Int. Journal of Math. Analysis, Vol. 3, 2009, no. 40, 1997 - 2007

Nonlinear Three-Point Boundary Value Problems for First Order Impulsive Integro-Differential Equations of Mixed Type[1]

Xiaofei He, Jingli Xie and Guoping Chen

Department of Mathematics, Jishou University
Jishou, Hunan 416000, P.R. China
xiejingli721124@163.com

Abstract

In this paper, the method of upper and lower solutions and monotone iterative technique are employed to the study of nonlinear three-point boundary value problems for a class of first order impulsive integro-differential equations of mixed type. Sufficient conditions for the existence of extreme solutions are obtained.

Mathematics Subject Classification: 34B37

Keywords: Impulsive integro-differential equation; Monotone iterative technique; Three-point boundary value problem

1. Introduction

In this paper, we study the following impulsive integro-differential problem:

$$\begin{cases} x'(t) = f(t, x(t), x(\theta(t)), (Kx)(t), (Hx)(t)),\ t \neq t_k,\ t \in J = [0, T],\ T > 0, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m, \\ g(x(0), x(\eta), x(T)) = 0, \end{cases} \tag{1.1}$$

where $f \in C(J \times R^4, R), \theta \in C(J, J),\ I_k \in C(R, R),\ g \in C(R^3, R),\ 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T, \eta \in J,\ \Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump of $x(t)$ at $t = t_k$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Denote $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$. The integral part in Eq.(1.1) is defined by

$$(Kx)(t) = \int_0^{\gamma(t)} k(t, s)x(\delta(s))ds, (Hx)(t) = \int_0^T h(t, s)x(\sigma(s))ds,$$

[1]This Project is supported by Provincial Natural Science Foundation of Hunan (No. 09JJ6010), P.R. China.

where $\gamma, \delta, \sigma \in C(J, J), k \in C(D, R^+), D = \{(t, s) \in J \times J : 0 \le s \le \gamma(t) \le T\}, h \in C(J \times J, R^+), R^+ = [0, \infty), K_0 = \max\{k(t, s) : (t, s) \in D\}, H_0 = \max\{h(t, s) : (t, s) \in J \times J\}$.

Let $PC(J) = \{u : J \to R,\ u$ is continuous for $t \in J, t \ne t_k, u(t_i^+), u(t_i^-)$ exist and $u(t_i^-) = u(t_i),\ i = 1, 2, \ldots, m\}$. $PC^1(J) = \{u \in PC(J) : u$ is continuously differentiable for $t \in J, t \ne t_k\}$. $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$||u||_{PC(J)} = \sup\{|u(t)| : t \in J\}, ||u||_{PC^1(J)} = \max\{||u||_{PC(J)}, ||u'||_{PC(J)}\}.$$

By a solution of (1.1) we mean a $u \in PC^1(J)$ for which problem (1.1) is satisfied.

Note that (1.1) has a very general form, as special instances resulting from (1.1), one can have impulsive differential equations with deviating arguments and impulsive differential equations with the Volterra or Fredholm operators. For example, if f does not include Kx and Hx, and $g(u, v, w) \equiv g(u, v), \theta(t) = t$, then (1.1) reduces to

$$\begin{cases} x'(t) = f(t, x(t)),\ t \in J, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m, \\ g(x(0), x(T)) = 0. \end{cases} \tag{1.2}$$

Two point boundary value problem with nonlinear boundary conditions for impulsive ordinary differential equations, which is discussed in several papers [13-15].

when f does not include Kx and Hx, and $g(u, v, w) \equiv g(u, v)$, then (1.1) reduces to the following two boundary value problem with nonlinear boundary conditions for impulsive function differential equation

$$\begin{cases} x'(t) = f(t, x(t), \theta(t)),\ t \in J, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m, \\ g(x(0), x(T)) = 0. \end{cases} \tag{1.3}$$

Which is discussed in [10].

When $g(u, v, w) \equiv g(u, v)$, (1.1) is the following nonlinear two boundary problem for impulsive integro-differential equation

$$\begin{cases} x'(t) = f(t, x(t), (Kx)(t), (Hx)(t)),\ t \ne t_k,\ t \in J = [0, T],\ T > 0, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m, \\ g(x(0), x(T)) = 0. \end{cases} \tag{1.4}$$

Which is also discussed in [16].

It is well known that the monotone iterative technique offer an approach for obtaining approximate solutions of nonlinear differential equations, for details, see [6] and the references therein. There also exist several works devoted to the applications of this technique to boundary value problems of impulsive differential equations, see, for example, [1-5,7-11]. In this paper, we consider (1.1) by using the method of upper and lower solutions combined with monotone iterative technique. This technique plays important roles in constructing monotone sequences which converge to the solutions of our problems. In presence of a lower solution α and an upper solution β with $\alpha \le \beta$, we show under suitable conditions the sequences converge to the solutions of (1.1) by using the method of upper and lower solutions and monotone iterative technique.

This paper is organized as follows. In section 2, we establish a new comparison principle and give a proof for the existence lemma related to a linear problem associated to Eq.(1.1). In section 3, we first introduce the concept of lower and upper solutions, and then obtain the existence of extreme solutions for (1.1) by using the method of upper and lower solutions and monotone iterative technique.

