As many data-driven fields, finance is rich in problems requiring high computational power and intelligent systems techniques. In particular, the problem of selecting an optimal financial portfolio can be conveniently represented as a constrained optimization problem or a decision-making problem. The aim of this paper is to show how to express the optimal portfolio selection problem from a decision-theoretic perspective and show how to address this problem using fuzzy measures and fuzzy integrals. © 2010 Wiley Periodicals, Inc.

1. INTRODUCTION

Over the past four thousand years, finance has been studied from various perspectives ranging from basic arithmetic to probabilistic techniques and stochastic modeling, and machine learning approaches. This evolution, from understanding simple one-asset investments to creating financially optimal portfolios, has required the development of modern techniques to collect, validate, analyze, and integrate data that dynamically change. Therefore, the need for advanced mathematical and computational techniques is now a necessity in understanding modern finance, in particular, pricing, investments, and evolution of financial markets.

Numerous problems in the area of finance use computational techniques (e.g., data mining, machine learning, stochastic differential equations) to solve problems such as option pricing and optimal portfolio selection. There is a clear need for the development of intelligent system techniques to analyze large financial data sets and extract relevant information for portfolio selection (see, e.g., Refs. 2 and 3). Option pricing is another problem in finance that heavily relies on the use of computational method to predict the future behavior of investment assets. Pricing theory is tackled...
mostly from a stochastic perspective, using models such as Black–Scholes (see, e.g., Ref. 4).

The problem that we are particularly interested in this paper is the problem of optimal portfolio selection. We first present a set of computational techniques that have been used to select optimal portfolios, along with the hypotheses made on the portfolios. We also present the advantages of disadvantages of each approach. Then we present decision theory from a utility perspective, and how fuzzy measures and fuzzy integrals can be used to identify optimal decisions. Finally, we reframe the problem of optimal portfolio selection in a decision-theoretic context and show a new method based on fuzzy integration to solve this problem.

2. OPTIMAL PORTFOLIO SELECTION

A portfolio is a distribution of wealth among several investment assets such as stocks, bonds, and their derivatives. Stock holders are the owners of the company, so they receive a part of the earning of the company in terms of dividends—monthly or yearly payments that can last indefinitely. On the other hand, bond holders are lenders to the company. They get compensated for lending money to the company through the interest until the maturity date when all borrowed money is to be returned by the company. Derivatives are financial contracts between two parties and come in several flavors. Futures, options, and swaps are the most common derivatives of stocks or bonds. Both future and option contracts are set for a simple asset (stock or bond) to be bought or sold on (or until) a specific date in future at a predefined price. The only difference is that there is an obligation to buy/sell the future, and there is no obligation but rather an option to buy/sell the option. While futures and options are contracts that involve selling or buying a simple asset, swap is an agreement between two parties to exchange one stream of cash flow for another stream. The best known swap contract is the international currency exchange.

For a rational investor, the leitmotiv to ensure an increase in wealth is to diversify risk. With numerous possible combinations of assets, the goal is to select the optimal portfolio, where the optimality is defined based on the investor’s goal. The two most commonly sought goals are maximization of the return for a given acceptable level of risk and minimization of risk to obtain a predefined level of return. Typically, the higher the value of the expected return, the higher the value of risk associated with the asset, since one of the implicit rules of investments is that it is not possible to increase wealth without taking risks, and arbitrage ensures that the rule holds.

Moreover, regardless of the objective, there are usually several constraints imposed on the solution. Time horizon is certainly one of the constraints since there is usually time period allocated to accomplish a defined goal. Another constraint involving time is time to maturity, which is the remaining time until the money could be taken out of the assets in which it is invested. The fee that needs to be paid to transfer wealth from one asset to another, called transaction fee, complicates the selection of optimal portfolio even more since it can be defined in different ways. The transaction cost might be represented as a fixed amount of wealth or an amount of
wealth proportional to the amount being transferred. Finally, sometimes individuals have preferable portfolio structure. For example, a person close to retirement is usually risk averse, so this individual would prefer to invest money in less risky government bonds. On the other hand, an individual that needs to gain a large amount of money in a short period of time might be risk prone and want to invest in at least one highly profitable but also highly risky stock. Time horizon, time to maturity, transaction cost, and preferable portfolio structure are only a few of numerous possible constraints to selection of optimal portfolio.

