The History of Trigonometry

David Bressoud
Macalester College, St. Paul, MN

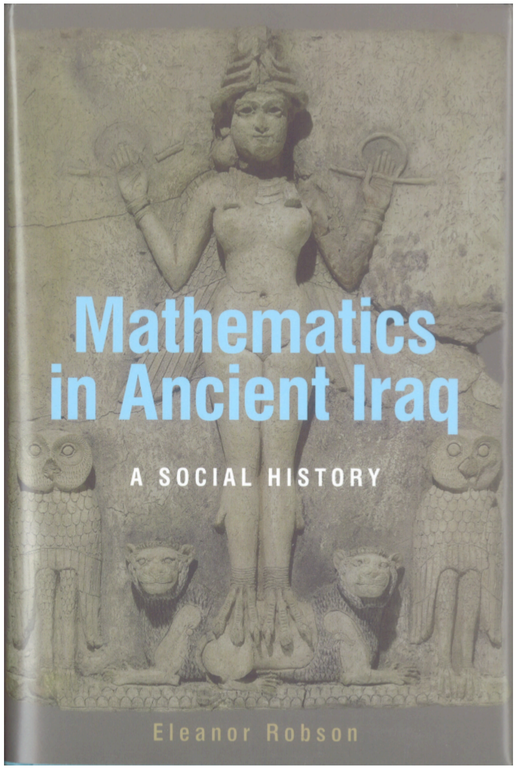
North Hennepin Community College
Brooklyn Park, MN
March 30, 2011

These slides are available at
www.macalester.edu/~bressoud/talks

"The task of the educator is to make the child's spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide."

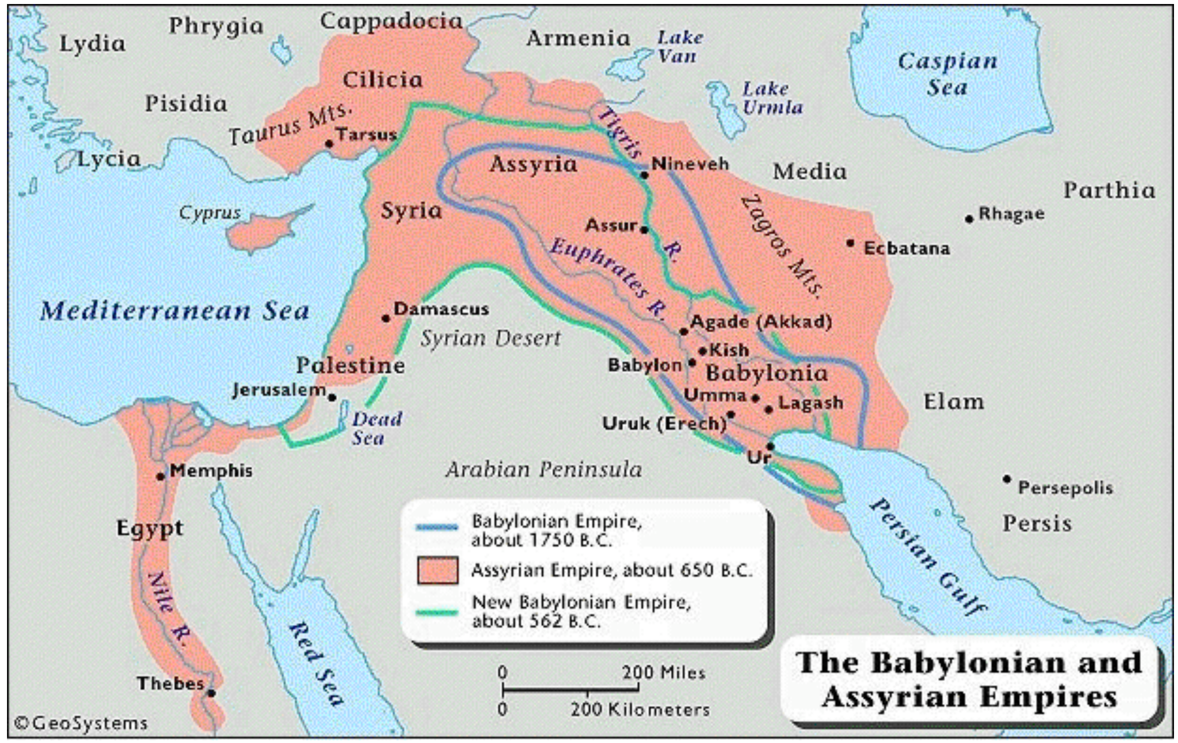
Henri Poincaré

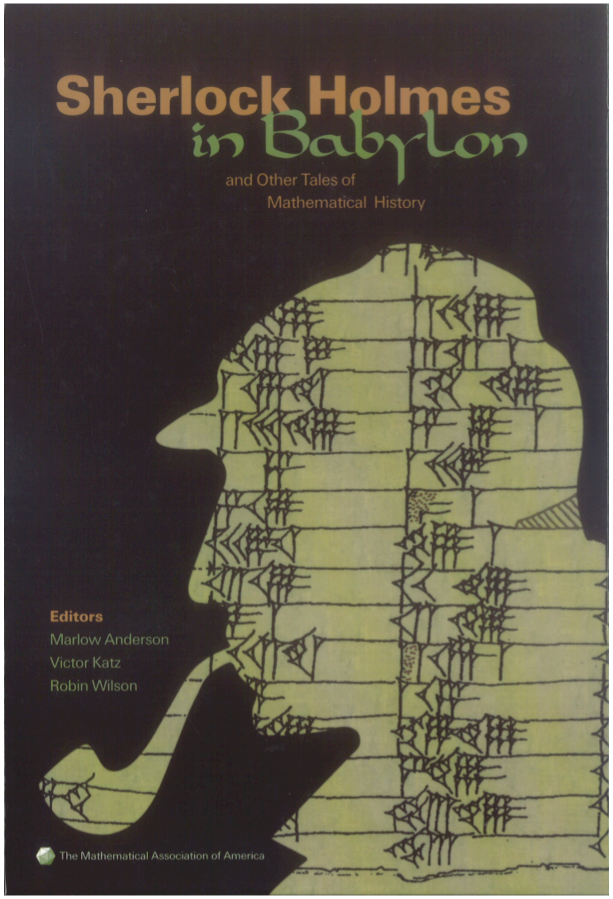
1854–1912



Princeton University Press, 2008

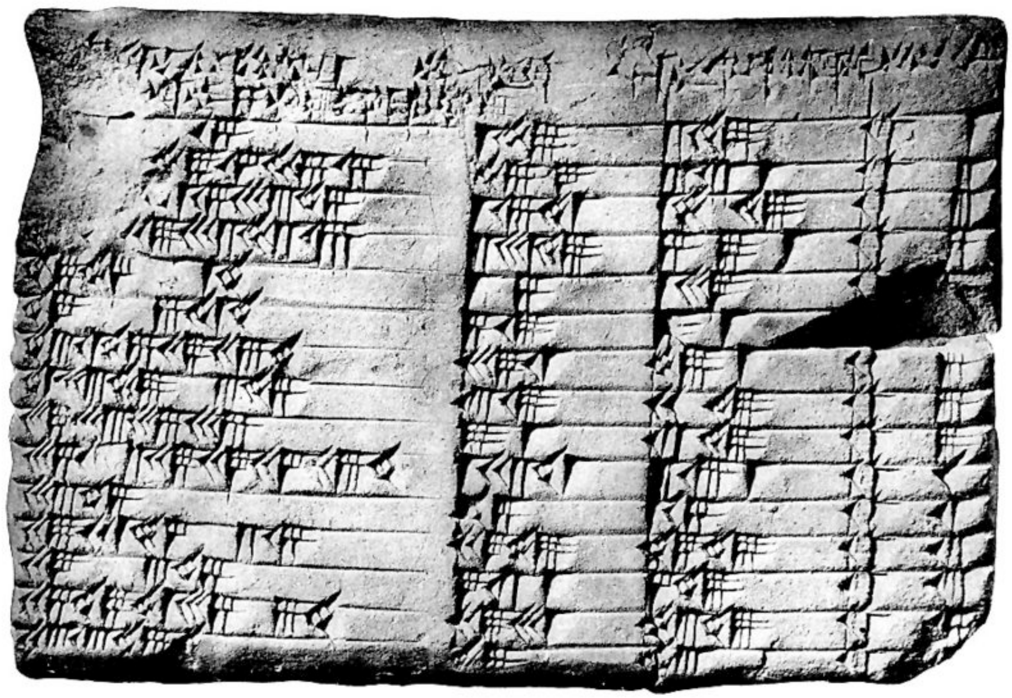
Old Babylonian Empire, 2000–1600 BCE

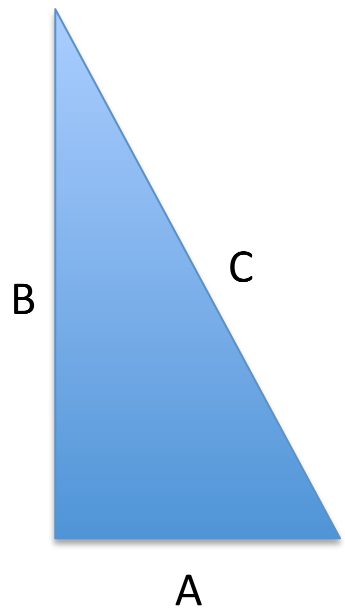




Mathematical Association of America, 2004

Plimpton 322 (Columbia University), table of Pythagorean triples, circa 1700 BCE, about 9 by 13 cm



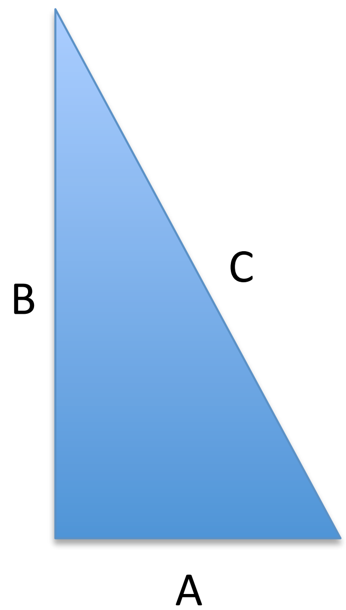


Simple Pythagorean triples:

$$3^2 + 4^2 = 5^2$$

$$5^2 + 12^2 = 13^2$$

$$8^2 + 15^2 = 17^2$$

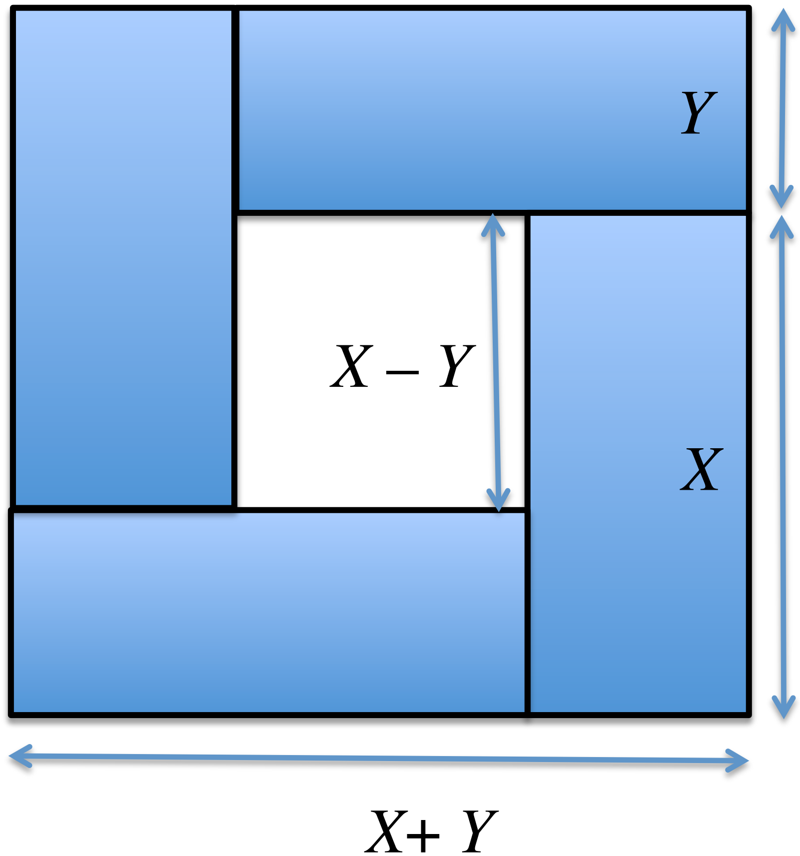


A	B	C
119	120	169
3367	3456	4825
4601	4800	6649
12709	13500	18541
65	72	97
319	360	481
2291	2700	3541
799	960	1249
481	600	769
4961	6480	8161
45	60	75
1679	2400	2929
161	240	289
1771	2700	3229
28	45	53