2. Some lemmas

To obtain our main results, we need the following lemmas.

Lemma 2.1([12]). Assume that $s \in [0,T), x \in PC^1(J), p, q \in PC(J), a_i \ge 0, b_i, i = 1, 2, \cdots, m$, are constants such that

$$\begin{cases} x'(t) \le p(t)x(t) + q(t), \quad t \in [s,T),\ t \ne t_k, \\ x(t_k^+) \le a_k x(t_k) + b_k,\ t_k \in [s,T),\ k = 1, 2, \cdots, m, \end{cases}$$

then

$$\begin{aligned} x(t) \le\ & x(s^+)\left(\prod_{s<t_k<t} a_k\right)\exp\left(\int_s^t p(u)du\right) + \sum_{s<t_k<t}\left(\prod_{t_k<t_i<t} a_i\right)\exp\left(\int_{t_k}^t p(\tau)d\tau\right) b_k \\ & + \int_s^t \left(\prod_{u<t_k<t} a_k\right)\exp\left(\int_u^t p(\tau)d\tau\right) q(u)du,\ t \in [s,T]. \end{aligned}$$

Lemma 2.2. Assume that $x \in PC^1(J), M_1 > 0, M_i \ge 0, i = 2, 3, 4, 0 \le L_k < 1, (k = 1, 2, \cdots, m)$ such that

$$\begin{cases} x'(t) \le -M_1 x(t) - M_2 x(\theta(t)) - M_3(Kx)(t) - M_4(Hx)(t), t \in J, t \ne t_k, \\ \Delta x(t_k) \le -L_k x(t_k), \quad k = 1, 2, \cdots, m, \\ x(0) \le 0, \end{cases} \tag{2.1}$$

and

$$\begin{aligned} &\textstyle\int_0^T \prod_{s<t_k<T}(1-L_k)(M_2 e^{M_1(t-\theta(t))} + M_3\int_0^{\gamma(s)} k_1(s,\sigma)d\sigma \\ &\textstyle +M_4\int_0^T h_1(s,\sigma)d\sigma)ds \le \prod_{j=1}^m (1-L_k). \end{aligned} \tag{2.2}$$

where

$$k_1(t,s) = e^{M_1(t-\delta(s))}k(t,s), h_1(t,s) = e^{M_1(t-\sigma(s))}h(t,s).$$

then $x(t) \le 0$ on J.

Proof. Let $v(t) = x(t)e^{M_1 t}$, then $v \in PC^1(J)$ and

$$\begin{cases} v'(t) \le -M_2 e^{M_1(t-\theta(t))}v(\theta(t)) - M_3(K_1 v)(t) - M_4(H_1 v)(t), t \in J', \\ \Delta v(t_k) \le -L_k v(t_k), k = 1, 2, \cdots, m, \\ v(0) \le 0, \end{cases} \tag{2.3}$$

where

$$(K_1v)(t)=\int_0^{\gamma(t)}k_1(t,s)v(\delta(s))ds,(H_1v)(t)=\int_0^T h_1(t,s)v(\sigma(s))ds,$$

$$k_1(t,s)=e^{M_1(t-\delta(s))}k(t,s),h_1(t,s)=e^{M_1(t-\sigma(s))}h(t,s).$$

Obviously, $v(t)\le 0$ implies $x(t)\le 0$. To show $v(t)\le 0$, suppose on the contrary, that $v(\tilde{t})>0$ for some $\tilde{t}\in J$. Then there are two possible cases:
(1) $v(t)\ge 0$ for all $t\in[0,\tilde{t}]$;
(2) there exists $t_1^*\in[0,\tilde{t})$ such that $v(t_1^*)<0$.

In case (1), (2.3) implies that $v'(t)\le 0$ for $t\ne t_k$ and $\Delta v(t_k)\le 0$,here $t,t_k\in[0,\tilde{t}]$, hence $v(t)$ is nonincreasing in J, then $v(0)\ge v(\tilde{t})>0$, which is a contradiction.

In case (2), let $\inf_{t\in[0,\tilde{t}]}=-p$,then $p>0$, and there exists $0\le t_i<t_2^*\le t_{i+1}<\tilde{t}$ for some $i\in\{0,1,\cdots,m-1\}$ such that $v(t_2^*)=-p$ or $v(t_i^+)=-p$. We may assume that $v(t_2^*)=-p$(since,in case of $v(t_i^+)=-p$, the proof is similar). From (2.3), we get

$$v'(t)\le p(M_2e^{M_1(t-\theta(t))}+M_3\textstyle\int_0^{\gamma(t)}k_1(t,s)ds+M_4\int_0^T h_1(t,s)ds),t\ne t_k,t\in[0,\tilde{t}]. \tag{2.4}$$

