper it is assumed that the matrix $S$ is left invertible.

Until now we have used the synthesis design approach as presented in [6]. Although this approach works well for linear time-invariant systems it has weaknesses when used for gain scheduling. First, the design method introduces closed loop zeros into the system without allowing their placement. This can have a detrimental effect on performance, and in the gain scheduled case these zeros will be functions of the scheduling variable. This can be seen by observing that the design selects $N_c$ and $D_c$ in equation (3.4). Even though the closed loop poles lie within desired regions changes in these zero locations with the scheduling variable will influence the system time response. The method in [6] eliminates steady-state error by increasing the desired input to the system inversely proportional to the steady-state error. Although this method works well for linear time-invariant systems, it only magnifies the effect of the migrating closed loop zeros of a gain scheduled system. A simple solution to both problems is to augment the discrete time plant transfer function $G_p(z,w)$ to include a discrete time integrator in cascade with the digital to analog converter, plant, and analog to digital converter. Thus a controller can be designed to reduce the effect of closed loop zeros and increase the system type.
4. Condition for Pole Region Assignment

In this section, we apply the interval mathematics of Section 2 to develop conditions for the selection of gain scheduling operating points. Let the closed loop characteristic equation coefficient vector \( f \) be an interval vector corresponding to the desired regions for closed loop pole localization, and let \( x \) be an interval vector of controller coefficients. Then interval mathematics can be used to find an allowable range for \( x \) which will place the closed loop poles in those regions defined by \( f \). We find a thin interval vector \( \text{mid}(x) \) representing the ideal values of the controller coefficient vector, and then test the solution for \( f \) using a linearly interpolated approximation of \( \text{mid}(x) \).

It is shown in [1] and [2] that, for a given interval coefficient vector \( f \), the controller parameter vector \( \text{mid}(x) \) places the coefficients of the systems closed loop characteristic equation at the midpoint of the solution interval with

\[
\text{mid}(S^T)x(w) = \text{mid}(f) \quad (4.1)
\]

Since \( S \) is thin \( S^T = \text{mid}(S^T) \)

\[
\text{mid}(x(w)) = (S^T)^{-1}\text{mid}(f) \quad (4.2)
\]

Thus, for any \( S, f \) we can find a thin controller parameter vector \( x(w) = \text{mid}(x(w)) \) which places the system poles at their nominal locations.

In gain scheduling, the controller parameter vector \( x(w) \) varies with the scheduling variable \( w \). If the system changes sufficiently slowly then \( x(w) \) yields satisfactory performance for the original nonlinear system [12]. At each operating point \( w_i \), we compute a nominal controller design \( x(w_i), i=1,...,n_d \) where \( n_d \) is the number of operating points. A controller parameter vector \( x(w) \) can be determined which is only equal to the selected design \( x(w_i) \) at the operating points. Between operating points a linearly interpolated controller vector \( x^*(w) \) is used to approximate \( x(w) \).

If the condition

\[
S^T(w)x^*(w) \in f \quad (4.3)
\]
is satisfied, then the closed loop poles of the linearly interpolated controller will also lie within regions defined by $f$. If (4.3) is satisfied for all points within the expected range of $w$, then the gain scheduled controller will restrict the systems closed loop poles to their desired regions.

5. Procedure for Selecting Operating Points

In this section, we introduce a procedure for selecting a minimal set of operating points which ensures the linearly interpolated controller parameters will restrict closed loop system poles to a specified region. We use gridding to select operating points and check the location of the closed loop poles at each point using a linearly interpolated compensator. This is computationally feasible for two reasons. First; the scheduling variable of a gain-scheduled controller must change more slowly than the open-loop dynamics of the system for gain scheduling to be a design option. We therefore use the plant’s open loop dynamics to find a sampling interval short enough to avoid any aliasing of the scheduling variable provided that the effect of the scheduling variable on the open loop dynamics is considered when selecting a sampling interval. It is also possible to find a grid interval small enough so that it can be used for analysis of the closed loop system. Second; because the test condition in (4.3) relates the controller parameter vector to a vector of closed loop characteristic equation coefficients, we are testing a finite number of controller parameters. In general an $m^{th}$ order controller will require the testing of $2(m+1)$ parameters. For example, a second order controller will require the testing of only six controller parameters.