The difference between the square of side $X+Y$ and the square of side $X–Y$ is four rectangles of size XY.

$$(X+Y)^2-(X-Y)^2$$
$$=4XY$$

If $XY = ¼$, then $(X+Y)^2$ and $(X-Y)^2$ differ by 1.



$$X = 6/5, \quad Y = 5/24$$

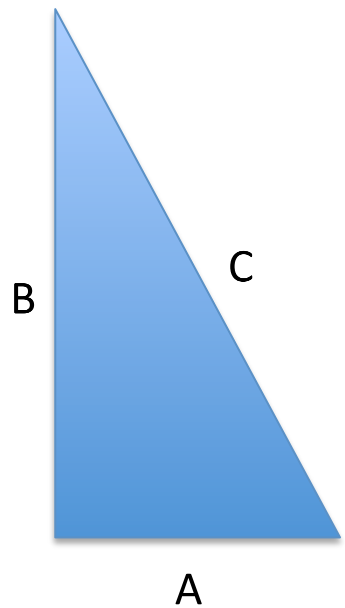
$$\frac{6}{5}\times\frac{5}{24}=\frac{1}{4}$$

$$\frac{6}{5}-\frac{5}{24}=\frac{119}{120},$$

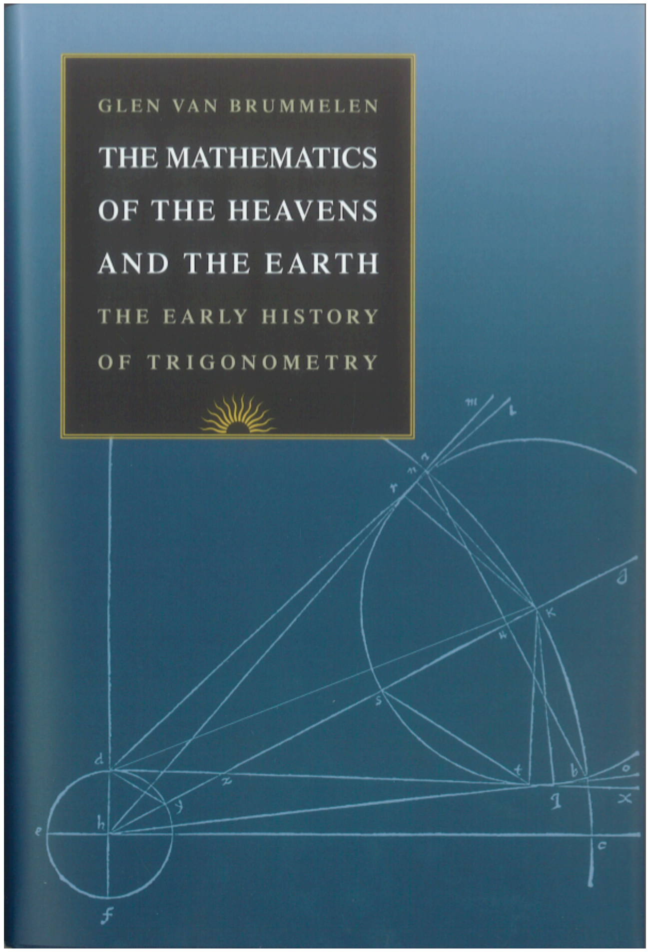
$$\frac{6}{5}+\frac{5}{24}=\frac{169}{120}$$

$$\left(\frac{169}{120}\right)^2-\left(\frac{119}{120}\right)^2=1$$

$$169^2=119^2+120^2$$



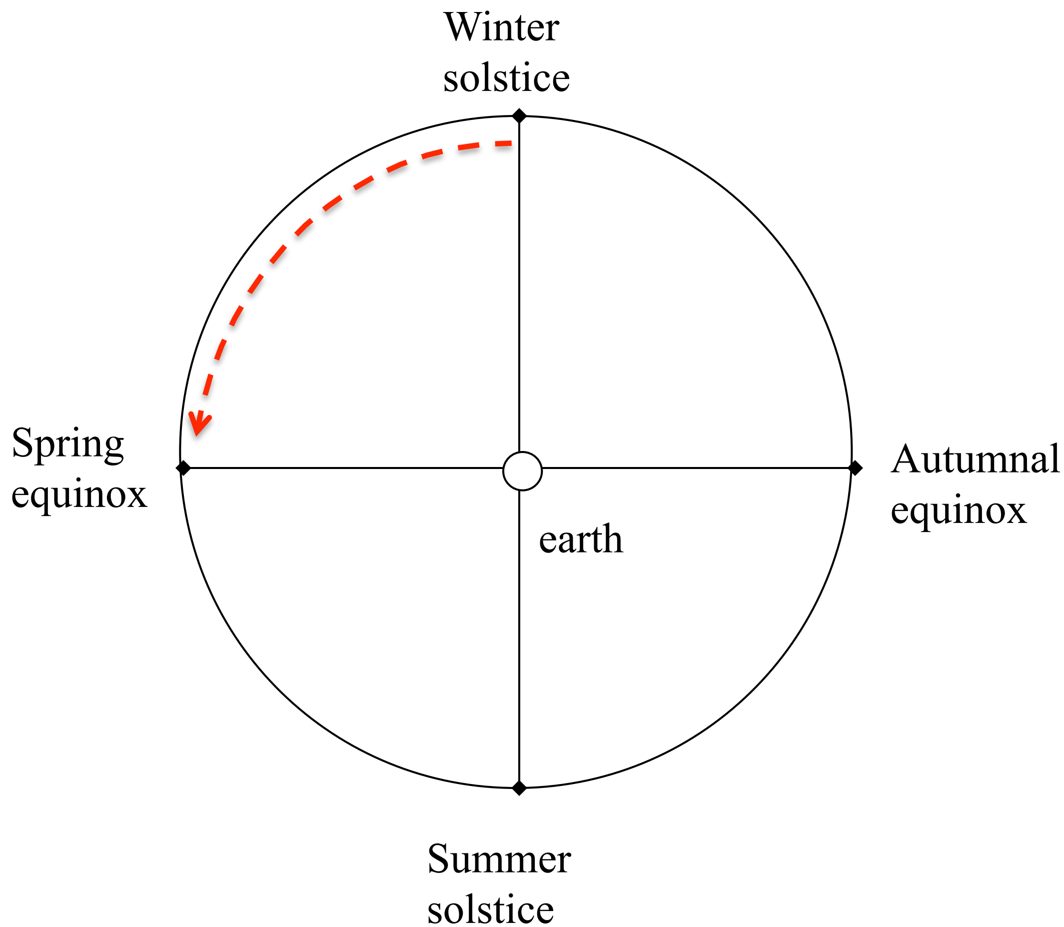
A	B	C	
119	120	169	6/5, 5/24
3367	3456	4825	32/27, 27/128
4601	4800	6649	75/64, 16/75
12709	13500	18541	125/108, 27/125
65	72	97	9/8, 2/9
319	360	481	10/9, 9/40
2291	2700	3541	27/25, 25/108
799	960	1249	16/15, 15/64
481	600	769	25/24, 6/25
4961	6480	8161	81/80, 20/81
45	60	75	1, 1/4
1679	2400	2929	24/25, 25/96
161	240	289	15/16, 4/15
1771	2700	3229	25/27, 27/100
28	45	53	9/10, 5/18

