The Shapley-Solidarity value for games with a coalition structure

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Abstract

A value for games with a coalition structure is introduced, where the rules guiding the cooperation among the members of the same coalition are different from the interaction rules among coalitions. In particular, players inside a coalition exhibit a greater degree of solidarity than they are willing to use with players outside their coalition. The Shapley value [Shapley, 1953b] is therefore used to compute the aggregate payoffs of the coalitions, and the Solidarity value [Nowak and Radzik, 1994] to obtain the payoffs of the players inside each coalition.

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1 Introduction

There are many settings in cooperative games where players naturally organize themselves into groups for the purpose of negotiating payoffs. This action can be modeled by

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including a coalition structure into the game, which consists of an exogenous partition of players into a set of groups or unions. These unions sometimes arise for natural reasons. Players are joined into groups of similar interests and characteristics in the case of trade unions, political parties, cartels, lobbies, etc. Another typical reason is due to geographical location as in the case of cities, states and countries.

When groups are formed, the agents interact at two levels: first, bargaining takes place among the unions, and then, bargaining occurs inside each union in order to share what the union has obtained. Owen (1977) was the first to follow this approach. In his coalitional value, unions play a quotient game among themselves and then each union receives a payoff that is shared among its players in an internal game. The payoffs at each level are given by the Shapley value (Shapley, 1953b). Thus, the same properties (axioms) that govern the interaction between groups also operate among the players of each group. The basic principle behind the Shapley value is to pay players according to their productivity. It can be expressed formally by the marginality axiom (see Young, 1985), i.e. if the marginal contributions of a player in two games are the same then his value should be the same. Alternatively, it can be expressed by the null player axiom, that is, if all the marginal contributions of a player in a game are zero, then the player should obtain zero. The Owen value applies this productivity principle when sharing rewards at both levels: between unions and within unions.

An direct consequence to apply the productivity principle redistributing rewards within unions is that null players always receive zero, doesn’t matter they are alone or inside a union. Nevertheless, it could be questioned if it is a legitimate point of view that a coalitional value must follow the same behavior at both levels of bargaining. A greater degree of solidarity existing between members of the same group than in the interaction between players of different groups seems also natural. It is very easy to find real life examples where the groups formed want to protect their weaker members giving them a share of the gains obtained by the group.

Our goal is to develop a coalitional value where the rule followed to share the payoffs within each union is less competitive that the rule used in the bargaining between unions.

For that purpose we follow the same Owen’s approach. In the first step, unions play a quotient game among themselves and each union receives a payoff given by the Shapley value. This value express the competing principle of paying unions according with their
productivity. In the second step, the same rule is done for any subcoalition of a union, replacing in the coalition structure the union for this subcoalition. In this way we find the payoff that the subcoalition could obtain if its complementary in the union withdraw the game. This is the internal game that Owen uses to reward players within the union applying the Shapley value to the internal game. At this point we leave Owen’s approach, replacing the Shapley value in the internal game by another value which takes into account not only productivity principles but also some degree of cohesion, or solidarity, between their members.

Many values can be chosen at this point: The Kernel (Davis and Maschler, 1965), the nucleolus (Schmeidler, 1969), the equal division solution\(^1\), the egalitarian Shapley values\(^2\) (Joosten, 1996, and van den Brink, Funaki and Ju, 2011), the consensus value\(^3\) (Ju, Borm and Ruys, 2007), and the weighted coalitional Lorenz solutions (Arin and Feltkamp, 2002) among others.

Our election is the \textit{solidarity value}, introduced by Sprumont (1990) as an example of a population monotonic allocation scheme and characterized axiomatically by Nowak and Radzik (1994). This value is a good compromise between the productivity and solidarity principles: it takes into account the productivity principle as well, as the players’ marginal contributions are used in the calculation. However, it also exhibits a redistribution effect, as it not only takes into account his own marginal contribution to the coalition that player belongs to, but also the marginal contributions of the remaining players, in such a way that the own marginal contribution is replaced in the computation of the value by the average of the marginal contributions of all players in the coalition. According to this approach, the value is obtained in two steps. First, unions play a quotient game among themselves and each union receives a payoff given by the Shapley value; and second, the outcome obtained by the union is shared among its members by paying the solidarity value in the internal game. We call it as the \textit{Shapley-solidarity value}.

We start by offering a new axiomatic support to the solidarity value. We take as reference the Myerson (1980) characterization of the Shapley value by means of the \textit{balanced contributions} axiom. This property states that for any two players, the amount that each player would gain or lose by the other player’s withdrawal from the game should be equal.

\(^1\) This value shares the payoffs equally between the members of the coalition.
\(^2\) Convex combinations between the Shapley value and the equal surplus solution.
\(^3\) A convex combination between the Shapley value and the CIS value (Driessen and Funaki, 1991).
This express the competing principle that each pair of players are in balance because the loss in the payoff that each player can inflict the other by withdraw the game is the same (as a consequence if they are equally productive they receive the same payoffs). Myerson shows that the Shapley value is the unique value which is efficient and satisfies balanced contributions.

We present a way to formulate the solidarity idea that all players are "in the same boat" as follows: Suppose that every player has the same chance to leave the game, and compute for a player the average variation in her payoff when every remaining player can leave the game. Then we say that a value satisfy the equal average gains axiom if these expected payoff variations are the same for all players. In Theorem 3 we prove that the solidarity value is the unique value that satisfies efficiency and equal average gains.

With the help of this axiom we are ready to offer the axiomatic characterization of the Shapley-solidarity value on the family of games with a coalition structure. The competing principle of interaction among unions is expressed by an axiom of balanced contributions between unions, and the solidarity among the members inside a union, by an axiom of equal average gains between the members of the same union. In Theorem 5 we prove that the Shapley-solidarity value is the unique value in games with coalition structures which satisfies efficiency, balanced contributions between unions and equal average gains between the members of the same union.

This result allows an easy and direct comparison with the Owen value. In this value the competing principle guide the interaction between unions and also the interaction between the members of the same union. Accordingly the Owen value is the unique coalitional value that satisfies efficiency, balanced contributions between unions and balanced contributions between the members of the same union (see Calvo, Lasaga and Winter, 1996, and Amer and Carreras, 1995).

The rest of the paper is organized as follows. Section 2 is devoted to definitions and notation. Section 3 introduces the new coalitional value. We provide the axiomatic characterization of this value in Section 4. Section 5 is devoted to the comparison with other coalitional values existing in the Literature. This is done with the help of an example and looking the differences between the axioms which characterize these values. The conclusions are presented in Section 6.
2 Notation and definitions

Let \( U = \{1, 2, \ldots\} \) be the (infinite) set of potential players. A cooperative game with transferable utility (TU-game) is a pair \((N, v)\) where \( N \subseteq U \) is a nonempty and finite set and \( v : 2^N \rightarrow \mathbb{R} \) is a characteristic function, defined on the power set of \( N \), satisfying \( v(\emptyset) = 0 \). An element \( i \) of \( N \) is called a player and every nonempty subset \( S \) of \( N \) a coalition. The real number \( v(S) \) is called the worth of coalition \( S \), and it is interpreted as the total payoff that the coalition \( S \), if it forms, can obtain for its members. Let \( \mathcal{G}^N \) denote the set of all cooperative TU-games with player set \( N \).

For each two games \((N, v)\) and \((N, w)\) \(\in \mathcal{G}^N\), the game \((N, v + w)\) is defined as \((v + w)(S) = v(S) + w(S)\) for each \( S \subseteq N \). For all \( S \subseteq N \), we denote the restriction of \((N, v)\) to \( S \) as \((S, v)\). For simplicity, we write \( S \cup i \) instead of \( S \cup \{i\} \), \( N \setminus i \) instead of \( N \setminus \{i\} \), and \( v(i) \) instead of \( v(\{i\}) \).

Two players \( \{i, j\} \subseteq N \) are symmetric in \((N, v)\) if, for each \( S \subseteq N \setminus \{i, j\} \): \( v(S \cup i) = v(S \cup j) \).

Player \( i \in N \) is a null player in \((N, v)\) if \( v(S \cup i) = v(S) \) for all \( S \subseteq N \setminus i \).

