Natural landmark-based autonomous vehicle navigation

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Abstract

This article describes a natural landmark navigation algorithm for autonomous vehicles operating in relatively unstructured environments. The algorithm employs points of maximum curvature, extracted from laser scan data, as point landmarks in an extended Kalman filter. A curvature scale space algorithm is developed to locate points of maximum curvature. The location of these points is invariant to view-point and sensor resolution. They can therefore be used as stable and reliable landmarks in a localization algorithm. This information is then fused with odometric information to provide localization information for an outdoor vehicle. The method described is invariant to the size and orientation of the range images under consideration (with respect to rotation and translation), is robust to noise, and can reliably detect and localize naturally occurring landmarks in the operating environment. The method developed is generally applicable to a range of unstructured environments and may be used with different types of sensors. The method is demonstrated as part of a navigation system for an outdoor vehicle in an unmodified operating domain.

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1. Introduction

Localization, the process of determining the position and orientation (pose) of a platform in its operating environment, is a fundamental ability required of an autonomous vehicle. Landmark-based navigation methods are one of the most common approaches to localization. Landmarks in the context of this article are distinct features that a vehicle can recognize reliably from its sensor observations. Landmark-based methods for navigation vary significantly based on the sensing used to identify the landmarks, the type of landmarks and the number of landmarks needed. Landmarks can be natural (like roof-edges or any structure that is already present) or artificial (reflective stripes, active beacons, transmitters, etc.) having a fixed and known position. Several landmark-based (feature-based) methods for autonomous navigation can be found in the literature [6,9,20]. Natural landmark-based methods for autonomous localization have become increasingly popular as they do not require infrastructure or other external information.
to be provided to the vehicle for operation. Such methods require that natural landmarks can be robustly detected in sensor data, that the localization algorithm can reliably associate landmarks from one sensor observation to the next, and that the method can overcome problems of occlusion, ambiguity and noise inherent in observation data.

In indoor environments, features such as walls (line-segments), corners (diffuse points) or doorways (range discontinuities) are used as landmarks. In outdoor environments however, similar simple features are sparse and infrequently observed. In unstructured or natural outdoor environments a more general notion of a landmark or navigation feature is required. In such domains, the landmark detection and data association algorithms should be robust to false alarms so as to prevent catastrophic failure. Thus the effectiveness of outdoor feature-based navigation approaches depends on the reliability of feature extraction, availability of sufficient number of features and the complexity in feature association procedures. Given a range scan of an environment, one natural measure of feature or landmark significance is the local curvature in this range data. The range data in this article are obtained from a scanning laser rangefinder mounted on a moving vehicle that scans through a 180° arc to give the distance to objects in the environment at angular intervals of 0.5° with a maximum range of 50 m. Rapidly changing range data indicate either a significant physical discontinuity or a prominent geoform. The use of local curvature as an interest measure is well known in the computer vision community [2]. Further, and most important for navigation, curvature extrema are well known to be viewpoint invariant. Practically, this means that points of maximum curvature can be used as robust point landmarks in localization.

In this article, a general purpose, multiresolutional qualitative description called scale space [21] is utilized to analyze the range images from the laser rangefinder at varying levels of detail. In scale space methods, a qualitative description of the signal is obtained by studying the behavior of the extrema over the continuum of scales. Also the extrema that survive over larger smoothing extents are considered to be more significant than others. The Curvature Scale Space (CSS) algorithm [15] is developed to identify, extract and localize landmarks characterized by points of maximum curvature in laser scans. The proposed algorithm extracts dominant points (which correspond to curvature extrema) that are:

- invariant to the size and orientation of the signal under consideration,
- robust to noise, and
- reliably detected and localized.

By employing the CSS algorithm, dominant points are extracted based on the persistence of these points within a scan and over consecutive scans. Such persistent dominant points enable the construction of a natural landmark map. The landmark information is then fused with odometric information in an extended Kalman filter (EKF) to enable localization of a utility vehicle operating in an unmodified outdoor environment.

The article proceeds as follows: Section 2 describes the curvature scale space algorithm for extracting natural landmarks from scanning laser rangefinder images. Section 3 demonstrates outdoor localization of an utility vehicle using an a priori natural landmark map constructed from extracted landmarks resulting from the proposed CSS algorithm. Section 4 summarizes the contributions and briefly outlines the limitations of the mapping and localization algorithms proposed in this article.