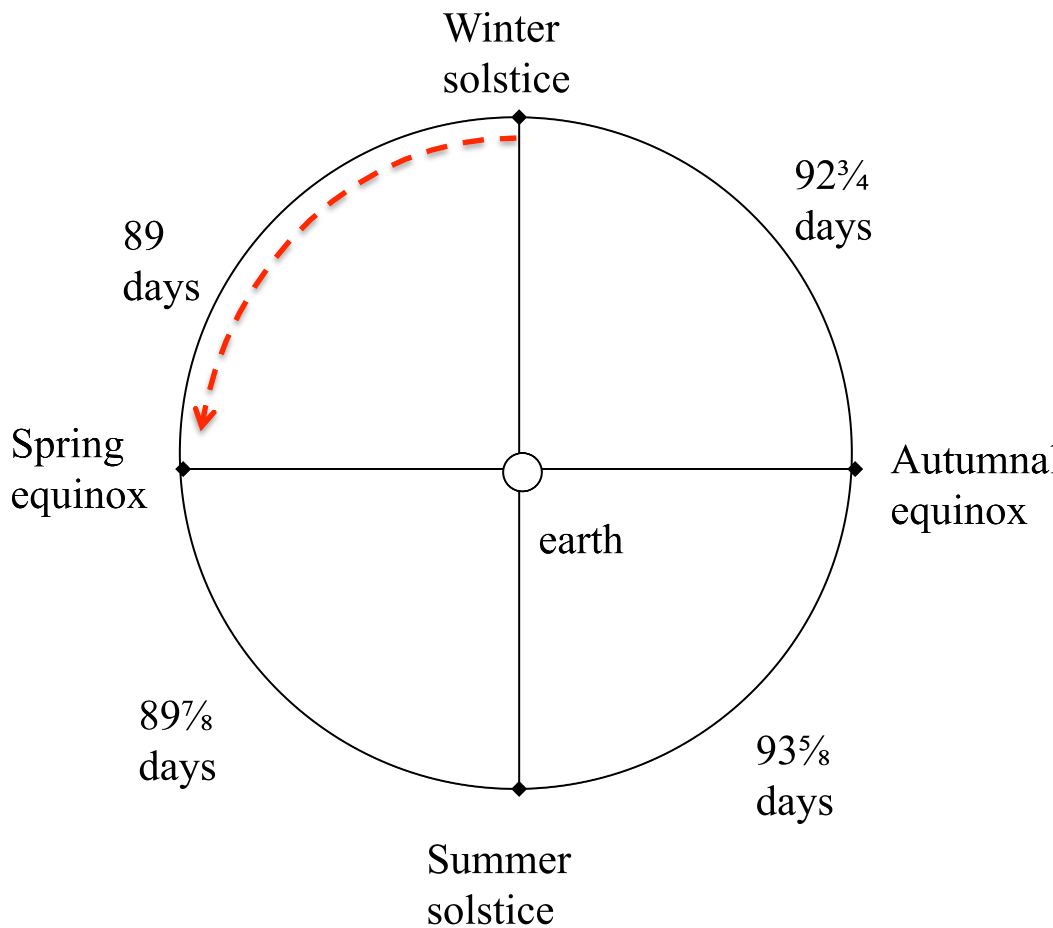


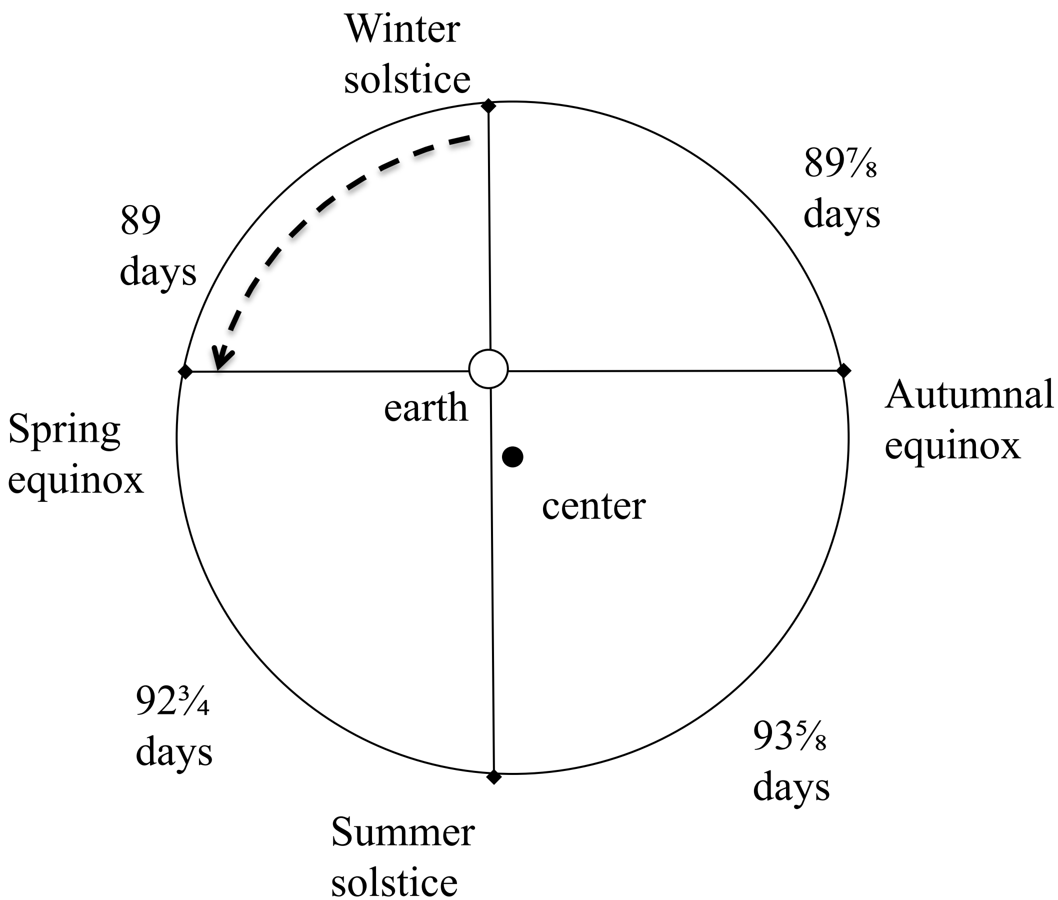
Hipparchus of Rhodes

Circa 190–120 BC

Princeton University Press, 2009


Winter
solstice
Spring
equinox
Autumnal
equinox
earth
Summer
solstice

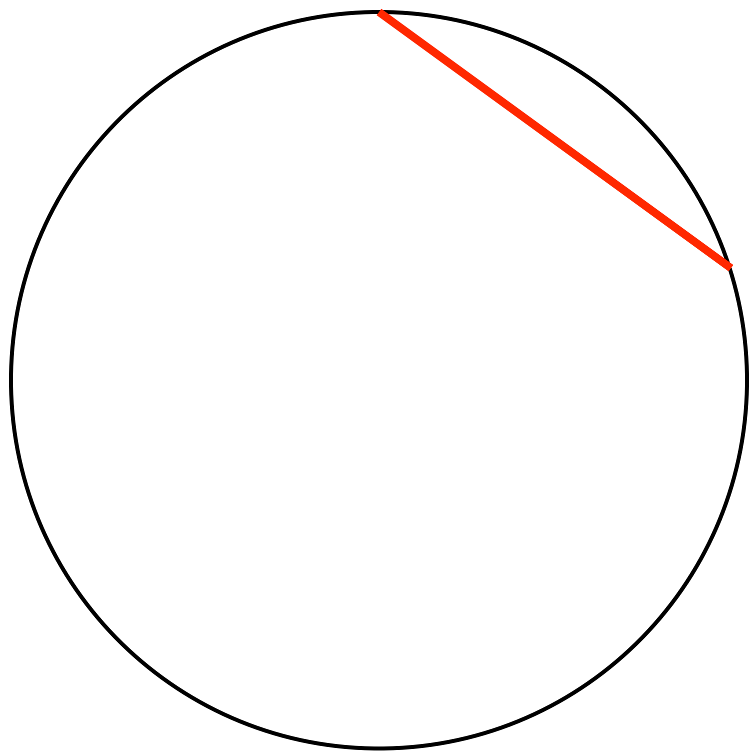

Winter solstice
92¾ days
89 days
Spring equinox
Autumnal equinox
earth
89⅞ days
93⅝ days
Summer solstice



Winter solstice
89⅞ days
89 days
Spring equinox
earth
Autumnal equinox
center
92¾ days
93⅝ days
Summer solstice

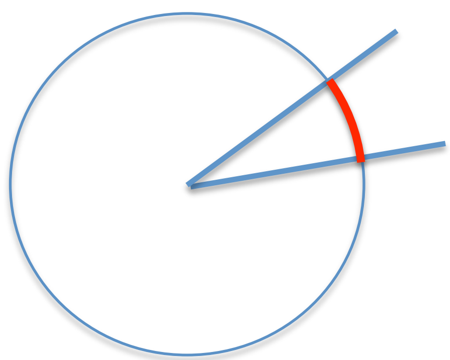
Basic problem of astronomy:

Given the arc of a circle, find the length of the chord that subtends this arc.



To measure an angle made by two line segments, draw a circle with center at their intersection.

The angle is measured by the distance along the arc of the circle from one line segment to the other.

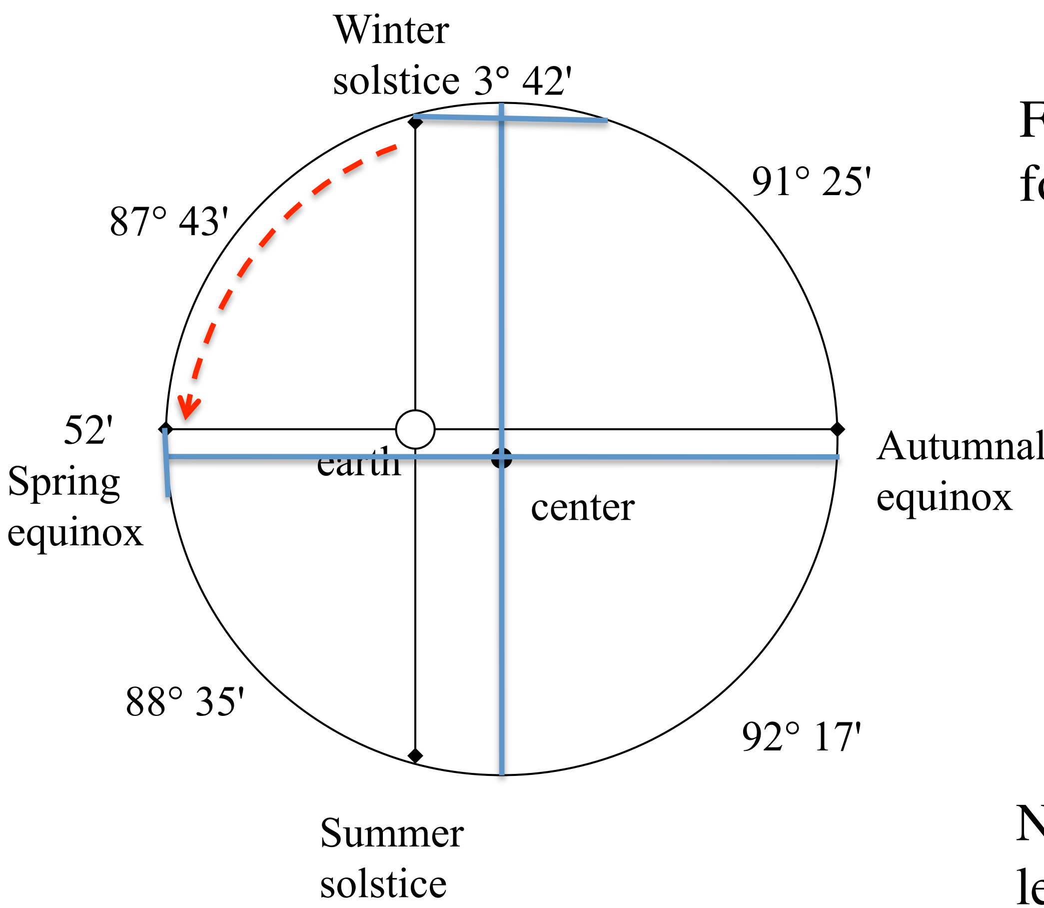


The length of the arc can be represented by a fraction of the full circumference: $43° = 43/360$ of the full circumference.