A value is a function \( \gamma \) which assigns to every TU-game \((N, v)\) and every player \( i \in N \), a real number \( \gamma_i(N, v) \), which represents an assessment made by \( i \) of his gains from participating in the game.

Let \((N, v)\) be a game. For all \( S \subseteq N \) and all \( i \in S \), define

\[ \Delta^i(v, S) := v(S) - v(S \setminus i). \]

We call \( \Delta^i(v, S) \) the marginal contribution of player \( i \) to coalition \( S \) in the TU-game \((N, v)\). The Shapley value (Shapley, 1953b) of the game \((N, v)\) is the payoff vector \( Sh(N, v) \in \mathbb{R}^N \) defined by

\[ Sh_i(N, v) = \sum_{S \subseteq N; i \in S} \frac{(n - s)! (s - 1)!}{n!} \Delta^i(v, S), \quad \text{for all } i \in N, \]

where \( s = |S| \) and \( n = |N| \).

For all \( T \subseteq N \), the unanimity game of the coalition \( T \), \((N, u_T)\), is defined by

\[ u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise}. \end{cases} \]

It is well known that the family of games \( \{(N, u_T)\}_{T \subseteq N} \) is a basis for \( \mathcal{G}^N \). This allows an alternative definition of the Shapley value as the linear map \( Sh : \mathcal{G}^N \rightarrow \mathbb{R}^N \), which
is defined for all unanimity game \((N, u_T)\) as follows

\[
    Sh_i(N, u_T) = \begin{cases} 
        \frac{1}{|T|} & \text{for all } i \in T, \\
        0 & \text{otherwise}. \end{cases}
\]

For all finite set \(N \subseteq U\), a coalition structure over \(N\), i.e., \(B = \{B_1, B_2, \ldots, B_m\}\) is a coalition structure if it satisfies that \(\bigcup_{1 \leq k \leq m} B_k = N\) and \(B_k \cap B_l = \emptyset\) when \(k \neq l\). We also assume \(B_k \neq \emptyset\) for all \(k\). The sets \(B_k \in B\) are called “unions” or “blocks”. There are two trivial coalition structures: The first, which we denote by \(B^N\), where only the grand coalition forms, that is, \(B^N = \{N\}\); and the second is the coalition structure where each union is a singleton and it is denoted by \(B^n\), that is, \(B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}\). Denote by \(B(N)\) the set of all coalition structures over \(N\). A game \((N, v)\) with coalition structure \(B \in B(N)\) is denoted by \((B, N, v)\). Let \(\mathcal{CSG}^N\) denote the family of all TU-games with coalition structure with player set \(N\), and let \(\mathcal{CSG}\) denote the set of all TU-games with coalition structure.

We say that, for each \(\{k, l\} \subseteq M\), \(B_k\) and \(B_l\) are symmetric coalitions in \((B, N, v)\) if the players \(k\) and \(l\) are symmetric in the game \((M, v_B)\). We say that \(B_k \in B\) is a null coalition if player \(k \in M\) is a null player in the game \((M, v_B)\).

A coalitional value is a function \(\Phi\) that assigns a vector in \(\mathbb{R}^N\) to each game with coalition structure \((B, N, v) \in \mathcal{CSG}^N\).

One of the most important coalitional values is the Owen value (Owen, 1977). His approach resolves the problems of intercoalitional and intracoalitional bargaining by the same procedure. First, it is defined the game between unions, called the quotient game, as follows: For all game \((B, N, v) \in \mathcal{CSG}^N\), with \(B = \{B_1, B_2, \ldots, B_m\}\), the quotient game is the TU-game \((M, v_B) \in \mathcal{G}^M\) where \(M = \{1, 2, \ldots, m\}\) and \(v_B(T) := v \left( \bigcup_{i \in T} B_i \right)\) for all \(T \subseteq M\). That is, \((M, v_B)\) is the game induced by \((B, N, v)\) by considering the unions of \(B\) as players. Notice that for the trivial coalition structure \(B^n\) we have \((M, v_{B^n}) \equiv (N, v)\).

Second, for all \(k \in M\) and all \(S \subseteq B_k\), denote by \(B \mid_S\) the new coalition structure defined on \((\cup_{j \neq k} B_j) \cup S\), which appears when the complementary of \(S\) in \(B_k\) leaves the game. That is,

\[
    B \mid_S = \{B_1, \ldots, B_{k-1}, S, B_{k+1}, \ldots, B_m\}.
\]

The game \((M, v_{B\mid_S})\) describes what would happen in the quotient game if union \(B_k\) were
replaced by $S$, i.e.,

$$v_{B \mid S}(T) = v(\cup_{j \in T} B_j \setminus S') \quad \text{for all } T \subseteq M,$$

where $S' = B_k \setminus S$.

Owen (1977) defines an *internal game* $(B_k, v_k)$ by setting $v_k(S) = Sh_k(M, v_{B \mid S})$ for all $S \subseteq B_k$. Thus, $v_k(S)$ is the payoff to $S$ in $v_{B \mid S}$. The *Owen value* of the game $(B, N, v)$ is the payoff vector $Ow(B, N, v) \in \mathbb{R}^N$ defined by

$$Ow_i(B, N, v) := Sh_i(B_k, v_k), \quad \text{for all } k \in M \text{ and all } i \in B_k. \quad (1)$$

Thus, first each union $S \subseteq B_k$ plays the quotient game $(M, v_{B \mid S})$ among the unions, and the payoff obtained, $Sh_k(M, v_{B \mid S})$, determines the reward of coalition $S$ in the internal game $(B_k, v_k)$. The total reward of union $B_k$ is $Sh_k(M, v_B)$ and it is shared among its members, $i \in B_k$, applying again the Shapley value in the internal game $(B_k, v_k)$, that is $Ow_i(B, N, v) = Sh_i(B_k, v_k)$. In that sense, we can denote the Owen value as $Ow \equiv \Gamma^{(Sh, Sh)}$.

Note that the Owen value satisfies the *quotient game property*:

$$\sum_{i \in B_k} Ow_i(B, N, v) = Sh_k(M, v_B), \quad \text{for all } k \in M,$$

and for the trivial coalition structures $B^n$ and $B^N$, $Ow(B^N, N, v) = Ow(B^n, N, v) = Sh(N, v)$.

### 3 Definition of the Shapley-solidarity value

The usual motivation for incorporating a coalition structure into a game is that players are interested in joining a union in order to improve their bargaining position in the game. Hence, when a union is formed, all its members commit themselves to bargaining with the others as a unit. A critical question here is how to share the gains (or losses) obtained by the players in a union. The fact that the Owen value uses the Shapley value in the internal game makes this interpretation a bit problematic. We illustrate this question with the help of the next example.

**Example 1** Consider the player set $N = \{1, 2, 3, 4\}$ with the coalition structure $B = \{B_r = \{1\}, B_k = \{2, 3\}, B_t = \{4\}\}$, where players 2 and 3 form the union $B_k = \{2, 3\}$ and players 1 and 4 remain isolated. And let the unanimity game $u_T$, with $T = \{1, 2\}$.
In the game between unions \((M = \{r, k, t\}, (u_T)_B)\), unions \(B_r\) and \(B_k\) are symmetric players, i.e. both contribute the same in the quotient game, so the Shapley value yields \(1/2\) each one, and union \(B_t\) is a null player in the quotient game and then obtains zero. Inside union \(B_k = \{2, 3\}\), the internal game \((B_k, (u_T)_k)\) is the unanimity game given by

\[
(u_T)_k(\{2, 3\}) = 1/2, \quad (u_T)_k(\{2\}) = 1/2, \quad (u_T)_k(\{3\}) = 0.
\]

Player 3 is again a null player in the internal game \((B_k, (u_T)_k)\), hence his Shapley value is zero. Therefore, the payoffs associated to the Owen value in \((B, N, u_T)\) are

\[
Ow_1 = \frac{1}{2}, \quad Ow_2 = \frac{1}{2}, \quad Ow_3 = 0, \quad Ow_4 = 0.
\]

Therefore, there is no difference for player 3 between belonging to the union \(\{2, 3\}\) or being isolated, as in \(B^n = \{\{1\}, \{2\}, \{3\}, \{4\}\}\), because in this case \(Ow_3(B^n, N, u_T) = Sh_3(N, u_T) = 0\).