2. Natural landmark extraction via curvature scale space

Dominant points characterized by curvature extrema are useful descriptive primitives and are chosen as the candidates for features in this article. The motivation behind this selection is many fold. There is strong psychophysical, physiological as well as mathematical and computational support in favor of using curvature. Curvature extrema are also important descriptors used by the human visual system to segment contours into meaningful parts [5,10,11]. Several curvature-based corner detection algorithms have been reported in the literature. Most of these approaches use image features that are considered at a single level of resolution or scale. Single scale methods suffer from disadvantages of finding many unimportant details and at the same time missing vital information. Such methods
usually employ a parameter to determine the degree of smoothing to be imparted to the curve and are faced with the difficult problem of deciding what constitutes the best scale [7,18]. The value of the parameter is usually selected on the basis of trial and error with no a priori information on the feature size of the curve. If the curve consists of features of multiple size where large and small curvature regions coexist, then the selection of scale parameter becomes tedious. Teh and Chin [19] propose an algorithm for detecting dominant points in a digital curve. A region of support is first determined and then based on curvature these points are detected on a non-maxima suppression procedure. The resulting algorithm is known to produce many unwanted dominant points in the presence of noise. Polygonal approximation methods have also been used for the same purpose [17]. These methods suffer from the disadvantage of not being able to reliably extract dominant points.

2.1. Curvature scale space filtering

Scale space filtering is a qualitative signal description method that deals with the problem of scale by treating the size of the smoothing kernel as a continuous parameter. As the scale parameter is varied, the extrema or zero-crossings in the smoothed signal and its derivatives move continuously. The main idea of multiscale representation is to successively suppress fine scale information and thus to progressively remove the high frequency information which results in the signal becoming gradually smoother. The common way of constructing a scale space is by convolution with a Gaussian kernel. The essence of the Gaussian kernel is that a signal at a coarser (higher) scale level should contain less structure than at a finer (lower) level of scale.

2.1.1. Curvature scale space algorithm

The aim of the CSS algorithm is to identify dominant points in the range images obtained from a scanning laser rangefinder. The basic idea behind the algorithm is: Convolve a signal with a Gaussian kernel and impart smoothing at different levels of scale (the scale being proportional to the width of the kernel). From the resulting convolved signal, identify the dominant points that correspond to curvature extrema.

(A1) The range images are segmented according to the discontinuities present in the scanned environment. As the discontinuities on the scans give rise to unwanted and unreliable dominant point landmarks, the whole range image is not treated as one but as broken-up segments [13].

(A2) For various degrees of smoothing of the curve (the segmented range scan), it is desired to find the curvature extrema. A parametrization of the curve is necessary to compute the curvature at varying levels of detail. Such parametrization is possible by considering a path length variable along the curve and expressing the curve in terms of two functions \(x(s)\) and \(y(s)\) (see Fig. 1):

\[C = \{(x(s), y(s))\}\]

The uniqueness of the Gaussian kernel for scale space filtering is well known. The interested reader is referred to [3] for additional details.
with $s$ being a linear function of the path length ranging over the closed interval $[0, 1]$. The simplest way to parametrize a curve is to use a path length parameter, $s$, and let it vary equidistantly. That is, let $s_i = i$ or $s_i = i/m$, where $i = 0, 1, \ldots, m$ and $s_m$ when $i = m$ being the total path length. It can be readily seen that this form of parametrization is not preferable if the distances between consecutive points vary much along the curve and can lead to unpleasant and unexpected features [4]. A better and efficient way to do the parametrization is to allow the values of the parameter $s$ to depend on the distances between data points, i.e. path lengths in the curve. The following procedure is adopted:

Let
\[
d_0 \triangleq 0, \quad d_i \triangleq \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, \quad i = 0, 1, \ldots, m, \quad D_p \triangleq \sum_{i=1}^{m} d_i, \quad s_0 \triangleq 0, \quad s_i \triangleq \frac{1}{D_p} \sum_{j=1}^{i} d_j, \quad i = 0, 1, \ldots, m.
\]

The parametrized curve with \{(xi, si), (yi, si); i = 0, 1, \ldots, m\} with the path length parameter $s$ ranging from $s_0 = 0$ to $s_m = 1$ is now available.