The simplest and most natural way to represent this type of problems is as a constrained optimization problem. The objective is to maximize or minimize an objective function (usually maximization of return or minimization of risk) subject to constraints such as nonnegativity of weights (i.e., amount of money allocated to each asset), maximum amount of money available and (often) the requirement that exactly all money is invested, maximal level of risk acceptable, and minimal return required among others.

The problem can simply be represented as follows:

\[
\text{maximize (or minimize)} \quad \sum_{i=1}^{m} w_i x_i, \quad (1)
\]

subject to constraints such as

\[
w_i \geq 0 \quad \forall i \in \{1, 2, \ldots, m\} \quad (2)
\]

\[
\sum_{i=1}^{m} w_i r_i \leq \text{risk} \quad (3)
\]

\[
\sum_{i=1}^{m} w_i R_i \geq \text{return} \quad (4)
\]

\[
\sum_{i=1}^{m} w_i = 1 \quad (5)
\]

where \(x_i\) is either return (in maximization problems) or risk (in minimization problems), \(m\) is the number of investment assets, \(w_i\) is the proportion of wealth invested in the asset \(i\), \(R_i\) is the return rate of the asset \(i\), \(r_i\) is the risk of the asset \(i\), \(\text{return}\) is required level of return, and \(\text{risk}\) is the level of risk acceptable by the investor. As presented above, selecting the optimal portfolio seems to be a straightforward linear programming problem. However, this representation is just the simplest problem that we can face when looking for the optimal portfolio. In general, many more constraints are imposed on the solution. Moreover, the objective function and the constraints are usually much more complex if more information, such as transaction cost, time period, preferable portfolio structure, relationships between characteristics.
of assets, etc., are taken into consideration. A real-life problem of portfolio selection is most commonly a nonlinear optimization problem with constraints that is usually not (easily) solvable using general constraint solving techniques.

Regardless of the setting and its complexity, in the portfolio optimization problem, we aim at finding the vector of weights (i.e., amount of wealth allocated into each asset), \( w = (w_1, w_2, \ldots, w_m) \), given all the other parameters. However, the risk and the return of each asset are predicted rather than certain values, so the uncertainty of the values complicates even more the process of selecting the optimal portfolio.

### 2.1. Existing Techniques

Numerous techniques have been used to solve the problem of selecting the optimal portfolio including return-based strategies, methods involving stochastic processes, and intelligent systems techniques. Return-based strategies, as the name suggests, take into consideration only the return of assets and aim at maximizing the return. These methods usually use the simple linear programming techniques to find the optimal portfolio. Methods involving stochastic processes mostly focus on predicting the behavior of assets rather than finding the optimal distribution of wealth among the assets, but of course, predicting the return and the risk of assets is an important part in making the decisions of where to invest money. Intelligent systems techniques are heuristic algorithms that usually find a solution that is not necessarily the optimal one but is close to the optimal solution. These algorithms are used when linear programming and other simple methods are not applicable due to complexity of the objective function and the constraints. Commonly used intelligent system techniques are genetic algorithms,\(^5\)−\(^7\) rule-based expert systems,\(^8\)−\(^10\) neural networks,\(^11\)−\(^15\) and support vector machines.\(^16\)

In Ref. 7, the authors developed a genetic algorithm (GA) to select the highest performing asset among thousands of available assets. The selection of the best asset is based purely on the return of assets, which is measured by the return on capital employed, price per earning ratio, earning per share, and liquidity ratio. This idea is further used in the two-stage GA presented in Ref. 5, where a predefined number of highest performing assets is selected to proceed to the second stage and the second stage utilizes another GA to distribute the wealth among selected assets. This stage of the algorithm takes into consideration the trade-off between the risk and the return of assets. The algorithm was tested on data obtained from Shanghai Stock Exchange during a period ranging from January 2001 to December 2004 and has proven to outperform the equally weighted portfolio, which was used as a benchmark portfolio. With several modifications in selection, cross-over, and mutation stages of the genetic algorithm, a similar genetic algorithm was presented in Ref. 6. It was tested using data from the Australian Stock Exchange and showed to perform just slightly weaker but ran much faster than greedy algorithm, which was used as a benchmark algorithm.