So, according with the Owen value there are no incentives to form union \(\{2, 3\}\) in the game \(u_T\). This is because the Owen value rewards players in the internal game also according with their productivity, and the productivity of player 3 is zero.

However, we wish to stress that the Owen approach does not determine the value and the coalition structure simultaneously. On the contrary, the coalition structure is given a priory and fixed, before starting any computation of the value. The reasons for the existence of a coalition structure are varied and depend on the context at hand. Although we can agree that unions try to obtain as much as possible by not letting the others exploit their (individual) weaknesses when they are separated, that does not necessarily imply that the members of the union are interested to join with only productive ones. For example, we can imagine that the union is formed by a family with a child, that can be considered as a null member of the family during his childhood. In this context a positive reward for the child looks quite natural.

Our purpose is to consider coalitional values with some degree of solidarity in the interaction between players of the same union, in contrast with a competitive interaction between different unions. For that purpose we stick with the Shapley value at the first level of interaction among unions. But for the interaction level between players within the same union, we wish to apply a value with a greater degree of cohesion between their members than the Shapley value does.
There are several candidates to share the value \( Sh_k(M, v_B) \) between the players in \( B_k \). An extreme option could be the egalitarian rule, which gives \( Sh_k(M, v_B) / |B_k| \) to each player \( i \) in \( B_k \). In our Example 1 it yields \( 1/4 \) to each player 2 and 3. However, this seems rather unfair from the productivity point of view, as 3 is a null player that does not contribute to the rewards of the union \{2, 3\}. Although the redistribution of the gains obtained by the union is a good cohesion property, it seems desirable maintaining also at some degree the productivity principle as it is a good incentive rule. Can we make both principles compatible?

In this paper, we propose a new coalitional value which applies the solidarity value introduced by Nowak and Radzik (1994), in the internal game \((B_k, v_k)\). This value takes into account both principles (productivity and redistribution) in its definition. To see this we recall first the definition of the solidarity value in \( G \).

Let \((N, v)\) be a game. For all \( S \subseteq N \), define
\[
\Delta^{av}(v, S) := \frac{1}{s} \sum_{i \in S} \Delta^i(v, S).
\]
\( \Delta^{av}(v, S) \) is the average of the marginal contributions of players within coalition \( S \) in the game \((N, v)\).

**Definition 1** The solidarity value of the game \((N, v)\) is the payoff vector \( S_l(N, v) \in \mathbb{R}^N \) defined by
\[
S_l_i(N, v) = \sum_{S \subseteq N: i \in S} \frac{(n-s)! (s-1)!}{n!} \Delta^{av}(v, S), \quad \text{for all } i \in N.
\]

The productivity principle is taken into account, as the players’ marginal contributions are used in the calculation. Moreover, it also exhibits a redistribution effect, as it not only takes into account his own marginal contribution to the coalition that player belongs to, but also the marginal contributions of the remaining players of the coalition. In that way, even a null player, whose all \( \Delta^i(S, v) \) are zero, can still obtain positive rewards if the associated \( \Delta^{av}(S, v) \) are positive.

We now define the coalitional value in \( CSG \).

**Definition 2** For all game \((B, N, v) \in CSG^N\), for all \( k \in M \) and all \( i \in B_k \), the Shapley-Solidarity value of \((B, N, v)\) is the payoff vector \( \xi(B, N, v) \in \mathbb{R}^N \) defined by:
\[
\xi_i(B, N, v) := S_l_i(B_k, v_k), \quad (2)
\]
where \( v_k(S) = Sh_k(M, v_{B_k}) \) for all \( S \subseteq B_k \).

First union \( k \) plays the quotient game \((M, v_B)\) among the unions, and the payoff obtained (by the Shapley value) is shared among its members by computing the solidarity value in the internal game \((B_k, v_k)\). In this sense, we also denote the value \( \xi \) as \( \Gamma^{(Sh, Sl)} \).

As the solidarity value satisfies efficiency in the internal game \((B_k, v_k)\), it follows that \( \xi \) also satisfies the quotient game property:

\[
\sum_{i \in B_k} \xi_i(B, N, v) = Sh_k(M, v_B), \quad \text{for all} \ k \in M.
\]

Note that the Shapley-solidarity value applies different principles in the trivial coalition structures \( B^n \) and \( B^N \): As in \((B^N, N, v)\) all players are in the same union, \( \xi \) applies the cohesion principle and then \( \xi(B^N, N, v) = Sl(N, v) \); as in \((B^n, N, v)\) all players are isolated, \( \xi \) applies the productivity principle and then \( \xi(B^n, N, v) = Sh(N, v) \). This is in contrast with the Owen value which applies the same competing principle to reward unions and players inside unions: \( Ow(B^N, N, v) = Ow(B^n, N, v) = Sh(N, v) \).

In the game \((B, N, u_T)\) of Example 1, the payoffs obtained with the Shapley-Solidarity value are

\[
\xi_1(B, N, u_T) = \frac{1}{2}, \quad \xi_2(B, N, u_T) = \frac{3}{8}, \quad \xi_3(B, N, u_T) = \frac{1}{8}, \quad \xi_4(B, N, u_T) = 0.
\]

The principle of joining productivity and cohesion to reward players inside the union \( \{2, 3\} \) is expressed by the transfer of \( 1/8 \) from player 2 to the null player 3.

### 4 Axiomatic characterization

This section provides an axiomatic characterization of the Shapley-solidarity value \( \xi \). This approach helps us to clarify the differences and similarities with other coalitional values, by looking at the differences and similarities in the properties which characterize these values.

First, we have a look of the unique existent characterization of the solidarity value up to now. In Novak and Radzik (1994), a variation of the null player axiom is introduced as follows: Player \( i \in N \) is an \( A \)-null player in \((N, v)\) if \( \Delta^{av}(v, S) = 0 \) for all coalition \( S \subseteq N \) containing \( i \). The solidarity value satisfies the following axiom in \( \mathcal{G} \):
A-Null player axiom: For all \((N, v)\) and all \(i \in N\), if \(i\) is an A-null player, then \(\gamma_i(N, v) = 0\).

Consider the following properties of a value \(\gamma\) in \(G^N\):

Efficiency: For all \((N, v)\), \(\sum_{i \in N} \gamma_i(N, v) = v(N)\).

Additivity: For all \((N, v)\) and \((N, v')\), \(\gamma(N, v + v') = \gamma(N, v) + \gamma(N, v')\).

Symmetry: For all \((N, v)\) and all \(\{i, j\} \subseteq N\), if \(i\) and \(j\) are symmetric players in \((N, v)\), then \(\gamma_i(N, v) = \gamma_j(N, v)\).

Null player axiom: For all \((N, v)\) and all \(i \in N\), if \(i\) is a null player in \((N, v)\), then \(\gamma_i(N, v) = 0\).

The following theorem is due to Nowak and Radzik (1994).

**Theorem 1** (Nowak and Radzik, 1994) A value \(\gamma\) on \(G^N\) satisfies efficiency, additivity, symmetry and A-null player axiom if, and only if, \(\gamma\) is the solidarity value.

If we compare this theorem with the standard characterization of the Shapley value:

**Theorem 2** (Shapley, 1953b) A value \(\gamma\) on \(G^N\) satisfies efficiency, additivity, symmetry and null player axiom if, and only if, \(\gamma\) is the Shapley value.