(A3) The segmented and parametrized laser scan is then individually convolved with the Gaussian kernel pertaining to a particular value of the scale ($\sigma$) to obtain a value of the curvature at that scale as described in Appendix A.

(A4) The curvature values for a set of fine-to-coarse scales, starting from the finest scale corresponding to the finest level of detail desired ($\sigma = \sigma_{\text{min}}$) to the maximum scale ($\sigma = \sigma_{\text{max}}$) corresponding to the highest level of smoothing desired, are then similarly computed using Eq. (A.2) shown in Appendix A. A curvature is declared as a dominant curvature if and only if $|\kappa_i - 1| < |\kappa_i| < |\kappa_i + 1|$ and $|\kappa_{i-1}| < |\kappa_{i-2}|$, i.e. the absolute value of the curvature of the point $i$ in question should be greater than that of two of its immediate left and right points in the neighborhood.

(A5) Finally, the points corresponding to the detected local absolute curvature extrema at every level of $\sigma$ are plotted in the $s - \sigma$ space to form the scale space map where the $s$-axis represents the path length parameter $s$ and the $y$-axis represents the width of the Gaussian kernel. A multiscale representation of the curve from $\sigma = \sigma_{\text{min}}$ to $\sigma = \sigma_{\text{max}}$ has thus been constructed by convolution with the Gaussian kernel. The scale space map of the segmented laser scan under consideration is now said to be derived.

Asada and Brady [1] note that the curvature changes that are found at multiple scales are reliably localized at the finest scale and that are found only at the finest scale and do not persist across many scales are less compelling determiners of the global shape. The reliable determination of a dominant point is subject to two assumptions:

- The identification/detection assumption is made to satisfy the fact that an absolute curvature extremum observed over different scales in the scale space image is representative of an underlying event that has persisted and is indicative of a physical dominant point/structure observed right from the finest scale $\sigma_{\text{max}}$.
- The localization assumption is made so that the dominant point is unambiguously localized at the finest scale.

The identification is thus done at the coarsest scale, $\sigma = \sigma_{\text{max}}$, and reliably localized at the finest scale, $\sigma = \sigma_{\text{max}}$. When tracking an extremum, if a choice needs to be made between two extrema at the next finer scale, the extremum with the higher curvature value is chosen.

2.2. Natural landmark extraction results

Figs. 2 and 3 show the effects of scale space smoothing. The squares (□) and the asterisk (•) represent the start and end points of the segmented scans, respectively. The dots (•) represent the laser scan points. Figs. 2(b), (d)
Fig. 2. Dominant points characterized by curvature extrema. (a) and (c) show the points of maximum curvature of a segment of the laser scan pertaining to the smoothed curvature marked in the figures to the right. (b) and (d) show the corresponding extrema of the smoothed curvature $\kappa(s, \sigma)$ (computed using Eq. (A.2)) marked by crosses ($\times$). Figs. 2(a), (c) and 3(a), (c) show the dominant points (curvature extrema for those points marked in the figures to the right) marked on the segment of the laser scan as the smoothing progresses. It should be noted here that only a few selected snapshots (pertaining to arbitrarily selected scales) are presented in the figures for illustrative purposes. The scale space phenomenon can be clearly observed. At a finer scale, there are several dominant points but as the smoothing progresses proportional to the width of the Gaussian kernel, only the most dominant points survive this smoothing.

Fig. 4 shows the scale space map for the segment in Fig. 2. The horizontal axis is the path length of the curve at $\sigma = \sigma_{\text{max}}$. The vertical axis represents the scale that determines the degree of smoothing. In these scale space maps, the structure that survives the highest smoothing (at the coarsest scale $\sigma = \sigma_{\text{max}}$) on the top most level corresponds to the corner of the laser scan in Fig. 2(a). Notice the less relevant ‘features’ either dying out or merging as the scale increases. Attenuation of the noise in the signal is realized by convolving the signal with the Gaussian kernel. Features that exist only at finer scales and disappear at higher scales do not correspond to stable features and thus do not qualify as reliable candidates for subsequent detection and tracking during vehicle localization stages. All the dominant points that are above a selected scale $\sigma_k$, where $\sigma_{\text{min}} < \sigma_k < \sigma_{\text{max}}$ are tracked. Based on the persistence of these dominant points over consecutive scans, a particular dominant point is rejected or selected as a candidate for vehicle localization. Note that the persistence of a dominant feature within a scan is different from the persistence...
across consecutive scans. Both of these conditions need to be satisfied by a feature to qualify as a reliable landmark for localization.