If the radius of the circle is specified, the length of the arc can also be measured in the units in which the radius is measured:

Radius = 3438, Circumference = $60 \times 360 = 21600$

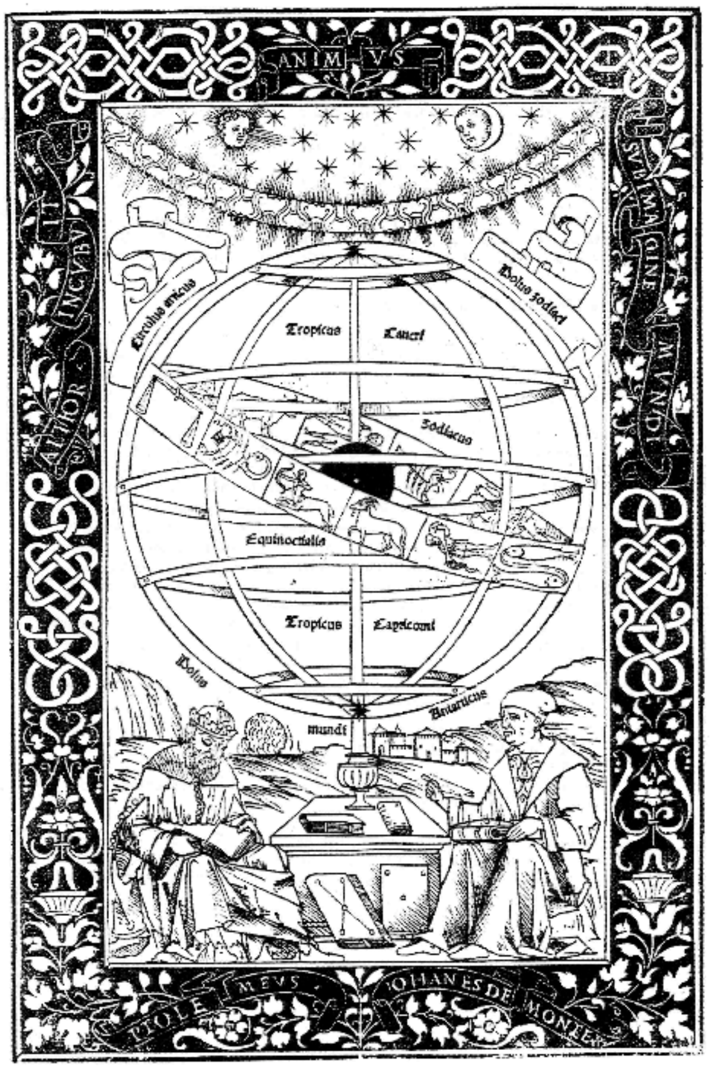
Radius = 1, Circumference = 2π



Find the chord lengths for 3° 42' and 52'.

The distance from the earth to the center of the sun's orbit is found by taking half of each chord length and using the Pythagorean theorem.

Note that the chord lengths depend on the radius.



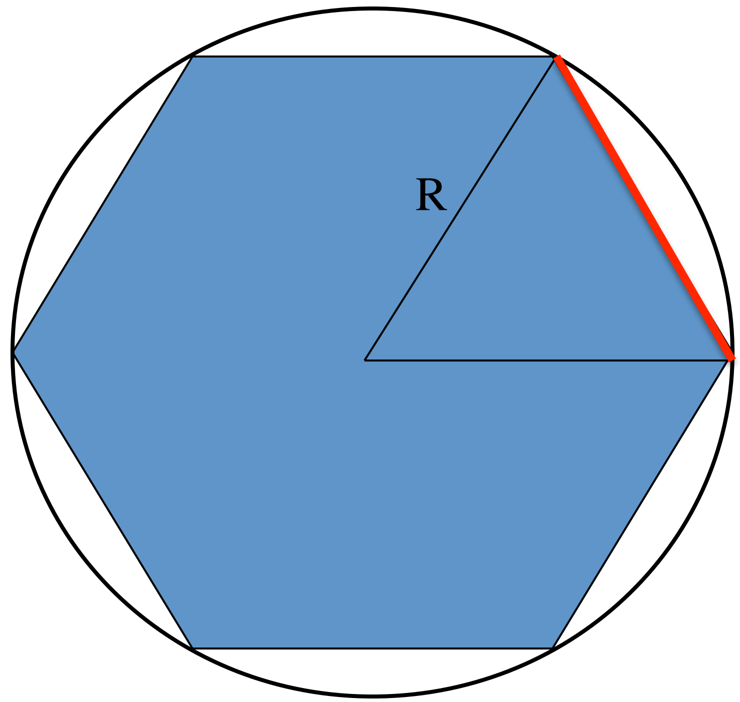
Frontispiece from Ptolemy's *Almagest*

Peurbach and Regiomantus edition of 1496

Ptolemy of Alexandria

Circa 85–165 CE

- •Constructed table of chords in increments of ½° and provided for linear interpolation in increments of ½ minute.
- •Values of chord accurate to 1 part in $60^3 = 216{,}000$ (approximately 7-digit accuracy).

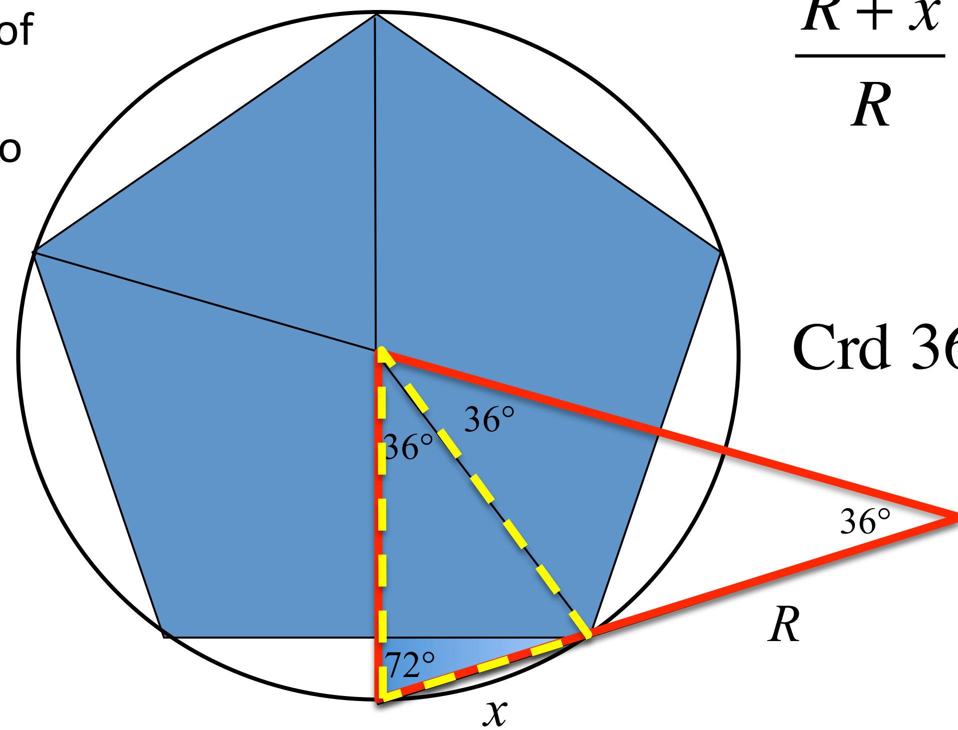


arc = 60°

chord = R

Euclid’s *Elements*
Book 13, Proposition 9

The ratio of the radius of a circle to the side of the regular inscribed decahedron is equal to the golden ratio (“mean and extreme proportion”).

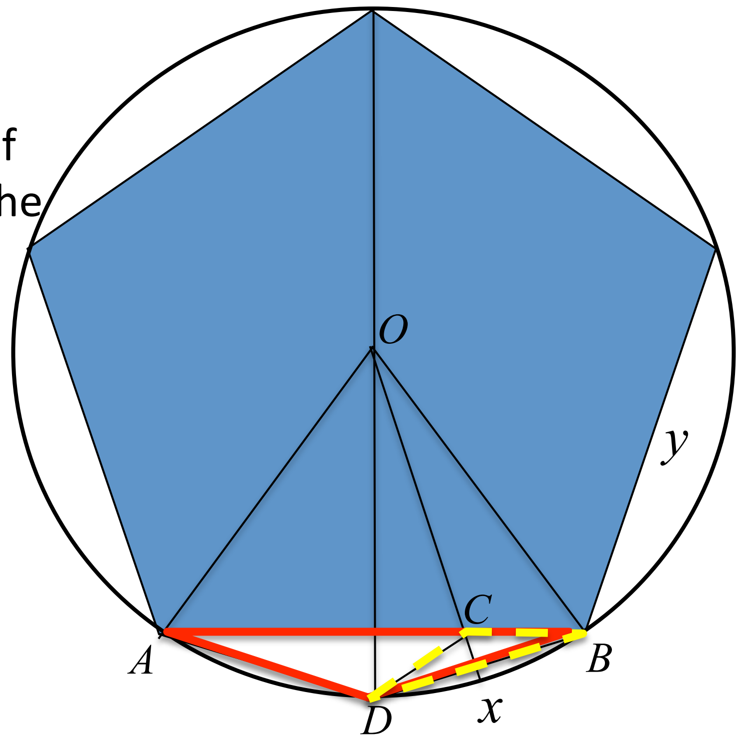


$$\frac{R+x}{R} = \frac{R}{x}$$

$$\text{Crd } 36^{\text{O}} = \frac{\sqrt{5}-1}{2} R$$

Euclid's *Elements*
Book 13, Proposition 10

The square of the side of the regular inscribed decahedron plus the square of the radius of the circle is equal to the square of the side of the regular inscribed pentagon.



$$\frac{R}{y} = \frac{\overline{AC}}{R} \Rightarrow R^2 = y \cdot \overline{AC}$$

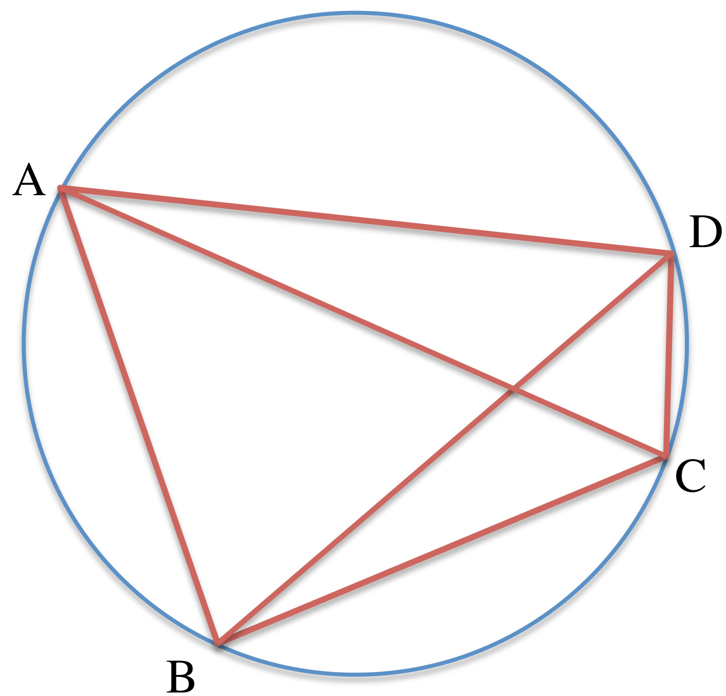
$$\frac{x}{\overline{BC}} = \frac{y}{x} \Rightarrow x^2 = y \cdot \overline{BC}$$

$$R^2 + x^2 = y^2$$

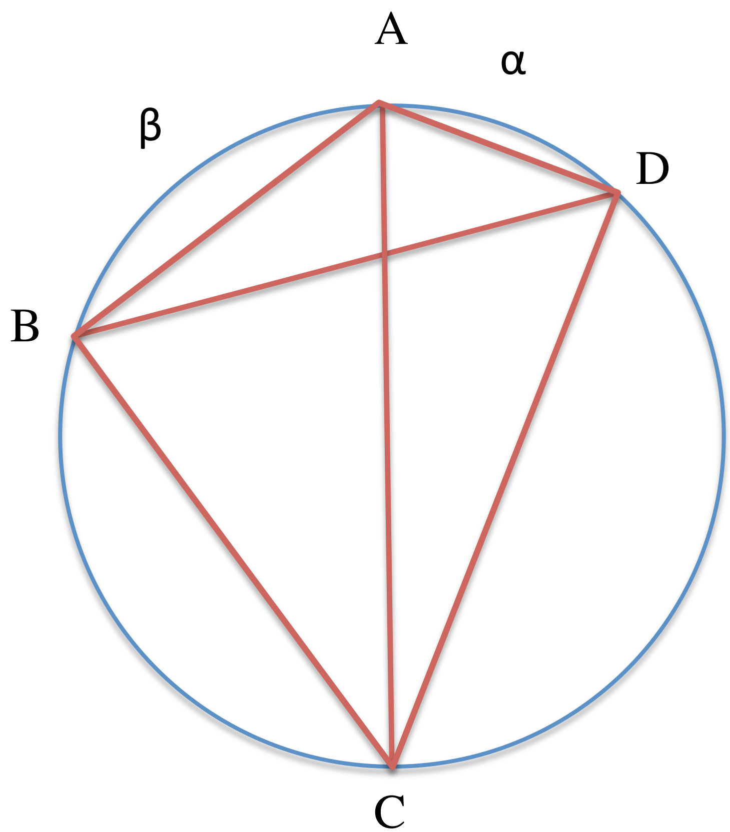
$$\text{Chord } 72^{\text{O}} = \sqrt{\frac{5+\sqrt{5}}{2}} R$$

Ptolemy's Lemma: Given any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.

$$\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC}$$



Ptolemy's Lemma: Product of the diagonals equals the sum of the products of the opposite sides.