It is clear that both values only differ in the treatment of the null players. The null player axiom says that if all the marginal contributions of a player in a game are zero (hence it is not a productive player), then the player should obtain zero. The interpretation of the A-null player is less evident. Notice that \(\Delta^{av}(v, S) = 0\) means that the expected productivity of the players in coalition \(S\) is zero, as

\[
\Delta^{av}(v, S) := \frac{1}{s} \sum_{i \in S} (v(S) - v(S \setminus i))
\]

is the expected variation in the worth of coalition \(S\) when every player in \(S\) has the same chance \(1/s\) to withdraw the game. The A-null player axiom says that when the average productivity of all coalitions to which the player belongs to are zero then he must receive zero. But notice that the spirit behind the solidarity value is based in a sort of cohesion principle which is difficult to express only in individual productivity terms. For that reason, we present an alternative way to formulate the idea that all players are "in the same boat".
Suppose that every player has the same chance to participate in the game. In that case we can interpret expression

\[ E\left[ \gamma_i(N, v) - \gamma_i(N \setminus k, v) \right] := \frac{1}{n} \sum_{k \in N} (\gamma_i(N, v) - \gamma_i(N \setminus k, v)) \]

as the expected variation in the payoff of player \( i \) when each of the players in coalition \( N \) have the same chance \( 1/n \) to withdraw the game\(^4\). We assume that when \( i \) leaves the game he obtains a payoff of zero, that is \( \gamma_i(N \setminus i, v) = 0 \). Then, we ask for a cohesion-type rule expressed by the equality in these expected payoff variations:

**Equal average gains.** For all \( (N, v) \) and all \( \{i, j\} \subseteq N \):

\[ E\left[ \gamma_i(N, v) - \gamma_i(N \setminus k, v) \right] = E\left[ \gamma_j(N, v) - \gamma_j(N \setminus k, v) \right], \]

for all \( \{i, j\} \subseteq N \).

We offer now a new characterization of the solidarity value with the help of this axiom.

**Theorem 3** A value \( \gamma \) on \( \mathcal{G} \) satisfies efficiency and equal averaged gains if, and only if, \( \gamma \) is the solidarity value.

**Proof.** Existence. It is well known that the solidarity value satisfies efficiency. Moreover, the solidarity value can be obtained recursively (see Calvo, 2008) by

\[ Sl_i(S, v) = \frac{1}{S} \Delta^{av}(v, S) + \sum_{j \in S \setminus i} \frac{1}{S} Sl_i(S \setminus j, v), \quad \text{for all } S \subseteq N \text{ and all } i \in S, \]

starting with

\[ Sl_i(\{i\}, v) = v(i), \quad \text{for all } i \in N. \]

Therefore, we have that for all \( \{i, j\} \subseteq N \):

\[ Sl_i(N, v) - \frac{1}{n} \sum_{k \in N \setminus i} Sl_i(N \setminus k, v) = Sl_j(N, v) - \frac{1}{n} \sum_{k \in N \setminus j} Sl_j(N \setminus k, v), \]

and this can be written as

\[ \frac{1}{n} \sum_{k \in N} (Sl_i(N, v) - Sl_i(N \setminus k, v)) = \frac{1}{n} \sum_{k \in N} (Sl_j(N, v) - Sl_j(N \setminus k, v)), \]

\(^4\) Obviously we are in the context of a transferable utility game where it is assumed that players are risk neutral.

\(^5\) If we are in a context where a player can guarantee \( v(i) \) by himself when he leaves the game, then we can restrict ourself to the setting of zero-monotonic games, that is, monotonic games with \( v(i) = 0 \) for all \( i \in N \).
where \( Sl_i(N \setminus i, v) := 0 \).

Thus, the solidarity value satisfies equal averaged gains.

**Uniqueness.** Let \( \gamma \) a value satisfying the above axioms and let \( (N, v) \in G^N \). We prove \( \gamma = Sl \) by induction over the number of players \( n \). If \( n = 1 \), by efficiency, \( \gamma(\{i\}, v) = Sl(\{i\}, v) = v(i) \) and hence the result holds. Assume that it is true for less than \( n \) players. We now prove it for \( n \) players. By equal averaged gains, we have that for all \( \{i, j\} \subseteq N \):

\[
\frac{1}{n} \sum_{k \in N} (\gamma_i(N, v) - \gamma_i(N \setminus k, v)) = \frac{1}{n} \sum_{k \in N} (\gamma_j(N, v) - \gamma_j(N \setminus k, v)) .
\] (3)

By the induction hypothesis, \( \gamma_i(N \setminus k, v) = Sl_i(N \setminus k, v) \), for all \( \{i, k\} \subseteq N \). Therefore, following in (3):

\[
\gamma_i(N, v) - \gamma_j(N, v) = \frac{1}{n} \left[ \sum_{k \in N} (Sl_i(N \setminus k, v) - Sl_j(N \setminus k, v)) \right] .
\]

This expression yields \( (n - 1) \) linearly independent equations, which jointly with the efficiency,

\[
\sum_{i \in N} \gamma_i(N, v) = v(N),
\]

form an \( n \times n \) linear equations system. The matrix of this system is:

\[
A_n = \begin{bmatrix}
1 & -1 & 0 & 0 & \ldots & 0 \\
0 & 1 & -1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & 0 & 1 & -1 \\
1 & 1 & \ldots & 1 & 1 & 1
\end{bmatrix}
\]

We now prove that \(|A_n| = n\). Indeed, we proceed by induction. For \( n = 2 \), we have \(|A_2| = 2\). Assume that it is true for less than \( n \). We now prove it for \( n \). We develop \(|A_n|\) by the elements of the first column:

\[
|A_n| = |A_{n-1}| + (-1)^{n-1} = |A_{n-1}| + (-1)^{n-1} (-1)^{n-1} = n - 1 + 1 = n.
\]
Therefore, \( |A_n| \neq 0 \), which implies that the system has a unique solution. Thus, we conclude that \( \gamma(N, v) = SI(N, v) \).

The above characterization follows a similar approach to the Myerson (1980) balanced contributions characterization of the Shapley value. Myerson (1980) introduced this property as a way to express the principle that the contributions of the players to the game must be balanced. Formally:

**Balanced contributions.** For all \((N, v)\) and all \(\{i, j\} \subseteq N\),

\[
\gamma_i(N, v) - \gamma_i(N \setminus j, v) = \gamma_j(N, v) - \gamma_j(N \setminus i, v).
\]

This property states that for any two players, the amount that each player would gain or lose by the other player’s withdrawal from the game should be equal. In other terms, in the bargaining over the surplus, every pair of players \(\{i, j\}\) are in balance because the loss in the payoff of player \(j\) that a player \(i\) can inflict by withdraw the game is the same that \(j\) can inflict to \(i\).

Then we have:

**Theorem 4 (Myerson, 1980)** A value \(\gamma\) on \(\mathcal{G}\) satisfies efficiency and balanced contributions if, and only if, \(\gamma\) is the Shapley value.

Note that applying \(BC\) over all players (see Hart and Mas-Colell, 1996) we have that

\[
\frac{1}{n - 1} \sum_{k \in N \setminus i} (\gamma_i(N, v) - \gamma_i(N \setminus k, v)) = \frac{1}{n - 1} \sum_{k \in N \setminus i} (\gamma_k(N, v) - \gamma_k(N \setminus i, v)),
\]

which, assuming that \(\gamma_i(N \setminus i, v) = 0\), is equivalent to

\[
\frac{1}{n} \sum_{k \in N} (\gamma_i(N, v) - \gamma_i(N \setminus k, v)) = \frac{1}{n} \sum_{k \in N} (\gamma_k(N, v) - \gamma_k(N \setminus i, v)).
\]

Hence \(BC\) says that the average variation in the payoffs of player \(i\) when every remaining player can leave the game is the same that the average variation in the payoffs of the remaining players when \(i\) leaves the game. This makes more transparent the differences between the competing principle behind the Shapley value and the cohesion principle of the solidarity value.

We are ready now to offer the axiomatic characterization of the Shapley-solidarity value on the family of games with a coalition structure. The competing principle of interaction
among unions is expressed by an axiom of balanced contributions between unions, and
the solidarity within the members inside a union, by an axiom of equal average gains
between the members of the same union.

For all coalitional value $\Phi$ and all $S \subseteq N$, let $\Phi(B, N, v)[S] := \sum_{i \in S} \Phi_i(B, N, v)$. For
all $k \in M$, and all $i \in B_k$, define $B_{-i} := (B_1, ..., B_k \setminus i, ..., B_m)$, that is $B_{-i}$ is the new
coalition structure when player $i$ leaves the game.