3. Natural landmark-based localization

3.1. The trial vehicle and experimental setup

Fig. 5(a) shows the utility vehicle used in the experimental trials. The vehicle is equipped with a commercial time-of-flight laser mounted on the front of the vehicle, a Real Time Kinematic (RTK) Differential Global Positioning System (DGPS) system, and steering and wheel rotation transducers. In these trials, the DGPS provided accurate ground truth position estimates which were used to compare the proposed localization algorithm. The DGPS antenna is mounted directly above the laser as shown in the photograph. It should be noted that DGPS information is not used in the proposed localization algorithm.

Fig. 5(b) shows the test area within the campus of the University of Sydney. The test area was chosen to allow unobstructed DGPS information and to provide a sufficiently unstructured environment for evaluation of the algorithm. Other vehicles and people frequented the area while the trial was conducted. The vehicle speed was measured by a wheel encoder and the steering maneuvers of the vehicle were read by a Linear Variable Differential Transformer (LVDT).
3.2. Utility vehicle kinematic model

A model that sufficiently captures the vehicle motion is key to efficient use of sensor data and is central to successful autonomous navigation. The model geometry of the utility vehicle is shown in Fig. 6. The vehicle states
include the Cartesian position \((x_v, y_v)\) centered mid-way between the rear wheels and the orientation, \(\phi_v\) of the vehicle. \(V\) is the measured velocity of the rear wheels, \(V'\) is the velocity of the front wheels along the steered angle \(\gamma\) and \(B\) is the vehicle wheel-base. The continuous model of the vehicle with all quantities referenced to the center of the front axle of the vehicle can be written as

\[
\frac{dx_v}{dt} = V(t) \cos \phi_v(t), \quad \frac{dy_v}{dt} = V(t) \sin \phi_v(t), \quad \frac{d\phi_v}{dt} = \left[ \frac{1}{B} \right] V'(t) \tan \gamma(t).
\]

(1)

By integrating Eq. (1) using Euler approximation and assuming that the control signals, \(V\) and \(\gamma\), are approximately constant over the sample period, the nominal discrete process model equations at time instant \(k\) can be written as

\[
\begin{bmatrix}
  x_k \\
  y_k \\
  \phi_k
\end{bmatrix} = \begin{bmatrix}
  x_{k-1} \\
  y_{k-1} \\
  \phi_{k-1}
\end{bmatrix} + \Delta T \begin{bmatrix}
  V_k \cos \phi_{k-1} \\
  V_k \sin \phi_{k-1} \\
  \frac{1}{B} V_k \tan \gamma_k
\end{bmatrix},
\]

where \(\Delta T\) is the sampling interval between states at discrete time instants, \((k - 1)\) and \(k\).

All the quantities are referred to the location of the laser rangefinder as the DGPS was placed right on top of the laser. Taking this into account, the nominal discrete process model equations at time instant \(k\) can be written as

\[
\begin{bmatrix}
  x_k \\
  y_k \\
  \phi_k
\end{bmatrix} = \begin{bmatrix}
  x_{k-1} \\
  y_{k-1} \\
  \phi_{k-1}
\end{bmatrix} + \Delta T \begin{bmatrix}
  V_k \cos \phi_{k-1} - \frac{V_k}{B} k_x \tan \gamma_k \\
  V_k \sin \phi_{k-1} + \frac{V_k}{B} k_y \tan \gamma_k \\
  \frac{V_k}{B} \tan \gamma_k
\end{bmatrix},
\]

(2)

where

\[
k_x = a_L \sin \phi_{k-1} + b_L \cos \phi_{k-1}, \quad k_y = a_L \cos \phi_{k-1} - b_L \sin \phi_{k-1}.
\]

In Eq. (2), \((a_L, b_L)\) are laser offsets measured from the rear axle of the vehicle.
The control signals applied to the vehicle are
\[ u_k = [V_k, \gamma_k] \].