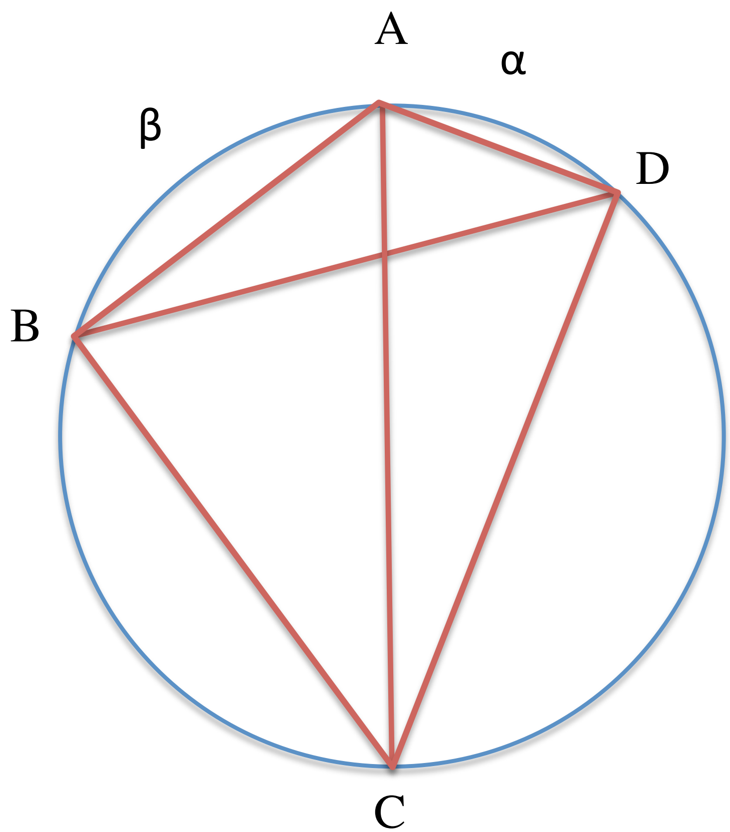


$$AC = 2R$$

$$\angle ADC = \angle ABC = 90^0$$

$$2R \cdot BD = AD \cdot BC + AB \cdot CD$$

Ptolemy's Lemma: Product of the diagonals equals the sum of the products of the opposite sides.



$$AC = 2R$$

$$\angle ADC = \angle ABC = 90^0$$

$$2R \cdot BD = AD \cdot BC + AB \cdot CD$$

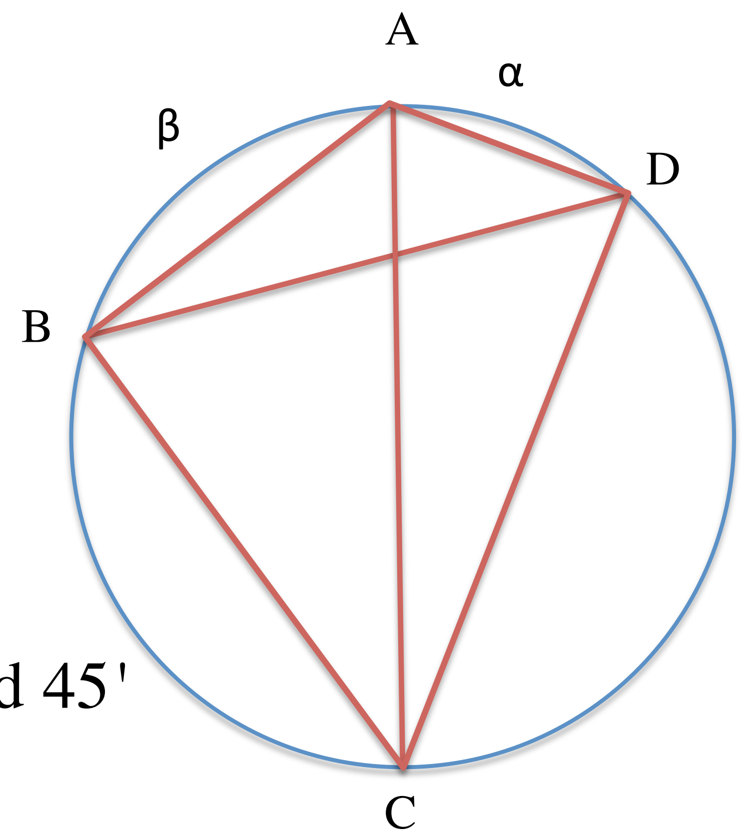
$$2R \cdot \mathrm{Crd}(\alpha + \beta)$$
$$= \mathrm{Crd}\,\alpha \cdot \mathrm{Crd}(180^0 - \beta)$$
$$+ \mathrm{Crd}\,\beta \cdot \mathrm{Crd}(180^0 - \alpha)$$

$$\text{Crd } 60^0 \text{ and Crd } 72^0 \Rightarrow \text{Crd } 12^0$$

If $\alpha < \beta < 90^0$, then

$$\frac{\text{Crd}\ \beta}{\text{Crd}\ \alpha} < \frac{\beta}{\alpha}$$

$$\frac{1}{\beta}\text{Crd}\ \beta < \frac{1}{\alpha}\text{Crd}\ \alpha$$



$$\frac{2}{3}\text{Crd}\ 1^0\ 30' < \text{Crd}\ 1^0 < \frac{4}{3}\text{Crd}\ 45'$$

$\Rightarrow \quad \text{Crd}\ 1^0$

$\Rightarrow \quad \text{Crd}\ 30'$

Kushan Empire

1st–3rd centuries CE

Arrived from Central Asia, a successor to the Seleucid Empire

Imported Greek astronomical texts and translated them into Sanskrit

Surya-Siddhanta

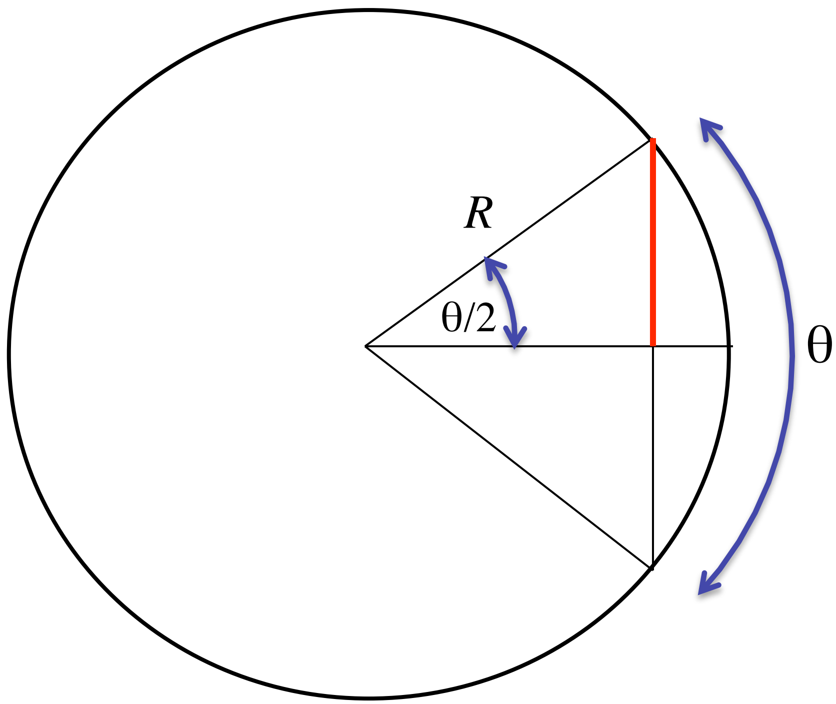
Circa 300 CE

Earliest known Indian work in trigonometry, had already made change from chords to half-chords