As an example of the process suppose one was trying to linearly approximate the nonlinear function shown in Figure 1. From $j = 0$ to $j = 15$ a linearly interpolated approximation is found which deviates little from the function. When $j = 30$ another linear approximation is shown, and another for $j = 45$. These approximations show a much greater difference between the nominal value and the approximation as the spacing between any two operating points is increased. As the difference between the nominal value, and the approximated value for any of the parameter functions at any value of the scheduling variable increases then there is a possibility of the failure of the condition of (4.3).
The procedure for selecting a minimal number of operating points is as follows:

1) Find a continuous linearized plant transfer function in terms of the scheduling variable $w$.

2) Start with two operating points at the minimum, and maximum expected values of the scheduling variable. ($w_{\text{min}} = \text{min}(w), w_{\text{max}} = \text{max}(w)$)

3) Select a number of sample points ($N$) which will adequately characterize the behavior of the scheduling variable $w$. The grid spacing is a design parameter subject to sampling theorem requirements necessary to avoid aliasing in the sampling of the scheduling variable [8].

4) For $w_j = 1 \ldots N$, in the range $[w_{\text{min}}, w_{\text{max}}]$ discretize the system then form $S(w_j)$ and calculate $x(w_j)$ using (3.9) and (3.12) respectively.

5) Assume that an operating point will be placed at each $w_j$. Starting at $j=2$, find a linearly interpolated approximation of the controller parameter vector $x'(w)$ in the range $[w_{\text{min}}, w_j]$, and test for any violation of (4.3). Increment $j$, and repeat until a failure of (4.3) occurs. If no violation occurs over the entire interval $[w_{\text{min}}, w_{\text{max}}]$ then no additional operating points are necessary, and we proceed to step 8.

6) At the value of $w_j$ when failure of (4.3) first occurs reduce $j$ until there is no failure on the interval $[w_{\text{min}}, w_j)$, and add the largest value of $w_j$ in this interval as a new operating point.
7) Repeat steps 5 and 6 for the interval \([w_j, w_{\text{max}}]\) until no failure occurs over the entire range \([w_{\text{init}}, w_{\text{max}}]\).

8) Simulate the system over various rates of change of \(w\) to verify performance and stability.

Because the gain scheduled system approximates a non-linear system with a linear parameter varying system there may be cases where the global system is unstable even though the poles of the closed loop characteristic equation lie within a range known to be stable [12].

6. Example

Consider the system

\[
\frac{d^2 y}{dt^2} + \frac{23}{4} \left(\frac{237}{100} - \sin(w)\right) \frac{dy}{dt} + 9y = 6u + \frac{du}{dt}
\]  

(6.1)

where \(w\) varies from \(\frac{\pi}{2}\) to \(\frac{\pi}{2}\) sufficiently slowly to allow gain scheduling.

A linearized transfer function with the scheduling variable as a parameter is

\[G(s, w) = \frac{s + 6}{s^2 + \frac{23}{4} \left(\frac{237}{100} - \sin(w)\right)s + 9}
\]  

(6.2)

A sample interval of 0.2 seconds is chosen using the open loop dynamics of \(G(s,w)\). The plant transfer function can now be discretized using a zero-order hold equivalent, augmented to include a discrete time integrator \(\frac{T_s}{z-1}\), and written in the form of (3.3) [7].

\[
G_p(z, w) = \frac{n_{p0}(w) + n_{p1}(w)z + n_{p2}(w)z^2 + n_{p3}(w)z^3}{d_{p0}(w) + d_{p1}(w)z + d_{p2}(w)z^2 + d_{p3}(w)z^3}
\]  

(6.3)

An appropriate controller for this system is of the form

\[
G_c(z, w) = \frac{n_{c0}(w) + n_{c1}(w)z + n_{c2}(w)z^2}{d_{c0}(w) + d_{c1}(w)z + d_{c2}(w)z^2}
\]  

(6.4)
In this example, $S$ is a 6x6 thin interval real matrix, $x$ is a 6x1 interval vector, the closed loop system is fifth order, and $f$ is a 6x1 interval vector.