The errors due to the control signals \( V \) and \( \gamma \) are modeled as simple additive noise sources, \( \delta V \) and \( \delta \gamma \) about their respective means \( \bar{V} \) and \( \bar{\gamma} \) as below:
\[
V_k = \bar{V} + \delta V_k, \quad \gamma_k = \bar{\gamma} + \delta \gamma_k.
\]

The error source vector is defined as
\[ \delta w_k = [\delta V_k, \delta \gamma_k]^T, \]
and is a direct effect of the associated modeling errors and uncertainty in control. The source errors \( \delta V \) and \( \delta \gamma \) are assumed to be zero-mean, uncorrelated Gaussian sequences with constant variances \( \sigma_V^2 \) and \( \sigma_\gamma^2 \), respectively. The variances were determined experimentally to reflect true noise variances. The relation between the L VDT output to the steer angles and that of the wheel encoder to the vehicle speed were experimentally determined by calibration trials on the utility vehicle. For additional details, see [8]. Fig. 7 shows data obtained from the velocity and steering sensors during the trials.

### 3.3. Natural landmark observation model

The predicted range and bearing for each natural landmark \( j \) at discrete time instant \( k \) is given by
\[
R_{k+1}^j = \sqrt{(x_{Ej}^j - x_{Ej})^2 + (y_{Ej}^j - y_{Ej})^2}, \quad \psi_{k+1}^j = \arctan \left( \frac{y_{Ej}^j - y_{Ej}}{x_{Ej}^j - x_{Ej}} \right) - \phi_{ej}. \]

where \((x_{Ej}^j, y_{Ej}^j)\) is the Cartesian location of landmark \( j \) and \((x_{Ej}, y_{Ej})\) is the location of the laser rangefinder on the vehicle.

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**Fig. 7.** Speed and steer angle outputs of the utility vehicle. The processed sensor data after utilizing the appropriate gains and calibrations is shown. The wheel encoder velocity is in m/s and the steering angle output of the L VDT is in degrees.
The observation model for a landmark is thus given by

\[
\mathbf{Z}_{j}^{n} = \begin{bmatrix} \mathbf{R}_{j}^{n} \\ \mathbf{\theta}_{j}^{n} \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{\mathbf{R}}^{n} \\ \mathbf{v}_{\mathbf{\theta}}^{n} \end{bmatrix},
\]  

(3)

where \( \mathbf{v}_{\mathbf{R}}^{n} \) and \( \mathbf{v}_{\mathbf{\theta}}^{n} \) refer to the uncertainty present in the range and bearing observations and are modeled as zero-mean uncorrelated Gaussian sequences with constant variances, \( \sigma_{\mathbf{R}}^{2} \) and \( \sigma_{\mathbf{\theta}}^{2} \), respectively.

3.4. Natural landmark-based localization results

An EKF was used to fuse the observations from the DGPS, the LVDT and the velocity encoder to provide vehicle pose estimates. These estimates then served as ground truth for the construction of the natural landmark map and also for calculating the accuracy of natural landmark-based localization. The EKF utilized the process model given by Eq. (2) for the prediction stage of the filter. The observations from the DGPS reported positions were then incorporated to update the states of the vehicle [8].

When landmarks are used for localization, it is essential that the extracted landmarks from a scan are reliably matched to those in the map. Whenever a range and bearing laser scan is received, the dominant point landmarks are extracted using the CSS algorithm. Locations of all the extracted dominant points above a certain scale level are then compared with the natural landmark map. Two landmarks that differ only slightly in their position should be
identified as distinct within the limits imposed by sensor noise. Such discrimination is achieved by checking if the computed innovation sequences pertaining to both the range and bearing of a matched natural landmark fall within the normalized innovation gate by using the natural landmark observation model given in Eq. (3). If the observations fall within the prescribed limits, then the landmark is accepted as a match. Subsequently, this observation is used to update the vehicle states. The predict–observe–validate–update estimation cycle [14] of the EKF for localization of the utility vehicle is described in Appendix B.

Fig. 8 shows the laser scan profile of the field trial area. It should be noted that the laser profile is a cumulative representation of the environment. The ‘‘T’’ and ‘‘L’’ structures due to the parked cars are easily extractable as landmarks using the CSS algorithm as these structures give rise to unique dominant points. All laser observations less than 30 m in front of the vehicle are used during both the mapping and localization stages. Fig. 8 also shows the extracted landmarks from laser scans. It is clear the CSS extracted natural landmarks correspond to dominant points characterized by curvature extrema.