Ardha-jya = half bowstring

Became *jya* or *jiva*

Chord θ = Crd θ = 2 Sin $\theta/2$ = $2R \sin \theta/2$



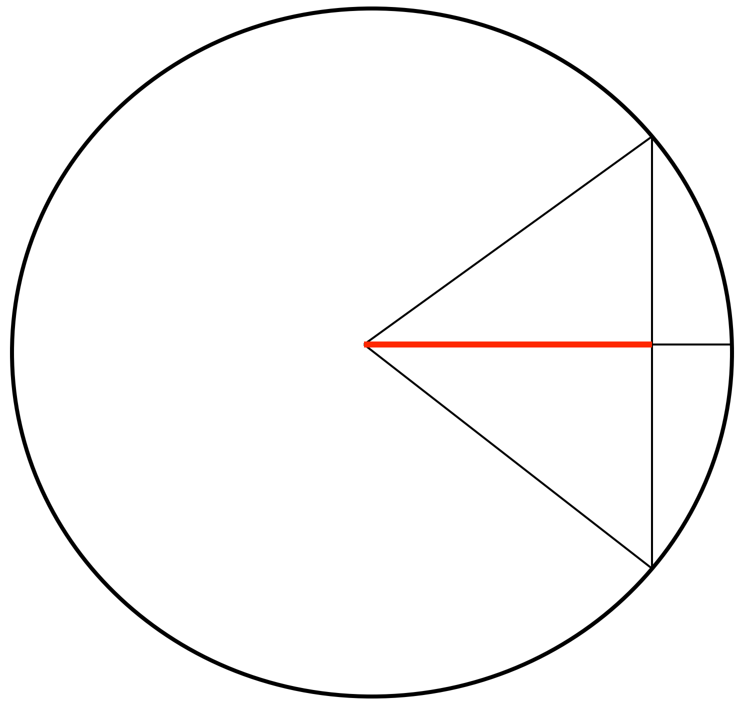
Jiya – Sanskrit

Jiba (jyb) – Arabic

jyb → jaib, fold or bay

Sinus, fold or hollow – Latin

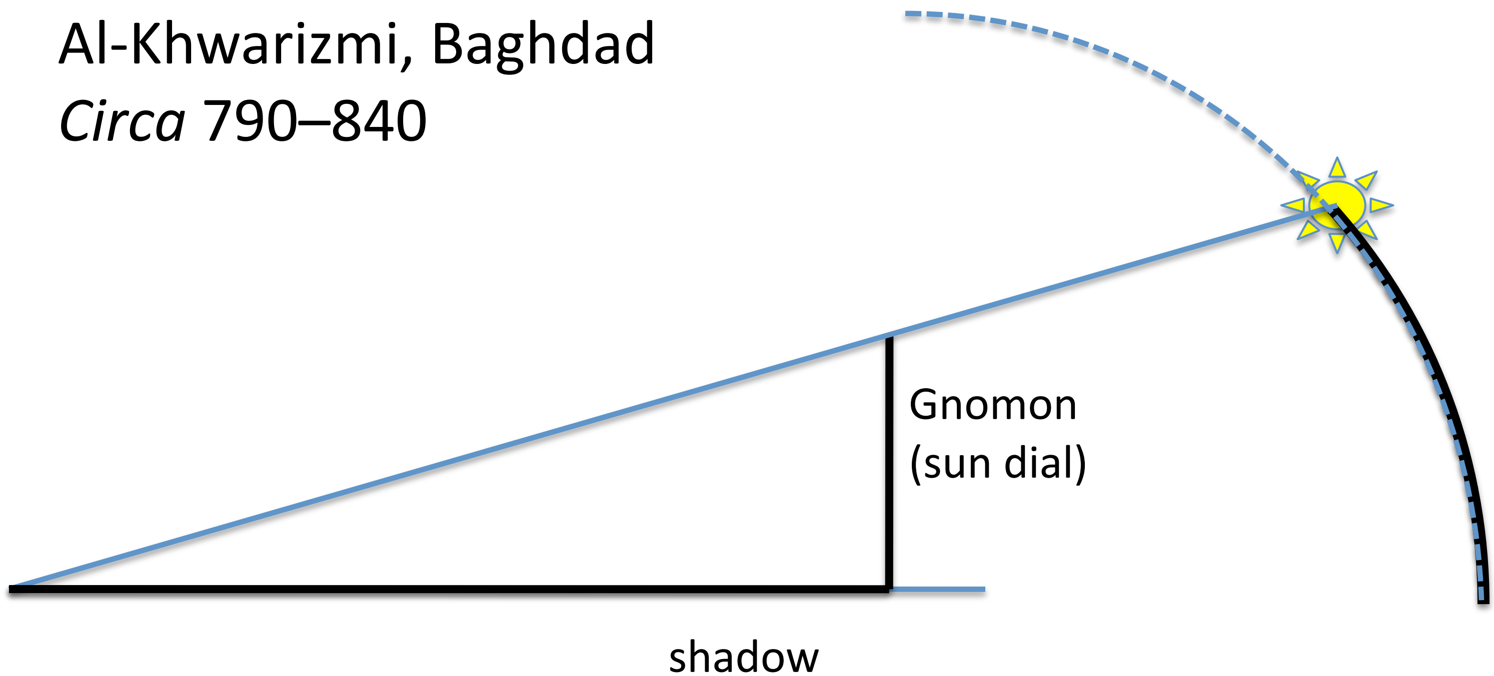
Sine – English



kotijya = cosine

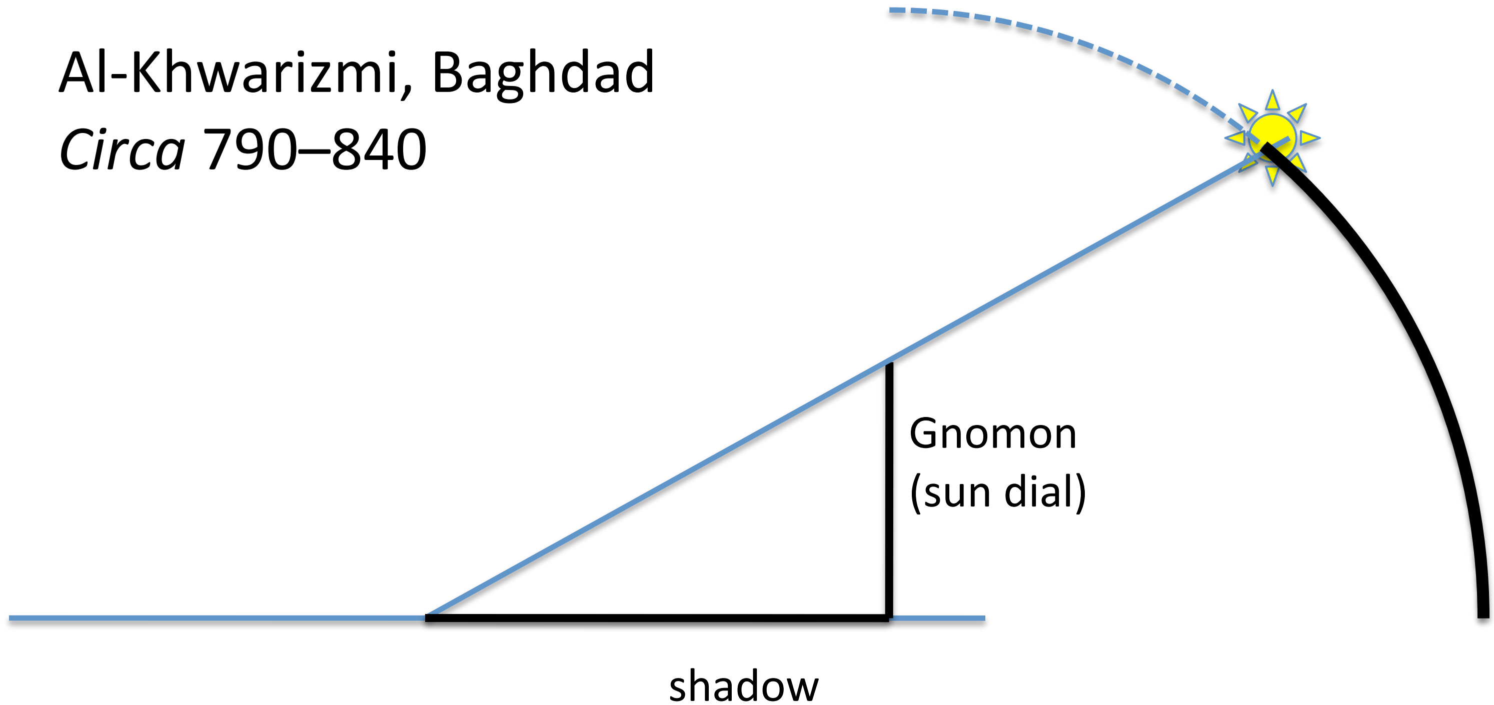
Al-Khwarizmi, Baghdad

Circa 790–840



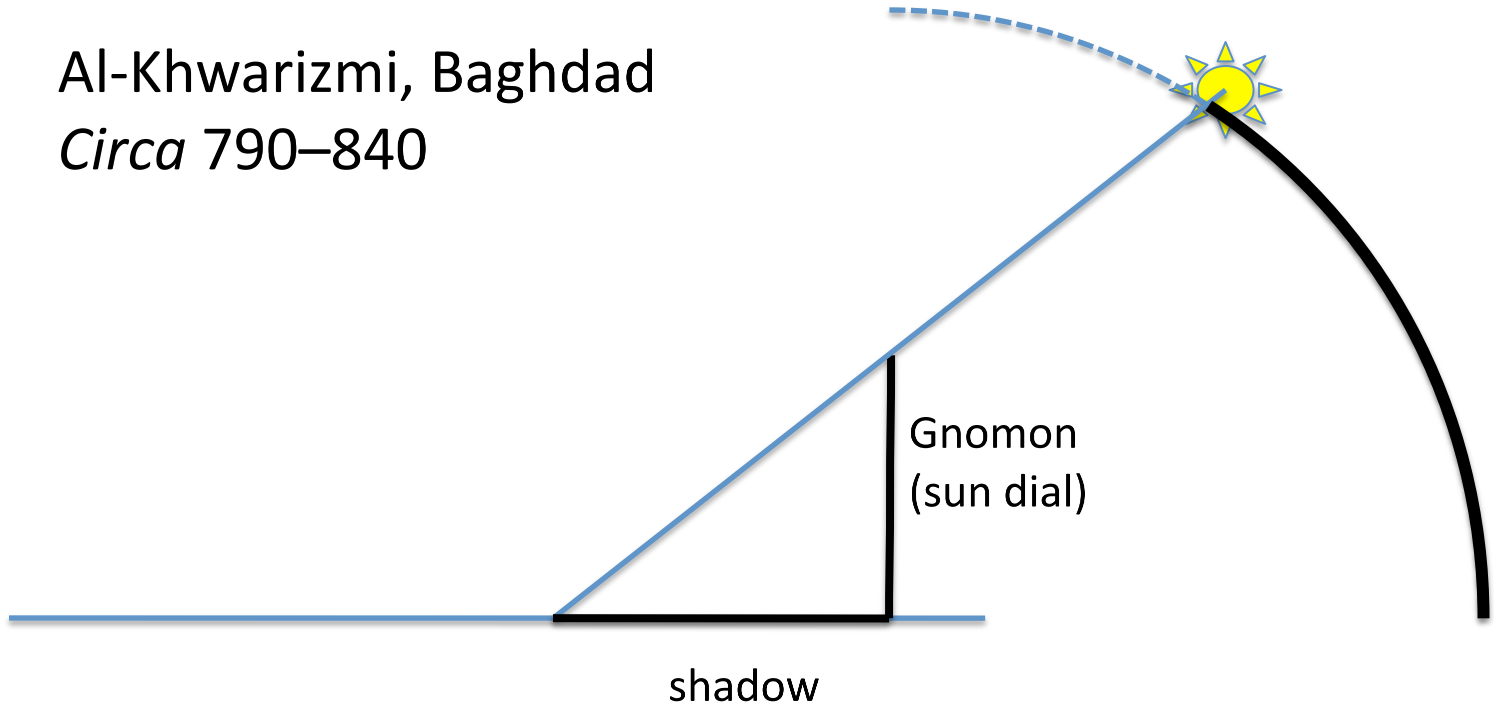
Earliest known reference to shadow length as a *function* of the sun's elevation