A rule-based expert system has found its first application in portfolio selection in Ref. 8, where the developed expert system interviews the investor through a set
of predefined questions, infers the goals of the investor based on answers during the interview, and calculates the best distribution of wealth based on the inferred goals. A more complex system was proposed in Ref. 10 and implemented in Ref. 17, where a multiagent system for portfolio monitoring utilizes several agents with different duties including performing interview, finding relevant information in online news, monitoring portfolio risk, and suggesting to the investor the need for modification of the portfolio. The rule-based expert systems models have not been tested on real data, but they give a framework that represents all the steps used by investment consultants.

Neural networks (NNs) have found several applications in portfolio management ranging from forecasting the behavior of investment assets to optimizing the distribution of wealth among assets. Optimal distribution of wealth has been determined in Ref. 13 by risk minimization, where the risk is defined as the deviation of the predicted risk from the actual risk. Its performance was tested on stocks in the U.K. market in 1989. A different approach to optimal portfolio selection was presented in Ref. 11, where all the money is invested in one of three classes of assets: stocks, bonds, and money markets. A neural network was designed to predict which class would outperform the other two classes to invest money in the best performing class for a time interval. At the end of the time interval, the behavior of assets is predicted again and the money is invested accordingly. The prediction of the behavior is based on factors such as earnings, price per earning ratio, interest rate, and inflation. Data from S&P500 Index were used to test the performance of the algorithm, which outperformed buy-and-hold strategy which was used as a benchmark algorithm. Another example of forecasting ability of a NN was tested in Ref. 15, where the forecasting is based on the previous state of the asset, external influences, and the difference between the predicted output and the observed output at the previous iteration. Furthermore, the excess return of one asset over another asset is calculated and used to determine the optimal distribution of wealth. The model was tested on G7 countries markets in the period from July 1993 to May 1995, and it outperformed the benchmark. The asset allocation step of this algorithm was improved in Ref. 14 by incorporating the risk of an asset through calculating risk-adjusted excess return of one asset over another asset. This algorithm outperformed the benchmark portfolio when tested on the German stock market data from July 1999 to June 2000. Finally, a NN model for wealth distribution presented in Ref. 12 was built to satisfy some predefined preferred characteristics of portfolio. The model outperformed the benchmark portfolio when tested on Toronto Stock Exchange market data from January 1971 to July 1996.

Support vector machine has found implementation in classification of stocks into one of two classes: the stocks with exceptional high returns and the stocks with unexceptional returns.16 The algorithm was designed to minimize the generalization error rather than the empirical error. For testing purposes, factors such as return on capital, profitability, leverage, investment, growth, short-term liquidity, return on investment, and risk have been used. The stocks were classified based on data from Australian Stock Exchange between 1992 and 2000, and the stocks classified in the group with exceptional return were included in the equally weighted portfolio which outperformed the benchmark portfolio.
2.2. Drawbacks of the Existing Techniques

We have presented several attempts to use intelligence systems in portfolio management. Genetic algorithms, rule-based expert systems, neural networks, and support vector machines have all contributed toward finding the optimal distribution of wealth among available assets. However, with the exception of genetic algorithm, all other methods are based on the ability to learn from examples and approximation of algorithm’s parameters due to training samples. This could lead to overfitting of the parameters to specific type of data or a specific sample, which might not be applicable in other situations.