Fig. 9(a) shows the path of the vehicle estimated by the landmark algorithm. It can be seen that the estimated path is in close agreement with the ground truth.\(^2\) In Fig. 9(b), the traversed path runs from bottom to top and the dots represent the natural landmark locations. The vehicle starts at \((-2.2, -0.65\) m). The extracted landmarks provide sufficient external resetting to correct for the model-based encoder estimates. Fig. 9(b) shows the orientation estimated by the CSS algorithm which also agrees well with that of the ground truth.

Fig. 10(a) shows the 2σ confidence bounds for the absolute pose errors computed using the ground truth. It can be seen that the errors are clearly bounded and are consistent. A closer look at the absolute errors reveals that peak position estimate errors of less than 25 cm were achieved with a corresponding error in orientation being less than 2°. This is notable in light of external disturbances (e.g. moving cars, walking people), associated systematic

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\(^2\) As the estimates and their corresponding ground truth are very close, extra effort is required on the part of the reader to distinguish between the two.
Fig. 10. (a) The pose errors and the 2σ confidence bounds computed using the covariance estimate for the error in x, y and φ compared to the actual error computed with the corresponding ground truth estimates are shown. (b) The range and bearing innovation sequences for the natural landmarks matched during the trial. The 1σ and 2σ confidence bounds are also shown.
errors and the absence of other relative aiding sources. The $1\sigma$ and $2\sigma$ bounds for the validated range and bearing innovation sequences of the natural landmarks is shown in Fig. 10(b). It can be seen that the innovation sequences are zero-mean and bounded by their estimated variances. The CSS natural landmark localization algorithm therefore shows the potential to achieve reliable and robust localization in real world environments.

4. Conclusions

A multiscale curvature scale space algorithm for identifying dominant point landmarks in range images obtained from a scanning laser rangefinder was developed in this article. Such extracted point landmarks were then combined in an extended Kalman filter to demonstrate the localization of an outdoor vehicle operating in relatively unmodified environments. Using a landmark map, the CSS extracted landmarks during the localization stage were matched against those in the map to obtain an estimate of the vehicle pose using the EKF. In spite of external disturbances, the proposed natural landmark localization algorithm provided consistent position estimates with an error of less than 25 cm and an orientation error of less than 2° without the need for maintaining any additional artificial infrastructure, thus demonstrating the robustness of the proposed algorithm. The CSS algorithm has also been successfully employed for the terrain-aided localization of an underground mining vehicle leading to a substantial reduction in the number of artificial landmarks required for navigation[12].

The CSS-based localization algorithm requires the presence of unique natural landmarks in the operating environment. In the utility trial, this requirement was satisfied due to parked cars. In the absence or sparsity of such landmarks, this approach might fail as the external aiding information to compensate for the diverging dead-reckoning estimates cannot be guaranteed. Also changes to the environment after the landmark map is constructed may have an undesirable effect on the performance of the algorithm. These effects remain to be further investigated.

In this article, map-aided pose estimation and map building have been decoupled using ground truth. However, the competency of a robotic vehicle to start in an unknown location and unknown environment and then incrementally build a map of the environment while simultaneously using the same map to compute its position, would be highly desirable. This is termed as the Simultaneous Localization and Map-building (SLAM) problem (also referred to as Concurrent Mapping and Localization in the literature). The presence of unique landmarks and their reliable extractability throughout the intended navigation environment are critical to the practical implementation of SLAM. In field trials described in this article, even though the reliable extraction of natural landmarks was possible, the absence of landmarks in certain regions made it difficult to demonstrate the SLAM capability. This is another area that warrants further research.