Al-Khwarizmi, Baghdad
Circa 790–840



Earliest known reference to shadow length as a *function* of the sun's elevation

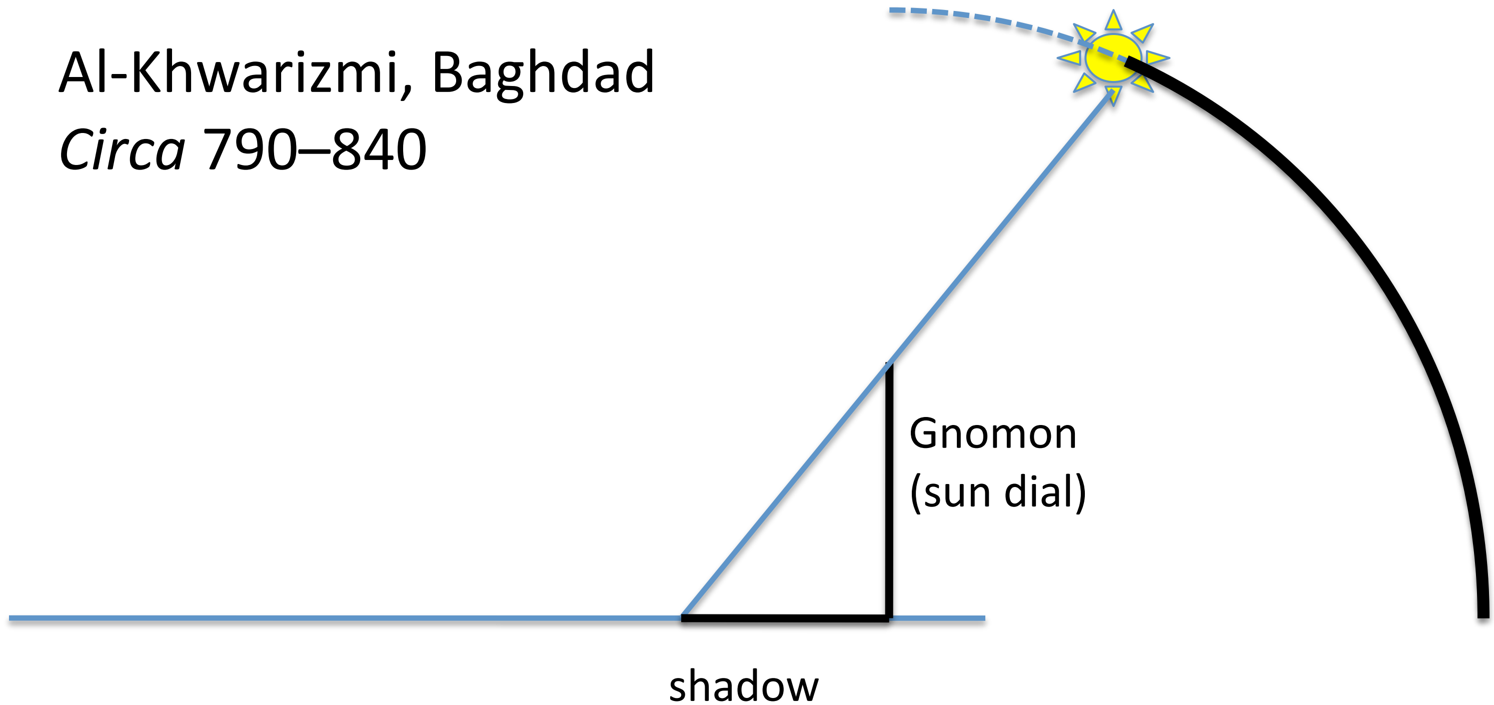
Al-Khwarizmi, Baghdad
Circa 790–840



Earliest known reference to shadow length as a *function* of the sun's elevation

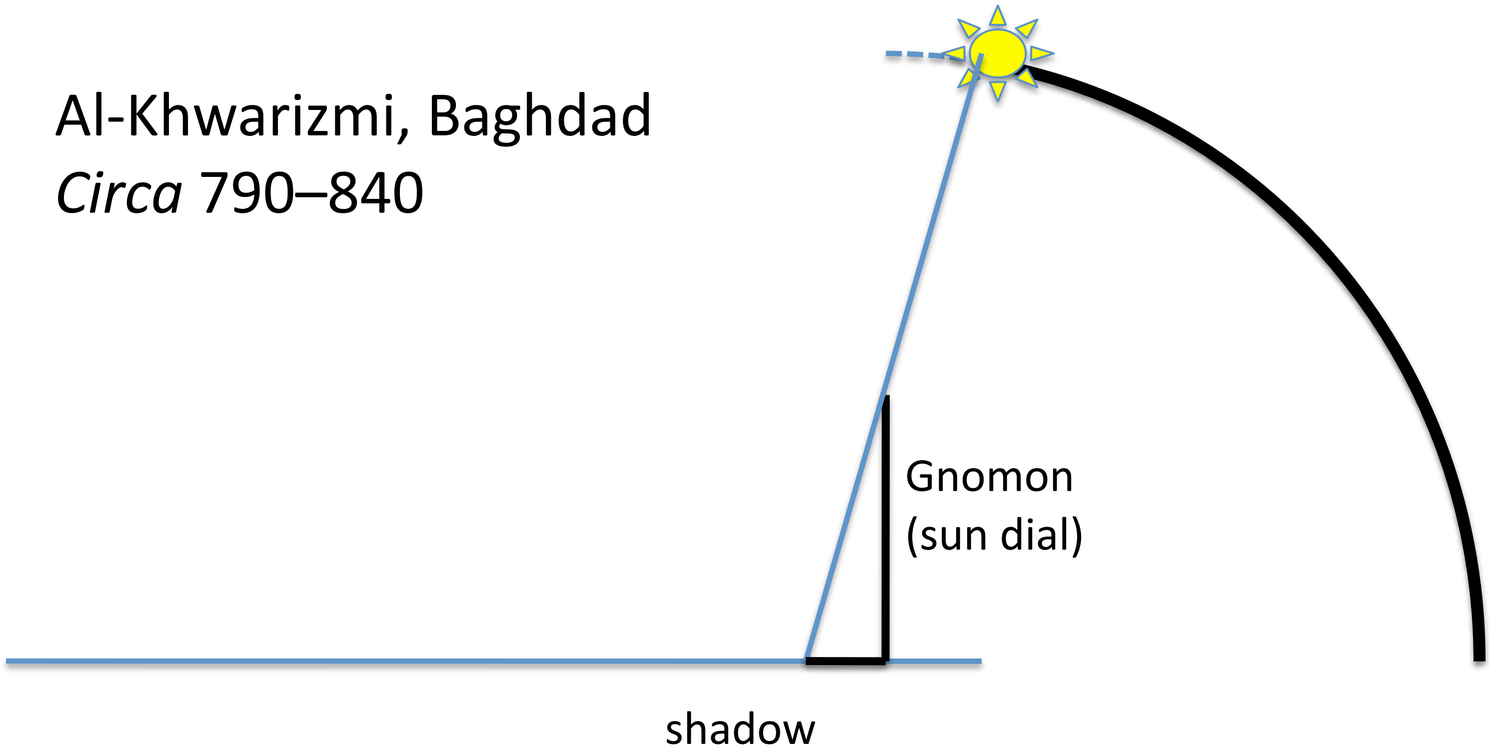
Al-Khwarizmi, Baghdad

Circa 790–840



Earliest known reference to shadow length as a *function* of the sun's elevation

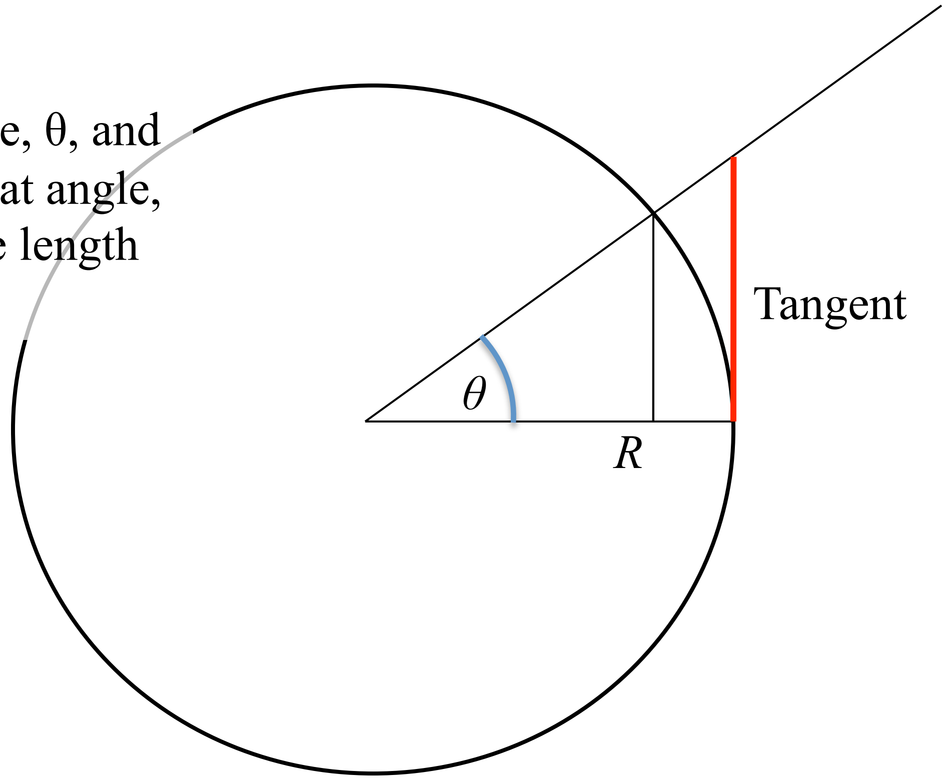
Al-Khwarizmi, Baghdad
Circa 790–840



Earliest known reference to shadow length as a *function* of the sun's elevation

Abu'l Wafa, Baghdad, 940–998

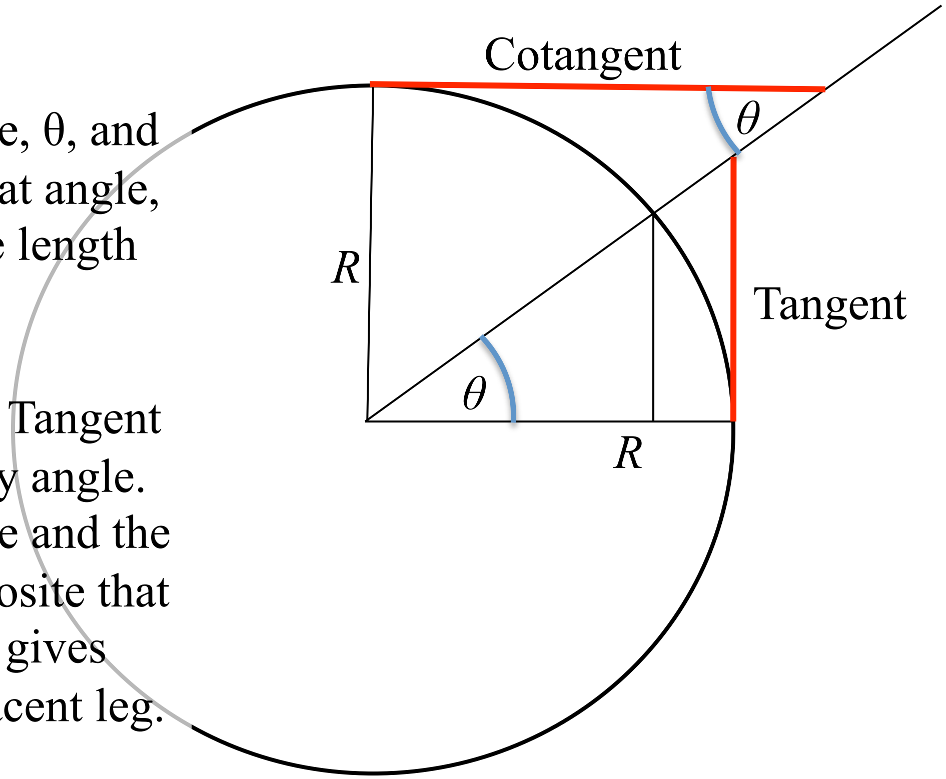
If you know the angle, θ, and the leg adjacent to that angle, the Tangent gives the length of the opposite leg.