$$
S = \begin{bmatrix}
    d_{p0}(w) & d_{p1}(w) & d_{p2}(w) & d_{p3}(w) & 0 & 0 \\
    n_{p0}(w) & n_{p1}(w) & n_{p2}(w) & n_{p3}(w) & 0 & 0 \\
    0 & d_{p0}(w) & d_{p1}(w) & d_{p2}(w) & d_{p3}(w) & 0 \\
    0 & n_{p0}(w) & n_{p1}(w) & n_{p2}(w) & n_{p3}(w) & 0 \\
    0 & 0 & d_{p0}(w) & d_{p1}(w) & d_{p2}(w) & d_{p3}(w) \\
    0 & 0 & n_{p0}(w) & n_{p1}(w) & n_{p2}(w) & n_{p3}(w)
\end{bmatrix} \quad (6.5)
$$

The controller parameter vector is

$$
x(w) = \begin{bmatrix}
    d_{e0}(w) \\
    n_{e0}(w) \\
    d_{e1}(w) \\
    n_{e1}(w) \\
    d_{e2}(w) \\
    n_{e2}(w)
\end{bmatrix} \quad (6.6)
$$

The closed loop poles of the system are specified by the discrete time characteristic equation

$$
D_f(z) = z^5 + \left[-2.002, -1.998\right]z^4 + \left[1.429, 1.441\right]z^3 + \left[-0.4775, -0.4635\right]z^2 + \left[0.0778, 0.0818\right]z + \left[-0.0048, -0.0042\right]
$$

Figure 2 shows the locations of the poles corresponding to (6.7) with nominal poles ($z = 0.7 \pm j0.05$, $z = 0.2 \pm j0.2$, $z = 0.1$). Note that our closed loop dynamics are limited by the selection of the sampling rate chosen from the open loop plant transfer function. A linear closed loop system with poles defined by (6.7) can be shown to be stable using the Jury test [7], and the interval mathematics of Section 2.

The desired closed loop characteristic equation yields the interval coefficient vector

$$
f = \begin{bmatrix}
    [-0.0048, -0.0042] & [0.0778, 0.0818] & [-0.4775, -0.4635] & [1.429, 1.441] & [-2.002, -1.998] & [1.1]
\end{bmatrix}^T \quad (6.8)
$$

with minimum,

$$
f_{\text{min}} = \begin{bmatrix}
    -0.0048 & 0.0778 & -0.4775 & 1.429 & -2.002 & 1
\end{bmatrix}^T \quad (6.9)
$$

and maximum
The design goal is to first find a controller parameter vector \(x\) that places the coefficient vector \(f\) at its midpoint over the entire design range of \(w\), and then to find a linearly interpolated controller parameter vector \(x^*\) such that

\[
S^T x^* \in f, \forall w \in [w_{\text{min}}, w_{\text{max}}] = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]
\]  

\(6.11\)

To characterize the scheduling variable we choose \(N=100\) sample points, and start with two operating points at \(w_1 = -\frac{\pi}{2}\) and \(w_{100} = \frac{\pi}{2}\). We then find \(w_j, S(w_j), \) and \(x(w_j)\) \(j=1\ldots N\), and save them for later use. At \(j = 2\) we assume that an operating point exists \(x_j^*\). We then test the condition of (4.3) for all \(x_j^*\) and \(S_j\) between \(j = 1\) and \(j = 2\). If there is no violation of (4.3) for any of these points then no operating point is necessary at \(j = 2\). The value of \(j\) is incremented to 3, and the test is repeated for all
points between \( j = 1 \) and \( j = 3 \). Again, there is no violation of (4.3). We continue to increment \( j \), and test (4.3). The first failure occurs at \( j = 21 \), so we place an operating point at \( w_{20} = -0.9679 \). We then repeat the process starting at \( j = 20 \). We now assume an operating point is necessary at \( j = 21 \) and test all points between \( j = 20 \) and \( j = 21 \). As before, \( j \) is incremented until a failure of (4.3) occurs. The next violation of (4.3) is at \( j = 40 \) and an operating point is placed at \( w_{39} = -0.3649 \). This process is again repeated starting at \( j = 39 \) until \( j = 100 \).