E Efficiency. For all $(B, N, v) \in \mathcal{CSG}^N$, $\Phi(B, N, v)[N] = v(N)$.

CBC Coalitional balanced contributions. For all $(B, N, v) \in \mathcal{CSG}^N$ and all $\{k, l\} \subseteq M$,

$$\Phi(B, N, v)[B_k] - \Phi(B \setminus B_l, N \setminus B_l, v) [B_k] = \Phi(B, N, v)[B_l] - \Phi(B \setminus B_k, N \setminus B_k, v) [B_l].$$

The CBC property states that, for all $\{k, l\} \subseteq M$, the contribution of $B_k$ to the total
payoff of the members in $B_l$ must be equal to the contribution of $B_l$ to the total payoff
of the members in $B_k$, hence balanced contributions is applied between unions.

IEAG Intracoalitional equal averaged gains. For all $(B, N, v) \in \mathcal{CSG}^N$, all $k \in M$ and all
$\{i, j\} \subseteq B_k$,

$$\frac{1}{|B_k|} \sum_{t \in B_k} (\Phi_i(B, N, v) - \Phi_i(B_{-t}, N \setminus t, v)) = \frac{1}{|B_k|} \sum_{t \in B_k} (\Phi_j(B, N, v) - \Phi_j(B_{-t}, N \setminus t, v)),$$

where $\Phi_i(B_{-i}, N \setminus i, v) := 0$ for all $i \in B_k$ and all $k \in M$.

The IEAG property states that the expected payoff variation of a player in a union
$B_k$, when every player of this union has the same chance to withdraw the game, is equal
for all players in $B_k$.

The characterization theorem is:

**Theorem 5** The Shapley-solidarity value $\xi$ is the only value in $\mathcal{CSG}$ that satisfies efficiency, coalitional balanced contributions, and intracoalitional equal averaged gains.

**Proof.** Existence. Let $(B, N, v) \in \mathcal{CSG}^N$ be a game. Since the Shapley value and the
solidarity value satisfy efficiency, for all $k \in M$ we have that $\sum_{i \in B_k} \xi_i(B, N, v) = v_k(B_k) = Sh_k(M, v_B)$, and then $\sum_{i \in N} \xi_i(B, N, v) = \sum_{k \in M} \sum_{i \in B_k} \xi_i(B, N, v) = \sum_{k \in M} Sh_k(M, v_B) = v(N).$
Thus \( \xi \) satisfies efficiency. Moreover, since \( \xi(B, N, v)[B_k] = \text{Sh}_k(M, v_B) \) for all \( k \in M \), then \( \xi \) satisfies \( \text{CBC} \) if and only if
\[
\text{Sh}_k(M, v_B) - \text{Sh}_k(M \setminus l, v_B) = \text{Sh}_l(M, v_B) - \text{Sh}_l(M \setminus k, v_B), \quad \text{for all } \{k, l\} \subseteq M.
\]

And this is true because the Shapley value satisfies balanced contributions.

Let \( k \in M \). Taking into account that \( \xi_i(B, N, v) = \text{Sl}_i(B_k, v_k) \) for each \( i \in B_k \), where \( v_k(S) = \text{Sh}_k(M, v_B|S) \) for each \( S \subseteq B_k \), then \( \xi \) satisfies \( \text{IEAG} \) if and only if
\[
\frac{1}{|B_k|} \sum_{t \in B_k} (\text{Sl}_i(B_k, v_k) - \text{Sl}_i(B_k \setminus t, v_k)) = \frac{1}{|B_k|} \sum_{t \in B_k} (\text{Sl}_j(B_k, v_k) - \text{Sl}_j(B_k \setminus t, v_k)), \quad \text{for each } \{i, j\} \subseteq B_k.
\]

And this is true because the solidarity value satisfies equal average gains.

Uniqueness. Let \( \Phi \) be a coalitional value satisfying the above axioms. Let \((N, v) \in \mathcal{G}^N\) be a game, applying \( \text{IEAG} \) for \( B = B^N \), we have that for all \( \{i, j\} \subseteq N \):

\[
\frac{1}{n} \sum_{t \in N} (\Phi_i(B^N, N, v) - \Phi_i(B^{N \setminus t}, N \setminus t, v)) = \frac{1}{n} \sum_{t \in N} (\Phi_j(B^N, N, v) - \Phi_j(B^{N \setminus t}, N \setminus t, v)).
\]

And due to Theorem 3, this expression jointly with efficiency imply that \( \Phi(B^N, N, v) = \text{Sl}(N, v) \) for all game \((N, v) \in \mathcal{G}^N\). Thus, \( \Phi \) is uniquely determined when \(|B| = 1|\).

We now use induction on \(|B|\). Let us assume that the uniqueness is established for \(|B| \leq m - 1\) and let \((B, N, v) \in \mathcal{CSG}^N\) be a game such that \(|B| = m\). By \( \text{CBC} \), for all \( \{k, l\} \subseteq M \):

\[
\Phi(B, N, v)[B_k] - \Phi(B, N, v)[B_l] = \Phi(B \setminus B_l, N \setminus B_l, v)[B_k] - \Phi(B \setminus B_k, N \setminus B_k, v)[B_l].
\]

(4)

The induction hypothesis yields
\[
\begin{align*}
\Phi(B \setminus B_l, N \setminus B_l, v)[B_k] &= \xi(B \setminus B_l, N \setminus B_l, v)[B_k] \\
\Phi(B \setminus B_k, N \setminus B_k, v)[B_l] &= \xi(B \setminus B_k, N \setminus B_k, v)[B_l].
\end{align*}
\]

And, because \( \xi \) satisfies \( \text{CBC} \), we have
\[
\xi(B \setminus B_l, N \setminus B_l, v)[B_k] - \xi(B \setminus B_k, N \setminus B_k, v)[B_l] = \xi(B, N, v)[B_k] - \xi(B, N, v)[B_l].
\]

Therefore, using (4):
\[
\Phi(B, N, v)[B_k] - \Phi(B, N, v)[B_l] = \xi(B, N, v)[B_k] - \xi(B, N, v)[B_l]
\]

implies that
\[
\Phi(B, N, v)[B_k] - \xi(B, N, v)[B_k] = \Phi(B, N, v)[B_l] - \xi(B, N, v)[B_l]
\]
for all \( \{k,l\} \subseteq M \). And then, by efficiency,
\[
\Phi(B, N, v) [B_k] = \xi(B, N, v) [B_k], \quad \text{for all } k \in M. \tag{5}
\]

Let \( k \in M \). We now prove that \( \Phi_i(B, N, v) = \xi_i(B, N, v) \) for all \( i \in B_k \), by induction over the number of players in \( B_k \). If \( |B_k| = 1 \), expression (5) means that \( \Phi_j(B, N, v) = \xi_j(B, N, v) \) for \( \{j\} = B_k \). Suppose that \( |B_k| \geq 2 \). By IEAG, we have for all \( \{i,j\} \subseteq B_k \):
\[
\sum_{t \in B_k} (\Phi_i(B, N, v) - \Phi_i(B_{-t}, N \setminus t, v)) = \sum_{t \in B_k} (\Phi_j(B, N, v) - \Phi_j(B_{-t}, N \setminus t, v)). \tag{6}
\]
By the induction hypothesis:
\[
\begin{aligned}
\Phi_i(B_{-t}, N \setminus t, v) &= \xi_i(B_{-t}, N \setminus t, v) \\
\Phi_j(B_{-t}, N \setminus t, v) &= \xi_j(B_{-t}, N \setminus t, v).
\end{aligned}
\]
Hence, using (6):
\[
\sum_{t \in B_k} (\Phi_i(B, N, v) - \Phi_j(B, N, v)) = \sum_{t \in B_k} (\xi_i(B_{-t}, N \setminus t, v) - \xi_j(B_{-t}, N \setminus t, v)) =
\sum_{t \in B_k} (\xi_i(B, N, v) - \xi_j(B, N, v)).
\]
This implies:
\[
\Phi_i(B, N, v) - \Phi_j(B, N, v) = \xi_i(B, N, v) - \xi_j(B, N, v) \Rightarrow \\
\Phi_i(B, N, v) - \xi_i(B, N, v) = \Phi_j(B, N, v) - \xi_j(B, N, v).
\]
And taking (5) into account, we conclude that \( \Phi_i(B, N, v) = \xi_i(B, N, v) \), for all \( i \in B_k \).