Moreover, all of the approaches do not consider the relationship between the characteristics of an asset. For example, return and risk are known to usually move in the same direction, that is higher the return of the asset, higher the risk of that asset. However, the presented approaches do not take into consideration this and many other existing relationships.

Furthermore, the return, the risk, and other characteristics of an asset are assumed to be precisely known for each asset in consideration. In reality, this is not always the case as the best we can do is to predict the future return and risk. Sometimes, these predictions are not correct, but all the presented techniques rely on precise knowledge of these values.

To face the drawbacks of the presented approaches, we propose a new approach to portfolio optimization. The novel approach is based on multicriteria decision making and fuzzy integration over intervals.

3. MULTICRITERIA DECISION MAKING

A multicriteria decision-making problem compares multidimensional alternatives to select the optimal one. As a simple illustration of the problem, we will consider the problem of selecting the best stock for investment, but a similar setting could be designed for investing money in several assets. It can formally be defined as a triple \((X, I, (\succeq_i)_{i \in I})\) where

- \(X \subseteq X_1 \times \cdots \times X_n\) is the set of alternatives with each set \(X_i\) representing a set of values of the attribute \(i\), e.g., \(X = \{\text{stock}_1, \text{stock}_2, \ldots, \text{stock}_N\}\).
- \(I\) is the (finite) set of criteria (or attributes), each containing a finite set of possible values. e.g., \(I = \{\text{risk, time to maturity, reputation of company}\}\) with \(\text{risk} = \{r_1, r_2, \ldots, r_{20}\}\), where \(r_1 = 0–5\%, r_2 = 5–10\%, \ldots, r_{20} = 95–100\\%\).
- \(\forall i \in I, \succeq_i\) is a preference relation (a weak order) over \(X_i\), e.g., \(\text{risk: } r_1 \succeq r_2 \succeq \cdots \succeq r_{20}\).

The first challenge of multicriteria decision-making problem is to “combine” the values of all criteria into a global value for the alternative such that the final order of the alternatives is in the agreement with the decision maker’s partial preferences. The natural way to construct a global preference is by using utility function for each criterion to reflect partial preferences of a decision maker, and then combine these monodimensional utilities into a global utility function using an aggregation operator. The utility functions \(u_i : X_i \to \mathbb{R}\) such that for all \(x_i, y_i \in X_i, u_i(x_i) \succeq u_i(y_i)\).
ui(y_i) if and only if \(x_i \succeq_i y_i\), scale the values of all attributes onto a common scale. The existence of monodimensional utility functions is guaranteed under relatively loose hypotheses by the work presented in Ref. 18.

Different approaches exist to combine monodimensional utilities into a global value. One of the simplest methods is maximax approach, which corresponds to optimistic situations in which we consider only the criterion with the highest utility and ignore all the other criteria. Similarly, maximin approach corresponds to a pessimistic situation and chooses the alternative with the least worst utility. Both approaches are very simple but usually not realistic approaches unless in critical situations where completely optimistic/pessimistic behavior is necessary.

Usually, we need to consider more complex aggregation operators that take into consideration all attributes. The simplest and most natural of them is a weighted sum approach, in which the decision-maker is asked to provide weights, \(w_i\), that reflect the importance of each criterion. Thus, the global utility of alternative \(x = (x_1, \ldots, x_n) \in X\) is given by

\[
    u(x) = \sum_{i=1}^{n} w_i u_i(x_i). 
\]  

(6)

The best alternative is the one that maximizes this value. Even though this approach is attractive due to its low complexity, it can be shown that using an additive aggregation operator, such as weighted sum, is equivalent to assuming that all the attributes are independent. In practice, this is usually not realistic and therefore, we need to turn to nonadditive approaches, that is to aggregation operators that are not linear combinations of partial preferences.

Before approaching nonadditive methods, we give the definition of a nonadditive measure, a tool for building non-additive aggregation operators.