Applying the algorithm shows that eight operating points are required to ensure the closed loop system poles will stay within the regions defined by (6.7). The operating points, along with the associated discrete time controllers for the augmented plant (approximate values) are shown in Table 1. Figure 3 shows the system closed loop poles as \( w \) varies over its expected range. The poles remain in a small neighborhood of their nominal locations even though the closed loop characteristic equation assigns them to a larger region. If any element of \( S^T \hat{x}^* \) is outside the bound of any element of \( f \) during the design phase we add an operating point to the system and bring the closed loop poles back to their nominal locations. Thus, we ensure that the system poles remain well within the interval vector \( f \). Since the vector \( \hat{x}^* \) will approximate \( x^* \) we expect for the poles to remain near their nominal locations as shown for closed loop coefficient \( d_{f3} \) in Figure 4. However, since we can only test the coefficients of the closed loop characteristic equation instead of the pole locations themselves a conservative result is necessary. Provided that the number of samples used for the test can adequately characterize the scheduling variable the operating points found will not change significantly, and the closed loop poles will remain within desired bounds. Additional tests were made with \( N = 50, 80, 250, \) and 500. When \( N \) was greater than or equal to 80 the number of operating points found did not change. When \( N \) was less than 80 an additional operating point was selected near the end of the sample space. We attribute this to a lack of resolution in the sampling of the scheduling variable.
### Table 1: Results of Operating Point Selection

<table>
<thead>
<tr>
<th>j</th>
<th>( w_j ) (radians, degrees)</th>
<th>( G_c(w) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-1.5708, -90)</td>
<td>( \frac{23.41z^2 - 20.98z + 0.9226}{z^2 - 0.0680z - 0.0865} )</td>
</tr>
<tr>
<td>20</td>
<td>(-0.9679, -55.45)</td>
<td>( \frac{22.08z^2 - 19.81z + 0.9413}{z^2 - 0.0678z - 0.0838} )</td>
</tr>
<tr>
<td>39</td>
<td>(-0.3649, -20.91)</td>
<td>( \frac{18.60z^2 - 16.83z + 1.018}{z^2 - 0.0635z - 0.0765} )</td>
</tr>
<tr>
<td>55</td>
<td>(0.1428, -8.18)</td>
<td>( \frac{14.98z^2 - 13.77z + 1.152}{z^2 - 0.0490z - 0.0685} )</td>
</tr>
<tr>
<td>71</td>
<td>(0.6505, -37.27)</td>
<td>( \frac{11.61z^2 - 10.87z + 1.256}{z^2 - 0.0184z - 0.0553} )</td>
</tr>
<tr>
<td>82</td>
<td>(0.9996, 57.27)</td>
<td>( \frac{9.37z^2 - 8.51z + 0.947}{z^2 + 0.00769z - 0.0281} )</td>
</tr>
<tr>
<td>90</td>
<td>(1.2535, 71.82)</td>
<td>( \frac{7.18z^2 - 5.40z - 0.0440}{z^2 + 0.023z + 0.0254} )</td>
</tr>
<tr>
<td>100</td>
<td>(1.5708, 90)</td>
<td>( \frac{4.40z^2 - 0.888z - 1.812}{z^2 + 0.0309z + 0.112} )</td>
</tr>
</tbody>
</table>

A simulation of the closed loop system over an entire period (100s) of the scheduling variable is shown in Figure 5. A series of step inputs is applied to the system to show the effect of closed loop zeros. These zeros often cause an overshoot in the step response, but since the poles are constrained to their desired regions the settling time is nearly uniform. It can be seen in Figure 5 that the periodic nature of the scheduling variable influences in the magnitude of the overshoot of each step. As the frequency of the scheduling variable increases so does the magnitude of the overshoot caused by the closed loop zeros.
7. Conclusions

Conditions under which a gain scheduled control system yields closed loop poles within desired regions were developed. This paper introduced a method which determines a near minimal set of operating points for a gain scheduled control system, while ensuring that the poles of the closed loop system remain within specified bounds. The number of operating points found can not be said to be a minimal set because the condition used for guaranteeing closed loop pole localization is only a sufficient condition, and the algorithm has not been evaluated to determine if a smaller set is of operating points is possible. Also, the number of operating points necessary may be decreased by changing either the location of the closed loop poles, or by increasing the radius of the closed loop character equation coefficients. These are design parameters which should be determined prior to application of the algorithm. The stability of the overall gain scheduled control system was not addressed, but can be inferred from extensive simulation or tested using methods of [14]. Future work includes extending the design method to include multivariate gain scheduling, and to multiple input/multiple output continuous and discrete time cases.

Figure 3: Closed Loop Poles from Linearly Interpolated Controller
Figure 4: Closed Loop Characteristic Equation Coefficient $d_3$

Figure 5: Simulation Using Gain Scheduled Controller
References