We have used the domain \( CSG \) as the player set \( N \) varies when the CBC and IAEG axioms are applied. Notice the advantage of this characterization over others that use the additivity axiom and a fixed player set \( N \), as it can be applied to any subdomain, provided only that such domain is closed under restrictions in the player set. By the contrary, there are subdomains that are not closed under addition of games, as for example simple games, and then axiom systems with additivity can fail to yield a unique value.

**Remark 1** The axiom system in Theorem 5 is independent. Indeed:

1. Let the coalitional value \( G \) be defined as \( G_i(B, N, v) = 0 \) for all \((B, N, v) \in CSG\) and all \( i \in N \). It satisfies all axioms except efficiency.

2. The Owen value satisfies all axioms, except IEAG.

3. The coalitional value \( \Gamma^{(\text{sl}, \text{st})} \), satisfies all axioms, except CBC.
5 Comparison with other coalitional values

Some other coalitional values have been defined in the literature. We make a sort revision of them.

5.1 The Owen value

As we have already seen, the Owen value was the starting point of coalitional values\(^6\). The main difference with the Shapley-solidarity value lies in that the competing principle to reward players by their productivity is applied not only between unions, but also between the members of the same union. This is expressed in the following axiom:

IBC  Intracoalitional Balanced Contributions. For all \((B, N, v) \in CSG^N\), all \(k \in M\) and all \(\{i, j\} \subseteq B_k\),

\[
\Phi_i(B, N, v) - \Phi_i(B_{-j}, N_{-j}, v) = \Phi_j(B, N, v) - \Phi_j(B_{-i}, N_{-i}, v).
\]

Hence, in the Owen value every null player always receives zero either being isolated or belonging to a union with more partners.

We can compare axiomatically \(Ow\) and \(\xi\) with the following characterization (see Calvo, Lasaga and Winter, 1996, and Amer and Carreras, 1995).

**Theorem 6** The Owen value \(Ow\) is the only value in \(CSG\) that satisfies efficiency, coalitional balanced contributions, and intracoalitional balanced contributions.

We can also mention two weighted versions of the Owen value. One is due to Levy and McLean (1989) and the other to Vidal-Puga (2012). In both versions the weighted Shapley value is applied in the game between unions with weights proportional to the size of the unions. Both values differ in the definition of the internal game.

\(^6\)Recall that we restrict our attention to coalitional values that satisfy efficiency. Hence, the coalition structure is an additional element which influence the way in which the worth of the grand coalition is shared among its members. This means that we left out of our analysis component-wise efficient values, i.e. values wich satisfy \(\sum_{i \in B_k} \Phi_i(B, N, v) = v(B_k)\) for all \(B_k \in B\) as, for example, the Aumann-Dreze (1974) value.
For all \( w \in \mathbb{R}^{N}_{++} \), the **weighted Shapley value** \( Sh^w \) is the linear map \( Sh^w : \mathcal{G}^{N} \rightarrow \mathbb{R}^{N} \), which is defined for each unanimity game \((N, u_T)\) as follows

\[
Sh_i^w (N, u_T) = \begin{cases} 
\frac{w_i}{\sum_{j \in T} w_j} & \text{for all } i \in T, \\
0 & \text{otherwise.}
\end{cases}
\]

In Levy and McLean, the internal game \((B_k, v^*_k)\) is defined by setting \( v^*_k (S) = Sh^w_k (M, v_{B|S}) \) for all \( S \subseteq B_k \), with weights \( w_r = |B_r| \) for all \( B_r \in B \). The coalitional value \( \Gamma^{(Sh^w, Sh)} \) is defined by

\[
\Gamma^{(Sh^w, Sh)}_i (B, N, v) := Sh_i (B_k, v^*_k), \quad \text{for all } k \in M \text{ and all } i \in B_k.
\]

In Vidal-Puga, the internal game \((B_k, v'_k)\) is defined by setting \( v'_k (S) = Sh^{w'}_k (M, v_{B|S}) \) for all \( S \subseteq B_k \), with weights \( w'_r = |B_r| \) for all \( B_r \in B \setminus B_k \), and \( w'_l = |S| \). The coalitional value \( \Gamma^{(Sh^{w'}, Sh)} \) is defined by

\[
\Gamma^{(Sh^{w'}, Sh)}_i (B, N, v) := Sh_i (B_k, v'_k), \quad \text{for all } k \in M \text{ and all } i \in B_k.
\]

In \( \Gamma^{(Sh^w, Sh)} \) the weight of every subcoalition \( S \subseteq B_k \) is always \( |B_k| \) and in \( \Gamma^{(Sh^{w'}, Sh)} \) this weight decreases with the size of \( S \).

These values try to prevent what is called the Harsanyi paradox. As Harsanyi (1977) points out, Owen’s approach assumes a symmetric treatment for each union and this procedure implies that, in unanimity games, players would be better off bargaining by themselves than joining forces. For example, consider \( N = \{1, 2, 3, 4\} \) and the unanimity game \( u_N \). By symmetry it holds that \( Ow_i (B^n, N, u_N) = 1/4 \) for all \( i \in N \). If players 2 and 3 join into union \( \{2, 3\} \) as in our Example 1, \( B = \{B_r = \{1\}, B_k = \{2, 3\}, B_l = \{4\}\} \), it holds that in the quotient game all unions are symmetric again and then the aggregated payoff corresponding to union \( \{2, 3\} \) is 1/3. Therefore the payoff of each player 2 and 3 will be 1/6, lower than their initial payoffs. On the contrary, if we apply the weighted Shapley value \( Sh^w \) in the quotient game \((M, (u_N)_B)\) with weights \( (w_r = 1, w_k = 2, w_l = 1) \), we obtain

\[
Sh^w_r (M, (u_N)_B) = Sh^w_i (M, (u_N)_B) = 1/4, \quad Sh^w_k (M, (u_N)_B) = 1/2,
\]

and by symmetry between 2 and 3, we back to the initial payoffs of 1/4 for each player.
Unfortunately this paradox cannot be always prevented. Consider the following symmetric\textsuperscript{7} monotonic game.

**Example 2** Let \((N, v)\) be the game where \(N = \{1, 2, 3, 4\}\) and

\[
v(S) = \begin{cases} 
9, & \text{if } |S| = 4, \\
5, & \text{if } |S| = 3, \\
4, & \text{if } |S| = 2, \\
2, & \text{if } |S| = 1.
\end{cases}
\]

When all players act as singletons, by symmetry, they obtain 2.25 each one: \(Ow_i(B^n, N, v) = 2.25, \ i \in N\). If players 2 and 3 joint into a union \(B_k = \{2, 3\}\), as in Example 1, their payoffs decrease with the Owen value:

\[Ow_i(B, N, v) = 2, \ i \in \{2, 3\}.\]

And it happens the same with the weighted versions:

\[
\Gamma_i^{(Sh^w, Sh)}(B, N, v) = \Gamma_i^{(Sh^w, Sh)}(B, N, v) = 2.083, \ i \in \{2, 3\}.
\]

It is also possible to find games in which coincides the Owen value with these two weighted versions: Let \((N, v')\) be the game where

\[
v'(S) = \begin{cases} 
8, & \text{if } |S| = 4, \\
4, & \text{if } |S| = 3, \\
4, & \text{if } |S| = 2, \\
1, & \text{if } |S| = 1.
\end{cases}
\]

Here \(Ow_i(B^n, N, v') = 2, \ i \in N\), and when 2 and 3 joint into a union \(B_k = \{2, 3\}\), it holds that

\[Ow_i(B, N, v) = \Gamma_i^{(Sh^w, Sh)}(B, N, v) = \Gamma_i^{(Sh^w, Sh)}(B, N, v) = 1.833, \ i \in \{2, 3\}.
\]

Hence, this type of paradox can only be solved by particular class of games, as convex games (see Proposition 3.1 in Vidal-Puga, 2012) or unanimity games; but not in general.