**DEFINITION 6.1. (Nonadditive measure).** Let \(I\) be the set of attributes and \(\mathcal{P}(I)\) the power set of \(I\). A set function \(\mu : \mathcal{P}(I) \to [0, 1]\) is called a nonadditive measure (or a fuzzy measure) if it satisfies the following three axioms:

1. \(\mu(\emptyset) = 0\) : the empty set has no importance.
2. \(\mu(I) = 1\) : the maximal set has maximal importance.
3. \(\mu(B) \leq \mu(C)\) if \(B, C \subset \mathcal{P}(I)\) and \(B \subset C\): a new criterion added cannot make the importance of a coalition (a set of criteria) diminish.

A nonadditive integral, such as the Choquet integral, is a type of a general averaging operator that can model the behavior of a decision maker. The decision maker provides a set of values of importance, this set being the values of the nonadditive measure on which the nonadditive integral is computed from. Formally, The Choquet integral is defined as follows:

**DEFINITION 6.2. (Choquet integral).** Let \(\mu\) be a nonadditive measure on \((I, \mathcal{P}(I))\) and an application \(f : I \to \mathbb{R}^+\). The Choquet integral of \(f\) w.r.t. \(\mu\) is defined by

International Journal of Intelligent Systems DOI 10.1002/int
\[(C) \int f \, d\mu = \sum_{i=1}^{n} (f(\sigma(i)) - f(\sigma(i-1)))\mu(A_{(i)}),\]

where \(\sigma\) is a permutation of the indices in order to have \(f(\sigma(1)) \leq \cdots \leq f(\sigma(n))\), \(A_{(i)} = \{\sigma(i), \ldots, \sigma(n)\}\), and \(f(\sigma(0)) = 0\), by convention.

It can be shown that many aggregation operators can be represented by Choquet integrals with respect to some fuzzy measure. Although the Choquet integral is well suited for optimal portfolio selection, it has a major drawback. The decision maker needs to input a value of importance of each subset of attributes, that is total of \(O(2^n)\) values, which is intractable. However, we can overcome intractability by using 2-additive measure to limit the complexity to a \(O(n^2)\) (as shown in Ref. 21 and still get accurate results.

Before giving the definition of a 2-additive measure, we need to define notion of interaction indices of orders 1 and 2. The importance of an attribute (or the interaction index of degree 1) is best described as the value this attribute brings to each coalition it does not belong to. It is given by the Shapley value:

**DEFINITION 6.3. (Shapley value).** Let \(\mu\) be a nonadditive measure over \(I\). The Shapley value of index \(i\) is defined by

\[v(i) = \sum_{B \subset I \setminus \{i\}} \gamma_I(B)[\mu(B \cup \{i\}) - \mu(B)]\]

with

\[\gamma_I(B) = \frac{(|I| - |B| - 1)! \cdot |B|!}{|I|!}\]

and \(|B|\) denoting the cardinal of \(B\).

While the Shapley value gives the importance of a single attribute to the entire set of attributes, the interaction index of degree 2 represents the interaction between two attributes and is defined by:

**DEFINITION 6.4. (Interaction index of degree 2).** Let \(\mu\) be a nonadditive measure over \(I\). The interaction index between \(i\) and \(j\) is defined by

\[I(i, j) = \sum_{B \subset I \setminus \{i,j\}} (\xi_I(B) \cdot (\mu(B \cup \{i,j\}) - \mu(B \cup \{i\}) - \mu(B \cup \{j\}) + \mu(B))\]

with \(\xi_I(B) = \frac{(|I| - |B| - 2)! \cdot |B|!}{(|I| - 1)!}\).

The interaction indices belong to the interval \([-1, +1]\) and
• \( I(i, j) > 0 \) if the attributes \( i \) and \( j \) are complementary;
• \( I(i, j) < 0 \) if the attributes \( i \) and \( j \) are redundant;
• \( I(i, j) = 0 \) if the attributes \( i \) and \( j \) are independent.