Abu'l Wafa, Baghdad, 940–998

If you know the angle, θ, and the leg adjacent to that angle, the Tangent gives the length of the opposite leg.

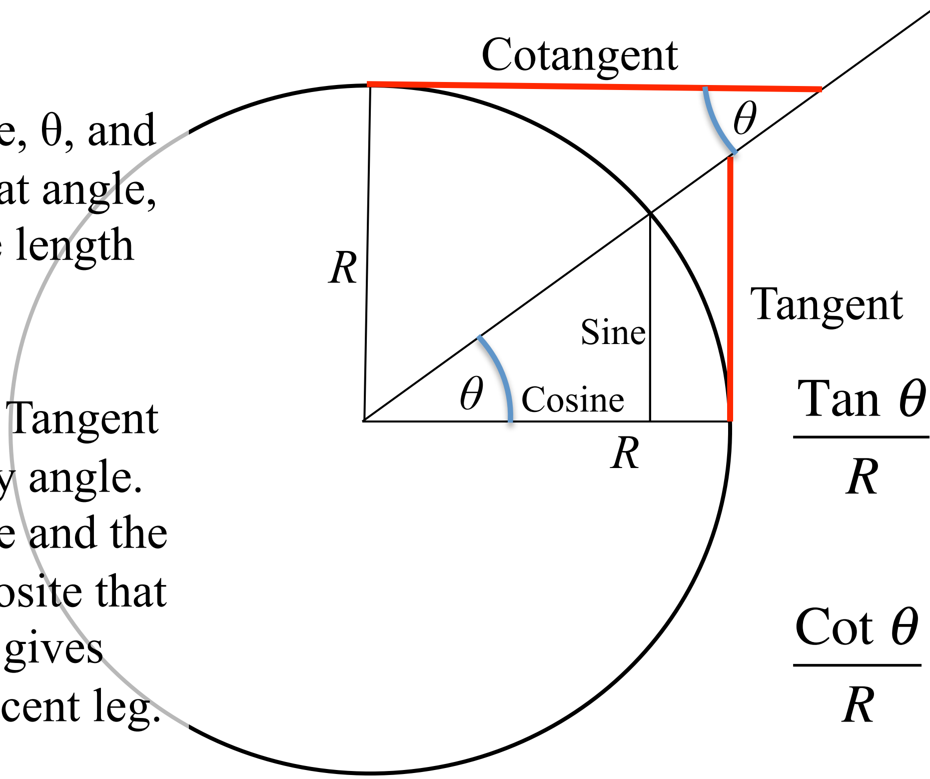
The Cotangent is the Tangent of the complementary angle. If you know the angle and the length of the leg opposite that angle, the Cotangent gives the length of the adjacent leg.



Abu'l Wafa, Baghdad, 940–998

If you know the angle, θ, and the leg adjacent to that angle, the Tangent gives the length of the opposite leg.

The Cotangent is the Tangent of the complementary angle. If you know the angle and the length of the leg opposite that angle, the Cotangent gives the length of the adjacent leg.

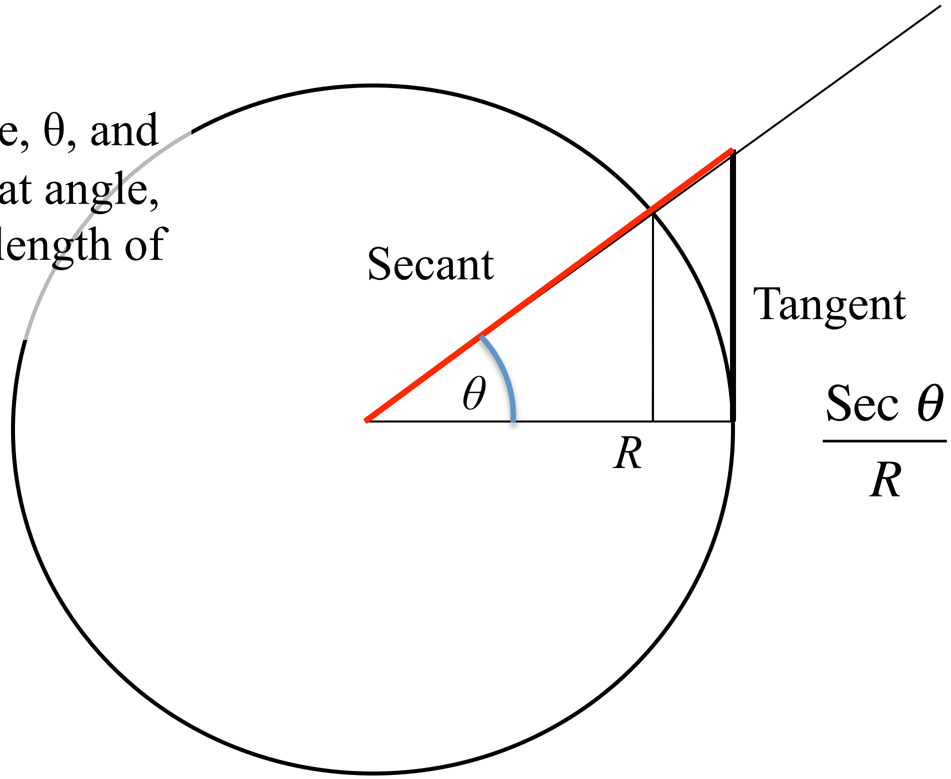


$$\frac{\text{Tan }\theta}{R} = \frac{\text{Sin }\theta}{\text{Cos }\theta}$$

$$\frac{\text{Cot }\theta}{R} = \frac{\text{Cos }\theta}{\text{Sin }\theta}$$

Abu’l Wafa, Baghdad, 940–998

If you know the angle, θ, and the leg adjacent to that angle, the Secant gives the length of the hypotenuse.

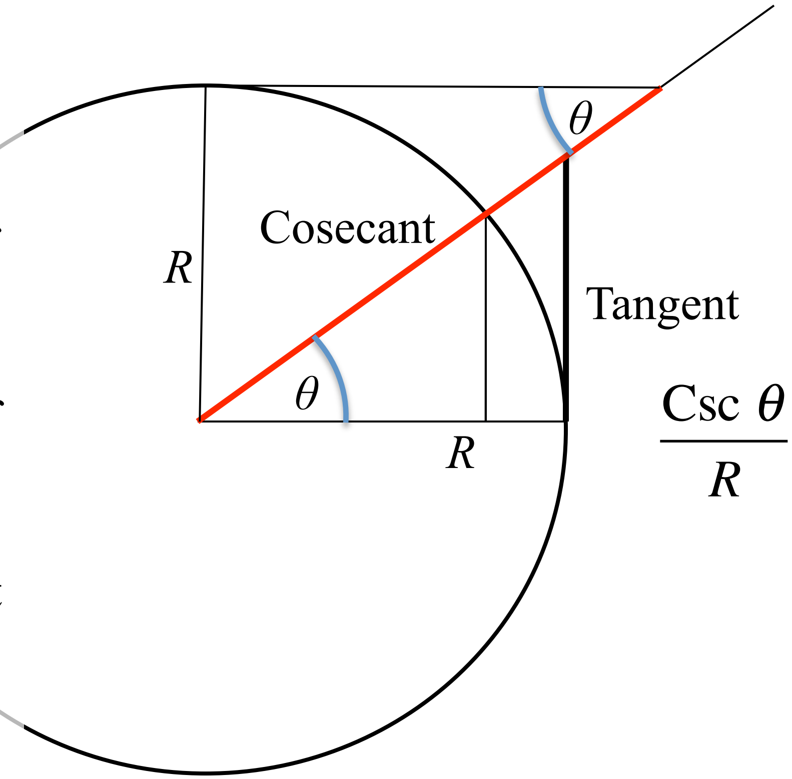


$$\frac{\text{Sec}\ \theta}{R} = \frac{R}{\text{Cos}\ \theta}$$

Abu'l Wafa, Baghdad, 940–998

If you know the angle, θ, and the leg adjacent to that angle, the Secant gives the length of the hypotenuse.

The Cosecant is the Secant of the complementary angle. If you know the angle and the length of the leg opposite that angle, the Cosecant gives the length of the hypotenuse.



$$\frac{\text{Csc } \theta}{R} = \frac{R}{\text{Sin } \theta}$$

Bartholomeo Pitiscus, 1561–1613, Grunberg in Silesia

1595 published *Trigonometria*, coining the term “trigonometry.”

According to Victor Katz, this was the first “text explicitly involving the solving of a real plane triangle on earth.”

Leonhard Euler, 1707–1783

Standardizes the radius of the circle that defines the angle to $R = 1$.

If we want to apply the tools of calculus, we need to measure arc length and line length in the same units, thus the circumference of the full circle is 2π.

Euler did *not* use *radians*. For him, trigonometric functions expressed the lengths of lines in terms of the length of an arc of a circle of radius 1.

1840–1890

During this half-century, trigonometry textbooks shift from trigonometric functions as lengths of lines determined by arc lengths to ratios of sides of right triangles determined by angles.

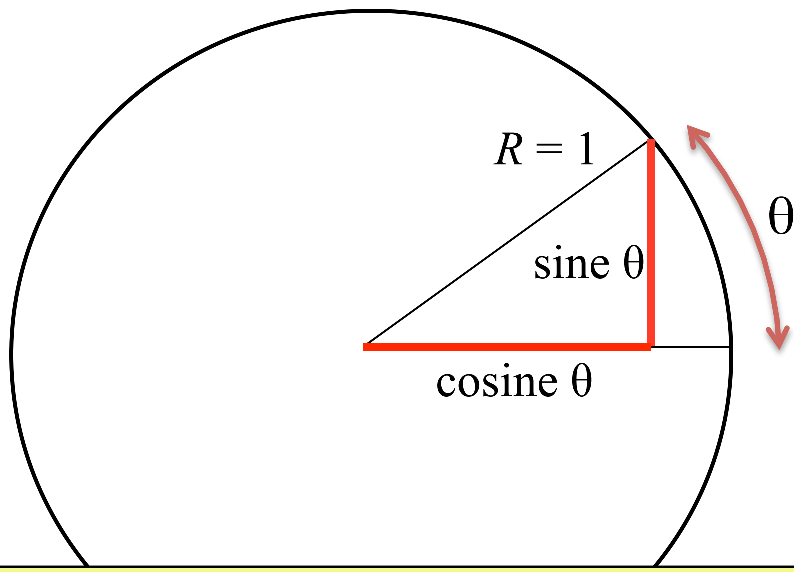
For the first time, this necessitates a name for the angle unit used in calculus: *radian*. Coined independently in the 1870s by Thomas Muir and William Thomson (Lord Kelvin).

Lessons:

1. There is a lot of good and interesting circle geometry that sits behind trigonometry.
2. Ratios are intrinsically hard. It is much easier and more intuitive to think of trigonometric functions as lengths of lines.
3. Rather than trying to deal with radians as the numerator in a fraction whose denominator is 2π, think of them as the distance traveled along the circumference of a circle of radius 1.

Lessons:

4. The circle definition of the sine and cosine is much closer to the way these functions have been defined and used throughout history than is soh-cah-toa.



This presentation is available at
www.macalester.edu/~bressoud/talks