Moreover, even in unanimity games, applying weighted versions yields problematic consequences. For example, in Example 1, in the unanimity game \(u_T\) with \(T = \{1, 2\}\), it holds that

\[Ow_1(B^n, N, u_T) = 1/2,\]

\textsuperscript{7}A symmetric game is a game in which the worth of a coalition is a function of its size.
and when union $B_k = \{2, 3\}$ is formed it holds that

$$
\Gamma_1^{(Sh^w, Sh)}(B, N, u_T) = \Gamma_1^{(Sh^{w'}, Sh)}(B, N, u_T) = 1/3.
$$

And if we add the null player 4 into the union, as $B' = \{B_r = \{1\}, B_k = \{2, 3, 4\}\}$, it holds that

$$
\Gamma_1^{(Sh^w, Sh)}(B, N, u_T) = \Gamma_1^{(Sh^{w'}, Sh)}(B, N, u_T) = 1/4.
$$

Player 1 decreases his bargaining power only because player 2 adds null players to his union, whereas 1 remains as a symmetric player in the quotient game between unions. When unions bargain over the surplus applying the productivity principle what is relevant should be the quotient game, which informs us about the *worth of the coalition of unions* and nothing else, being irrelevant the size of the unions that form such coalition. It is in the second stage, when the reward that a union has obtained must be shared between their members, when obviously the size of the union matters. If the size of a union should be taken into account in the quotient game, because we wish to prevent that players which belongs to unions of big size receive very little, perhaps should be better applying a different value in the quotient game so that this type of ethical considerations were incorporated in its definition.

Notice that the payoffs behavior of the members inside each union differs between $\Gamma^{(Sh^w, Sh)}$ and $\Gamma^{(Sh^{w'}, Sh)}$, because $\Gamma^{(Sh^w, Sh)}$ satisfies the null player axiom and $\Gamma^{(Sh^{w'}, Sh)}$ do not.

$$
\Gamma_2^{(Sh^w, Sh)}(B, N, u_T) = 2/3, \quad \Gamma_3^{(Sh^w, Sh)}(B, N, u_T) = 0,
$$

and

$$
\Gamma_2^{(Sh^{w'}, Sh)}(B, N, u_T) = 7/12, \quad \Gamma_3^{(Sh^{w'}, Sh)}(B, N, u_T) = 1/12.
$$

The reason why $\Gamma^{(Sh^{w'}, Sh)}$ yields a positive payoff to the null player 3 is because the bargaining power of a coalition changes when its size changes. This type of consideration is completely different than the cohesion principle between the members of the same union that inspires the Shapley-solidarity value.

From an axiomatic viewpoint, both values satisfy the same weighted version of the coalitional balanced contributions axiom, as each union receives the weighted Shapley value in the quotient game.

**CPBC**  *Coalitional per capita balanced contributions.* For all $(B, N, v) \in CSG^N$ and all
\{k, l\} \subseteq M, \\
\frac{1}{|B_k|} \left[ \Phi(B, N, v) [B_k] - \Phi(B \setminus B_l, N \setminus B_l, v) [B_k] \right] = \frac{1}{|B_l|} \left[ \Phi(B, N, v) [B_l] - \Phi(B \setminus B_k, N \setminus B_k, v) [B_l] \right].

The property says that the average amount that players in each union would gain or lose by the other union’s withdrawal from the game should be equal. The average is taken over the number of players in each union (this property was introduced with this name in Gómez-Rúa and Vidal-Puga, 2011).

The values differ in the internal game, therefore the competing principle is expressed in a different way.

For the Levy and McLean value, a slighted modification of balanced contributions is used, making a player as a null player instead of leaving him out the game: Given \((N, v)\) and \(i \in N\), define \((N, v^{-i})\) as \(v^{-i}(S) = v(S \cap (N \setminus i))\) for all \(S \subseteq N\).

**INBC Intracoalitional null balanced contributions.** For all \((B, N, v) \in \mathcal{CSG}^N\), all \(k \in M\) and all \(\{i, j\} \subseteq B_k\),

\[\Phi_i(B, N, v) - \Phi_i(B, N, v^{-j}) = \Phi_j(B, N, v) - \Phi_j(B, N, v^{-i}).\]

Moreover, a symmetry within a union is needed.

**IS Intracoalitional symmetry:** For all \((B, N, v) \in \mathcal{CSG}^N\), all \(k \in M\) and all \(\{i, j\} \subseteq B_k\),

if \(i\) and \(j\) are symmetric players in \((N, v)\), then \(\Phi_i(B, N, v) = \Phi_j(B, N, v)\).

Then, we have two characterizations.

**Theorem 7** The value \(\Gamma^{(Sh^w, Sh)}\) is the only value on \(\mathcal{CSG}\) that satisfies efficiency, coalitional per capita balanced contributions, intracoalitional symmetry and intracoalitional null balanced contributions.

**Theorem 8** The value \(\Gamma^{(Sh^w', Sl)}\) is the only value on \(\mathcal{CSG}\) that satisfies efficiency, coalitional per capita balanced contributions and intracoalitional balanced contributions\(^8\).

**Remark 2** Anyway, if one think that should be compulsory to apply a weighted Shapley value in the quotient game, then a weighted Shapley-solidarity value can also be defined by using one of the two weighted versions of the internal game: Either \(v^*_k\) or \(v'_k\). That is, it can be defined either \(\Gamma^{(Sh^w, Sl)}(B, N, v)\) or \(\Gamma^{(Sh^w', Sl)}(B, N, v)\).

---

\(^8\)The proof of Theorem 8 can be found in Gómez-Rúa and Vidal-Puga (2011).
5.2 Two-step Shapley value and collective value

Kamijo defined two new coalesional values, named the two-step Shapley value \((K)\) (Kamijo, 2009) and the collective value \((K^w)\) (Kamijo, 2011). At the first level, the Shapley value (respectively the weighted Shapley value) is used to determine the aggregate reward of each union in the quotient game. At the second level, within each union \(B_k\), players take as the status quo point the Shapley value of the game restricted to the union, that is \(Sh(B_k, v)\) (the productivity component of the rule); and the bargaining surplus of the union, \(Sh_k(M, v_B) - v(B_k)\) (respectively \(Sh^w_k(M, v_B) - v(B_k)\)), is shared equally between their members (the solidarity component).

For all game \((B, N, v) \in CSG^N\), the two-step Shapley value of \((B, N, v)\) is given by the formula:

\[
K_i (B, N, v) = Sh_i (B_k, v) + \frac{1}{|B_k|} [Sh_k (M, v_B) - v(B_k)], \text{ for all } k \in M \text{ and all } i \in B_k,
\]

and the collective value is given by

\[
K^w_i (B, N, v) = Sh_i (B_k, v) + \frac{1}{|B_k|} [Sh^w_k (M, v_B) - v(B_k)], \text{ for all } k \in M \text{ and all } i \in B_k,
\]

where \(w_k = |B_k|\) for all \(k \in M\).

In our Example 1, the two-step Shapley value, \(K\), yields

\[
K_1 = \frac{1}{2}, \quad K_2 = \frac{1}{4}, \quad K_3 = \frac{1}{4}, \quad K_4 = 0,
\]

and the collective value, \(K^w\), yields

\[
K^w_1 = \frac{1}{3}, \quad K^w_2 = \frac{1}{3}, \quad K^w_3 = \frac{1}{3}, \quad K^w_4 = 0,
\]

Player 3 now obtains the same as player 2 in both values. However, this egalitarian way to share the aggregated gains of a union seems rather unfair from the productivity point of view, as 3 is a null player that does not contribute to the rewards of the union \(\{2, 3\}\).

The following axioms can be used to characterize these values. Firstly, it is used the balanced contribution axiom but applied only to the trivial coalition structure \(B^N\).

BC\{N\} Balanced contributions in \(B^N\). For all \((B^N, N, v) \in CSG^N\), and all \(\{i, j\} \subseteq N\),

\[
\Phi_i (B^N, N, v) - \Phi_i (B^N_{-j}, N \setminus j, v) = \Phi_j (B^N, N, v) - \Phi_j (B^N_{-i}, N \setminus i, v).
\]
According to the next axiom, two players in distinct unions are affected equally by the deletion of the union associated with the other player.