Even though, we can define interaction indices of any order, defining the importance of attributes and the interaction indices between two attributes is generally enough in MCDM problems. Thus, 2-additive measures constitute a feasible and accurate tool in this setting. The formal definition of 2-additive measure follows:\(^{22}\)

**DEFINITION 6.5. (2-additive measure).** A nonadditive measure \( \mu \) is called 2-additive if all its interaction indices of order equal to or larger than 3 are null and at least one interaction index of degree two is not null.

We can also show\(^9\) that the Shapley values and the interaction indices of order two offer us an elegant way to represent a Choquet integral. Therefore, in a decision-making problem, we can ask the decision maker to give the Shapley values, \( I_i \), and the interaction indices, \( I_{ij} \), and then use the Choquet integral w.r.t. to a 2-additive measure, \( \mu \), to obtain the aggregation operator:

\[
(C) \int f \, d\mu = \sum_{I_{ij} > 0} (f(i) \wedge f(j))I_{ij} + \sum_{I_{ij} < 0} (f(i) \vee f(j))|I_{ij}| + \sum_{i=1}^{n} f(i) \left( I_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}| \right).
\]

This form of the Choquet integral is accurate and practical approach to many situation, one of them being portfolio management.

Even though the utility based multicriteria decision-making setting and the resulting solution of the Choquet integral with respect to 2-additive measure is a feasible and accurate solution for values given by the decision maker, this approach faces another problem. We cannot expect a decision maker to give precise values for the importance and interaction indices. As a simple example consider the decision maker who requires return of 10%. However, this same individual might be happy with gaining only 9.8% return if it can be obtained with 50% smaller risk or if it can be obtained by investing money in an asset (or a portfolio) that has much shorter time to maturity. To overcome this hurdle, it was shown\(^2\) that the use of intervals provides a nice solution in MCDM problem.

We use interval arithmetic to calculate the Choquet integral over intervals. The result of this calculation is an interval, which makes things more complicated since comparison of intervals is more complicated than comparison of real numbers. However, there exist techniques to compare intervals and determine which one is the best.\(^{25}\)
3.1. Application to Portfolio Selection

We propose two different algorithms that make use of presented multicriteria decision-making approach, fuzzy measures, and intervals to find the optimal portfolio allocation. A two-stage algorithm uses a multicriteria decision-making setting to rank all assets. Based on the rank, good assets are selected among thousands of assets that exist in market and wealth is invested in these selected assets only. The second step of the algorithm utilizes another MCDM setting to determine the exact wealth allocation among the assets to best suit the goals of the investor.

The second algorithm utilizes similar multicriteria decision-making settings but starts by clustering all assets into three groups based on their risk. Based on the investor’s acceptable level of risk, distribution of wealth among these three groups of assets is determined and MCDM setting is created to determine the exact allocation of wealth within each cluster.

We first define a multicriteria decision-making problem by considering the set of all asset as the set of alternatives. We determine a finite set of criteria that characterize investment assets—return ($R$), risk ($r$), time to maturity ($t$), transaction cost ($c$), etc., and define the utility functions for each of them. The simplest method to choose rational utility functions is to provide mappings from the values of an alternative onto the interval $[0, 1]$, $f : X_i \rightarrow [0, 1]$. For the return of an asset, this could mean that the highest realistic return is mapped into 1, the lowest return to 0, and the other returns are proportionally mapped into values between 0 and 1. The utility of risk could be defined in a similar fashion, but taking the reciprocal of the risk since a high value of risk is less desired than a low value of risk. Similar arguments hold for time to maturity and transaction cost. Once the utility function for each criterion is defined, we proceed to calculation of the value of each asset.

If the decision maker (the investor) is concerned only with the return or only with the level of risk, then the maximax strategy could be used to rank all the assets with a high importance given to return in the first case and to the risk in the second case, and low importance given to all the other attributes. However, usually an investor wants to maximize the return for a given level of risk or minimize the risk while attaining the required return level in a certain time period. Thus, all the criteria have some influence on the decision. The decision maker is asked to input the Shapley value of each criterion, that is the importance of each criterion relative to other criteria. Since the attributes are mutually dependent (e.g., higher return usually implies higher risk, longer time to maturity usually means higher return, etc.), the weighted sum approach does not promise to give accurate results. However, we can approximate the interaction indices for each pair of attributes by estimating the correlation between their values, and use the Choquet integral with respect to a 2-additive measure, defined by Shapley values and interaction indices of order 2, to calculate the global value of an asset. The Choquet integral values are used to order the assets giving higher rank to the assets with the higher value of the Choquet integral.