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Appendix A. Curvature computation

The curvature, $\kappa$, of a planar curve at a point $P$ on the curve is defined as the instantaneous rate of change of the slope angle $\Psi$ of the tangent at point $P$ with respect to the path length and is equal to the inverse of the radius of curvature, $\rho$ at point $P$ (see Fig. 1).

$$\kappa = \frac{d\Psi}{ds} = \frac{1}{\rho}$$
\( \kappa \) can be computed when it is expressed in terms of the derivatives of the functions \( x(s) \) and \( y(s) \) as below:

\[
\kappa(x, y) = \frac{\dot{y}^2 \ddot{x} - \dot{x}^2 \ddot{y}}{(1 + (\dot{y}(\dot{x}))^2)^{3/2}}.
\]

Define

\[
\dot{x} \triangleq \frac{\Delta x}{\Delta s}, \quad \dot{y} \triangleq \frac{\Delta y}{\Delta s}, \quad \ddot{x} \triangleq \frac{\Delta^2 x}{\Delta s^2}, \quad \ddot{y} \triangleq \frac{\Delta^2 y}{\Delta s^2}.
\]

Curvature can then be expressed in terms of \( \dot{x}, \dot{y}, \ddot{x} \) and \( \ddot{y} \) as

\[
\kappa(x, y) = \dot{x} \ddot{y} - \dot{y} \ddot{x}.
\]

It can be shown that \( x(s) \) and \( y(s) \) are related by

\[
\dot{x} = \cos \Psi, \quad \dot{y} = \sin \Psi,
\]

which gives the curvature expression as

\[
\kappa(x, y) = \dot{x} \ddot{y} - \dot{y} \ddot{x} = \kappa(x, y),
\]

(A.1)

To express the curvature at varying levels of detail, the coordinate functions \( x(s) \) and \( y(s) \) are convolved with the Gaussian kernel. In the following set of equations, \( \otimes \) denotes convolution and \( \sigma \), the standard deviation of the kernel.

The Gaussian kernel and its derivatives are given by

\[
g(s, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-s^2/2\sigma^2}, \quad \dot{g}(s, \sigma) \triangleq \frac{\Delta g(s, \sigma)}{\Delta s} = \frac{-1}{\sigma^3 \sqrt{2\pi}} s e^{-s^2/2\sigma^2},
\]

\[
g(s, \sigma) = \frac{\Delta^2 g(s, \sigma)}{\Delta s^2} = \frac{1}{\sigma^4 \sqrt{2\pi}} \left( \frac{s^2}{\sigma^2} - 1 \right) e^{-s^2/2\sigma^2}.
\]

The convolution with the Gaussian and its derivatives can be written as

\[
\begin{align*}
X(s, \sigma) &= x(s) \otimes g(s, \sigma) = \int_{-\infty}^{\infty} x(u) \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(s-u)^2}{2\sigma^2}} \, du, \\
Y(s, \sigma) &= y(s) \otimes g(s, \sigma) = \int_{-\infty}^{\infty} y(u) \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(s-u)^2}{2\sigma^2}} \, du, \\
\dot{X}(s, \sigma) &= \dot{x}(s) \otimes g(s, \sigma) = \frac{\dot{x}(s)}{\sigma} \int_{-\infty}^{\infty} x(u) \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(s-u)^2}{2\sigma^2}} \, du, \\
\dot{Y}(s, \sigma) &= \dot{y}(s) \otimes g(s, \sigma) = \frac{\dot{y}(s)}{\sigma} \int_{-\infty}^{\infty} y(u) \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(s-u)^2}{2\sigma^2}} \, du, \\
\ddot{X}(s, \sigma) &= \dot{x}(s) \otimes \dot{g}(s, \sigma) = \int_{-\infty}^{\infty} \dot{x}(u) \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(s-u)^2}{2\sigma^2}} \, du, \\
\ddot{Y}(s, \sigma) &= \dot{y}(s) \otimes \dot{g}(s, \sigma) = \int_{-\infty}^{\infty} \dot{y}(u) \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(s-u)^2}{2\sigma^2}} \, du,
\end{align*}
\]

The curvature \( \kappa(s, \sigma) \) is then computed by using Eq. (A.1) as

\[
\kappa(s, \sigma) = X(s, \sigma) \ddot{Y}(s, \sigma) - \ddot{X}(s, \sigma) Y(s, \sigma).
\]

(A.2)
Appendix B. Estimation cycle

B.1. State prediction

The state prediction takes place according to Eq. (2) as follows:

\[
\begin{bmatrix}
    x_v(k|k-1) \\
y_v(k|k-1) \\
\phi_v(k|k-1)
\end{bmatrix} =
\begin{bmatrix}
    x_v(k-1|k-1) \\
y_v(k-1|k-1) \\
\phi_v(k-1|k-1)
\end{bmatrix} + \nu \Delta \chi_T
\]

where the notation \( k|k-1 \) denotes the state estimate at time \( k \) based on information up to time \( (k-1) \).