\textbf{CllBC}  \textit{Collective Balanced Contributions.} For all \((B, N, v) \in \mathcal{CSG}^N\) with \(|B| \geq 2\), all \(\{k, h\} \subseteq M\ (k \neq h)\), all \(i \in B_k\) and all \(j \in B_h\),

\[\Phi_i(B, N, v) - \Phi_i(B \setminus B_h, N \setminus B_h, v) = \Phi_j(B, N, v) - \Phi_j(B \setminus B_k, N \setminus B_k, v).\]

An aggregated version of the above axiom is

\textbf{ABC}  \textit{Aggregate Balanced Contributions.} For all \((B, N, v) \in \mathcal{CSG}^N\) with \(|B| \geq 2\), all \(\{k, h\} \subseteq M\ (k \neq h)\), all \(i \in B_k\) and all \(j \in B_h\),

\[|B_k| \left[\Phi_i(B, N, v) - \Phi_i(B \setminus B_h, N \setminus B_h, v)\right] = |B_h| \left[\Phi_j(B, N, v) - \Phi_j(B \setminus B_k, N \setminus B_k, v)\right].\]

Then, we have:

\textbf{Theorem 9}  The two-step Shapley value \(K\) is the only value on \(\mathcal{CSG}\) that satisfies efficiency, balanced contributions in \(B^N\) and aggregated balanced contributions.

\textbf{Theorem 10}  The collective value \(K^w\) is the only value on \(\mathcal{CSG}\) that satisfies efficiency, balanced contributions in \(B^N\) and collective balanced contributions.

The proof of Theorem 10 can be found in Kamijo (2011) and the proof of Theorem 9 follows the same lines and it is left to the reader\(^9\).

\section{5.3 The Hamiache value}

In Hamiache (2006), a coalitional value which in unanimity games allocates to bigger unions a large share of the total worth is considered. What is relevant for our discussion is that this value yields a zero payoff for all null players. Moreover, the value satisfies what Hamiache called \textit{independence of irrelevant players}: The value does not change if we withdraw null players of the game. This implies that the payoffs in our Example 1 are

\[H_1(B, N, u_T) = H_2(B, N, u_T) = 1/2, \quad H_3(B, N, u_T) = H_4(B, N, u_T) = 0.\]

\(^9\)An alternative characterization of \(K\) can be found in Calvo and Gutiérrez (2010).
Hence, we can apply here the same criticism of a lack of solidarity with the null players of the union as in the Owen and the Levy and McLean values.

We summarize this section with two tables.

In the first table, we can compare the payoffs that the coalitional values yield in the unanimity game of Example 1.

<table>
<thead>
<tr>
<th>((B, N, u_T))</th>
<th>(\Phi_1)</th>
<th>(\Phi_2)</th>
<th>(\Phi_3)</th>
<th>(\Phi_4)</th>
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<tbody>
<tr>
<td>(\xi)</td>
<td>1/2</td>
<td>3/8</td>
<td>1/8</td>
<td>0</td>
</tr>
<tr>
<td>(Ow)</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma^{(Sh^w, Sh)})</td>
<td>1/3</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma^{(Sh^w, Sh)})</td>
<td>1/3</td>
<td>7/12</td>
<td>1/12</td>
<td>0</td>
</tr>
<tr>
<td>(K)</td>
<td>1/2</td>
<td>1/4</td>
<td>1/4</td>
<td>0</td>
</tr>
<tr>
<td>(K^w)</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>(H)</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the second table, we show the properties that these coalitional values satisfy (\(x^*\) means that the property is used in the characterization of the value)\(^{10}\).

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>CBC</th>
<th>CPBC</th>
<th>IEAG</th>
<th>IBC</th>
<th>INBC</th>
<th>IS</th>
<th>BC{N}</th>
<th>ABC</th>
<th>CllBC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi)</td>
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<td>x^*</td>
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<td>x^*</td>
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<td>-</td>
<td>x</td>
<td>-</td>
<td>-</td>
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</tr>
<tr>
<td>(Ow)</td>
<td>x^*</td>
<td>x^*</td>
<td>-</td>
<td>-</td>
<td>x^*</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(\Gamma^{(Sh^w, Sh)})</td>
<td>x^*</td>
<td>-</td>
<td>x^*</td>
<td>-</td>
<td>x^*</td>
<td>x</td>
<td>x</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(\Gamma^{(Sh^w, Sh)})</td>
<td>x^*</td>
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<td>x^*</td>
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<td>x</td>
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</tr>
<tr>
<td>(K)</td>
<td>x^*</td>
<td>x</td>
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<td>x</td>
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<tr>
<td>(K^w)</td>
<td>x^*</td>
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<td>x</td>
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<td>x</td>
<td>x</td>
<td>x^*</td>
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<td>x^*</td>
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</tbody>
</table>

6 Conclusion

We have presented in this work a new value for cooperative games with coalition structures. We have taken as starting point the Owen value, in which the competing principle to reward players by their productivity is applied between unions and between the members of the same union. In the rewards within members of the same union we have replaced the

\(^{10}\)There is not an equivalent characterization of the Hamiache value with variations of the balanced contributions axiom.
productivity principle by a new one, which exhibits a greater degree of solidarity among them. This principle is expressed formally by an axiom called \textit{intracoalitional equal average gains}. It says that the expected payoff variation of a player in a union, when every player of this union has the same chance to withdraw the game, is equal for all players of the union.

We have seen that this implies the use of the solidarity value in the internal game if we want to compute a coalitional value which satisfies efficiency, coalitional balanced contributions and intracoalitional equal averaged gains.

We argue that this value is a good compromise between the productivity and the solidarity principles: it takes into account the productivity principle as well, as the players’ individual marginal contributions are used in the calculation. Hence, if a player increases his productivity he will increases his payoffs. However, it also exhibits a redistribution effect, as it not only takes into account his own marginal contribution, but also the marginal contributions of the remaining players, in such a way that the own marginal contribution is replaced in the computation of the value by the average of the marginal contributions of all players in the coalition.

The redistribution effect inherent to the IEAG axiom is shown by the fact that null players can receive positive payoffs, as in the unanimity game considered in Example 1. This is in contrast with values that yield zero payoffs to null players, as the Owen (1977) value, the weighted version $\Gamma^{(\text{Sh}^w,\text{Sh})}$ of Levy and McLean (1989) and Hamiache (2006) value do. On the other hand, it still maintains incentives to the players’ productivity, as it differentiates between null and non null players within the union. This is in contrast with the two-step Shapley value (Kamijo, 2009) and the collective value (Kamijo, 2011), in which both type of players receive the same in this example. Only the value considered in Vidal-Puga (2012) yields a positive payoff to the null player, but less than the payoff of the productive player. However the reason to yield a reward to the null player in Example 1 is of different nature. In the Shapley-solidarity value comes from a solidarity behavior between the members of the union, in the Vidal-Puga’s value comes from the fact that adding the null player increases the size of the union which increases the bargaining power of the union. Whether or not the size of a union should be relevant for the bargaining in the quotient game is a bit controversial.

As it has been mentioned in the Introduction, there is open the possibility to consider
alternative values in the *internal game*, where the null player axiom is not satisfied. For example, the kernel (Davis and Maschler), the nucleolus (Schmeidler, 1969), the egalitarian Shapley values (Joosten, 1996), the consensus value (Ju, Borm and Ruys, 2007), and the weighted coalitional Lorenz solutions (Arin and Feltkamp, 2002). This approach can be object of further research.

We wish to finish mention that in a previous version of this work (Calvo and Gutiérrez, 2011) it can be found an alternative axiomatic characterization of the Shapley-solidarity value. In the set of axioms appears additivity and a consistency property. Then it is proved that the only difference between the Owen and the Shapley-solidarity value goes on in replacing the null player by the A-null player axiom in the axiom system. Moreover, it is also shown how to compute the value by using the random order approach, in a similar way as the Owen value does.

## References