Top $n$ assets are chosen to proceed to the second stage of the algorithm. The number $n$ is either pre-defined by the investor, or all the assets with the Choquet value above a threshold specified by the investor are selected. We denote the set of all assets
that are used to create portfolio by \( A \). The second stage of the algorithm tends to find the optimal distribution of wealth among \( n \) selected assets, \( w = (w_1, \ldots, w_n) \), by considering another multi-criteria decision making setting. The set of alternatives is defined as the set of all possible portfolios using only the assets selected into the set \( A \). The set of criteria is unchanged from the first stage of the algorithm. However, the values of the criteria are defined in terms of their values for each asset in the portfolio as follows:

- The return of the portfolio is
  
  \[ R(w) = \sum_{i=1}^{n} R_i w_i. \]  
  \( (9) \)

- The risk of the portfolio is
  
  \[ r(w) = \sum_{i=1}^{n} r_i w_i. \]  
  \( (10) \)

- Time to maturity of the portfolio, however, is not the weighted sum of the individual assets’ maturity times. It is the maximum time to maturity of all assets included in the portfolio:
  
  \[ r(w) = \max_j t_j, \]  
  where \( j \) is such that \( x_j \in A \).  
  \( (11) \)

Note that if all assets are included in every portfolio, then the time to maturity will be same for all portfolios.

- The transaction cost of the portfolio is
  
  \[ c(w) = \sum_{i=1}^{n} c_i v_i, \]  
  \( (12) \)

  where \( v_i = w_i \) if the transaction cost of the asset \( i \) is a proportion of wealth invested into the asset, and \( v_i = \text{constant } s \) if the transaction cost of the asset \( j \) is equal to \( s \) for any amount invested.

- Similarly, the values of other attributes characterizing a portfolio could be defined in terms of values of individual assets included into the portfolio.

Keeping the same Shapley values for all attributes and interaction indices of degree 2 for each pair of attributes as given in the first step of the algorithm, we maximize the Choquet integral of the alternatives. Thus, this stage of the algorithm reduces to an optimization constraint programming problem that finds the vector \( w = (w_1, \ldots, w_n) \) that maximizes the objective function

\[
\max_w \sum_{I_{ij} > 0} I_{ij}[u_i(x_i) \land u_j(x_j)] \\
+ \sum_{I_{ij} < 0} |I_{ij}| \cdot [u_i(x_i) \lor u_j(x_j)] \sum_{i=1}^{n} \left( u_i(x_i) - \frac{1}{2} \sum_{i \neq j} I_{ij} \right).
\]  
(13)
Here, $x_i$ and $x_j$ represent criteria of the portfolio (e.g., risk, return, time to maturity, and so on), which are defined in terms of $w_i$, and one of the characteristics of portfolio ($r_i$, $R_i$, $t_i$, $c_i$, or others).

The maximization problem is subject to the following constraints:

$$
\sum_{i=1}^{n} w_i R_i \geq R \text{ or } \sum_{i=1}^{n} w_i r_i \leq r \quad \text{(portfolio satisfies the main goal of the investor); (14)}
$$

$$
\sum_{i=1}^{n} w_i = 1 \quad \text{(exactly all wealth is invested); (15)}
$$

$$
w_i \geq 0 \quad \forall i = 1, \ldots, n \quad \text{(money can not be borrowed to be invested in an asset). (16)}
$$

This problem involving constraints could be solved using standard optimization techniques. Since all constraints are linear, the choice of the optimization technique depends on the form of the objective function. Using the simple utility functions described in this section, the objective function is linear as well, which allows us to use of the simplex method to determine the optimal solution. However, if some complex utility functions are used to execute the multicriteria decision process, the objective function might not be linear and other methods must be used to find the solution.