The prediction covariance can be computed using

\[
P(k|k-1) = \nabla f_{x_v} \nabla f_{x_v}^T + \nabla f_w k / \Sigma_1 c
\]

where \( \nabla f_{x_v} \) represents the Jacobian with respect to the states, \( \nabla f_w k \) is the Jacobian with respect to the error sources and \( / \Sigma_1 c \) is the noise strength matrix given by

\[
\begin{bmatrix}
    1 & 0 & -\Delta T_{11} \\
    0 & 1 & -\Delta T_{21} \\
    0 & 0 & -\Delta T_{22}
\end{bmatrix} \cdot \Sigma_e = \begin{bmatrix}
    \sigma_f^2 & 0 & \sigma_f^2 \\
    0 & \sigma_e^2 & 0
\end{bmatrix}
\]

where

\[
f_{11} = \frac{V}{3} \cos \phi_{k-1} - \frac{V}{3} \sin \phi_{k-1} \\
f_{12} = \frac{V}{3} \sin \phi_{k-1} + \frac{V}{3} \cos \phi_{k-1} \\
f_{21} = \frac{V}{3} \sin \phi_{k-1} - \frac{V}{3} \cos \phi_{k-1} \\
f_{22} = \frac{V}{3} \cos \phi_{k-1} + \frac{V}{3} \sin \phi_{k-1}
\]

\[
g_{11} = \frac{1}{\cos \gamma_k} \left[ aL \sin \phi_{k-1} - bL \cos \phi_{k-1} \right] \\
g_{12} = \frac{1}{\cos \gamma_k} \left[ aL \cos \phi_{k-1} + bL \sin \phi_{k-1} \right] \\
g_{21} = \frac{1}{\cos \gamma_k} \left[ aL \cos \phi_{k-1} - bL \sin \phi_{k-1} \right] \\
g_{22} = \frac{1}{\cos \gamma_k} \left[ aL \sin \phi_{k-1} + bL \cos \phi_{k-1} \right]
\]

B.2. Observation prediction and validation

Once the predicted states of the vehicle are available, the observations that will be made at that state can be predicted. From Eq. (3), the range and bearing to a landmark can be predicted. Observations of a landmark must be reliably associated with a landmark in the map giving rise to the observation. This is accomplished by utilizing
the predicted observation and innovation covariance computed using the observation Jacobians. An observation is accepted only if the observation falls inside the normalized innovation validation gate:

\[ v_k^T S_k^{-1} v_k \leq \epsilon \gamma \]

where \( v_k \) is the innovation defined as the difference between the actual and predicted observations. The value of \( \epsilon \gamma \) can be chosen from the fact that the normalized innovation sequence is a \( \chi^2 \) random variable with \( m \) degrees of freedom (\( m \) being the dimension of the observation) [14]. \( S_k \) is the innovation covariance given by

\[ S_k = \nabla h x_k^T P_{k|k-1} \nabla h + R_k \]

with the observation Jacobian \( \nabla h \) as below:

\[ \nabla h = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix} \]

where

\[ h_{11} = \frac{x_{\text{obs},d} - x_{\text{det}}}{d}, \quad h_{12} = \frac{y_{\text{obs},d} - y_{\text{det}}}{d}, \quad h_{13} = 0, \]

\[ h_{22} = \frac{x_{\text{obs},d} - x_{\text{det}}}{d^2}, \quad h_{23} = -1, \quad d^2 = (x_{\text{det}} - x_{\text{obs},d})^2 + (y_{\text{det}} - y_{\text{obs},d})^2. \]

The observation covariance, \( R_k \), is given by

\[ R_k = \begin{bmatrix} \sigma^2_{d,e} & 0 \\ 0 & \sigma^2_{e} \end{bmatrix} \]

### B.3. Update

Once a correct association with a unique landmark has been made, the state estimate and covariance updates are performed using the standard EKF update equations:

\[ x_{k|k} = x_{k|k-1} + W_k \cdot v_k, \quad P_{k|k} = P_{k|k-1} - W_k S_k W_k^T \]

where the Kalman gain matrix is given by

\[ W_k = P_{k|k-1} \nabla h S_k^{-1} \]

### References