To reduce the complexity of the algorithm, we propose another algorithm that utilizes MCDM setting in portfolio selection. It starts by ordering all assets based only on their risk, and using this ranking all the assets are clustered into three groups: high-, middle-, and low-risk assets. The clustering is performed such that the third of assets with the highest risk constitutes group 1, group 2 contains the middle risk assets, and group 3 is the third of assets with the lowest risk. Next, we calculate the Choquet integral of each asset following the same MCDM setting as in the first algorithm. We select top $n_1 > 0$, $n_2 > 0$, and $n_3 > 0$ assets, respectively, from high-, middle-, and low-risk clusters to be included into the portfolio. The values of $n_1$, $n_2$, and $n_3$ are either all equal and predetermined, or they are such that the values of assets selected from each cluster are higher than a predefined threshold value.

Based on the investor’s level of risk aversion, the proportion of wealth invested in each cluster is determined and denoted by $p_1$, $p_2$, and $p_3$, respectively, for groups 1, 2, and 3. If the decision maker is highly risk averse, $p_1$ will be much smaller than $p_2$ and $p_3$, whereas for a risk-prone individual, $p_3$ will be smaller than $p_1$ and $p_2$. However, none of the numbers will be equal to zero to diversify portfolio, which is necessary to reduce unsystemic risk, the risk that depends on the company.
Finally, the wealth allocated to each cluster is distributed among the assets that belong to the group, so that the optimal portfolio is selected. Each cluster is considered separately from the other two and the best distribution of wealth is determined by maximizing the Choquet integral of the portfolios built by selected assets in each group

\[
\max_w \sum_{I_{ij} > 0} I_{ij}[u_i(x_i) \land u_j(x_j)] + \sum_{I_{ij} < 0} |I_{ij}|[u_i(x_i) \lor u_j(x_j)] + \sum_{i=1}^{n} \left( u_i(x_i) - \frac{1}{2} \sum_{i \neq j}^{n} I_{ij} \right), \tag{17}
\]

subject to

\[
\sum_{i=1}^{n} w_i = 1 \text{ (exactly all wealth is invested)}; \tag{18}
\]

\[
w_i \geq 0 \forall i = 1, \ldots, n \text{ (money can not be borrowed to be invested in an asset)}. \tag{19}
\]

Note that this algorithm does not explicitly require satisfaction of the main goal of the investor (e.g., required return level, maximum risk rate, and so on), but this requirement is implicitly accounted for in the distribution of wealth among three clusters. We can again apply one of the standard optimization techniques to solve this problem.

4. CONCLUSION

We have seen that finance is an area that is well suited to computational intelligent approaches. Genetic algorithms, rule-based systems, neural networks, and support vector machines have been used in selection of optimal portfolio. We have presented yet another technique to select an optimal portfolio. We have shown that a utility-based approach to decision making offers a natural and logical framework to optimize portfolio selection and have shown how the Choquet integral (which generalizes a large class of aggregation operators in multicriteria decision making) and interval computation can be used to solve such problems and allow us to deal with both uncertain and imprecise data. This approach has not yet been tested on real data; however, the theoretical framework is sound and general enough to be applied successfully to real finance data sets. More specifically, the next step is to test our approach on current financial market data and see how our algorithm performs. This is necessary to truly evaluate the impact of our approach, and more meaningful than using old data and see how our strategy would perform.
now. Moreover, our approach addresses some of the issues pertaining to other computational technique approaches, such as overfitting of parameters, description of the dependencies between characteristics of an asset, imprecise data, etc.

Finally, although we have focused on portfolio management problems, it is quite possible to use similar computational technique approaches to pricing problems, as an alternative to the more traditional stochastic differential equations and stochastic integration approaches. This is also part of future work.

References


International Journal of Intelligent Systems DOI 10.1002/int