Without loss of generality,we suppose that $\tilde{t}\in(t_j,t_{j+1}]$ for some $j\in\{0,1,2,\cdots,m\}$. Obviously $i\le j$, from Lemma (2.1), we have

$$\begin{aligned}v(t)\ \le\ & v(t_2^*)\prod_{t_2^*<t_k<t}(1-L_k)+p\int_{t_2^*}^{t}\prod_{s<t_k<t}(1-L_k)(M_2e^{M_1(s-\theta(s))}+\\ & M_3\int_0^{\gamma(s)}k_1(s,\sigma)d\sigma+M_4\int_0^T h_1(s,\sigma)d\sigma)ds\end{aligned} \tag{2.5}$$

Let $t=\tilde{t}$ in (2.5),then

$$\begin{aligned}v(\tilde{t})\ \le\ & v(t_2^*)\prod_{t_2^*<t_k<\tilde{t}}(1-L_k)+p\int_{t_2^*}^{\tilde{t}}\prod_{s<t_k<\tilde{t}}(1-L_k)(M_2e^{M_1(s-\theta(s))}+\\ & M_3\int_0^{\gamma(s)}k_1(s,\sigma)d\sigma+M_4\int_0^T h_1(s,\sigma)d\sigma)ds\end{aligned} \tag{2.6}$$

Noting that $v(\tilde{t})>0$ and $v(t_2^*)=-p$,we obtain

$$\begin{aligned}0\ <\ & v(\tilde{t})\le -p\prod_{t_2^*<t_k<\tilde{t}}(1-L_k)+p\int_{t_2^*}^{\tilde{t}}\prod_{s<t_k<\tilde{t}}(1-L_k)(M_2e^{M_1(s-\theta(s))}+\\ & M_3\int_0^{\gamma(s)}k_1(s,\sigma)d\sigma+M_4\int_0^T h_1(s,\sigma)d\sigma)ds\end{aligned}$$

(2.7)

this yields

$$\begin{array}{l}\int_0^T \prod_{s<t_k<T}(1-L_k)(M_2e^{M_1(t-\theta(t))}dt + M_3\int_0^{\gamma(s)} k_1(s,\sigma)d\sigma \\ +M_4\int_0^T h_1(s,\sigma)d\sigma)ds > \prod_{j=1}^m(1-L_k).\end{array} \tag{2.8}$$

It contradicts (2.2).This completes the proof.

Lemma 2.3. Assume that $x \in PC^1(J), M_1 > 0, M_i \geq 0, i = 2,3,4, \eta \in PC(J), 0 \leq L_k < 1, k = 1,2,\cdots,m$. If

$$M_2\frac{e^{M_1T}-1}{M_1e^{M_1T}} + M_3TK_0\frac{e^{M_1T}-1}{M_1e^{M_1T}} + M_4TH_0\frac{e^{M_1T}-1}{M_1e^{M_1T}} + \sum_{k=1}^m L_k < 1, \tag{2.9}$$

then the impulsive differential problem

$$\begin{cases} x'(t) = -M_1x(t) - M_2x(\theta(t)) - M_3(Kx)(t) - M_4(Hx)(t) + \eta(t), t \in J, t \neq t_k, \\ \Delta x(t_k) = -L_kx(t_k) + d_k, \quad d_k \in R, \quad k = 1,2,\cdots,m, \\ x(0) = x_0, x_0 \in R, \end{cases} \tag{2.10}$$

has a unique solution.

Proof. Define a map $A : PC(J) \to PC(J)$ by

$$\begin{aligned}[]
[Ax](t) &= e^{-M_1t}x_0 + \int_0^T G(t,s)\left(\eta(s) - M_2x(\theta(s)) - M_3(Kx)(s) - M_4(Hx)(s)\right)ds \\
&\quad + \sum_{k=1}^m G(t,t_k)\left(-L_kx(t_k) + d_k\right),
\end{aligned}$$

where

$$G(t,s) = \begin{cases} e^{-M_1(t-s)}, & 0 \leq s < t \leq T, \\ 0, & 0 \leq t \leq s \leq T. \end{cases}$$

It is easy to verify that x is a solution of (2.10) if and only if x is a fixed point of A. For any $x, y \in PC(J)$, we have

$$\begin{aligned}
&|[Ax](t) - [Ay](t)| \leq |M_2\int_0^T G(t,s)(y(\theta(s)) - x(\theta(s)))ds| \\
&+|M_3\int_0^T\int_0^{\gamma(s)} G(t,s)k(s,u)(y(\delta(u)) - x(\delta(u)))duds| \\
&+|M_4\int_0^T\int_0^T G(t,s)h(s,u)(y(\sigma(u)) - x(\sigma(u)))duds| \\
&+|\sum_{k=1}^m G(t,t_k)L_k(y(t_k) - x(t_k))| \\
&\leq (M_2\frac{e^{M_1T}-1}{M_1e^{M_1T}} + M_3TK_0\frac{e^{M_1T}-1}{M_1e^{M_1T}} + M_4TH_0\frac{e^{M_1T}-1}{M_1e^{M_1T}} + \sum_{k=1}^m L_k)\|x-y\|,
\end{aligned}$$

this implies

$$\|[Ax]-[Ay]\| \le (M_2\frac{e^{M_1T}-1}{M_1e^{M_1T}}+M_3TK_0\frac{e^{M_1T}-1}{M_1e^{M_1T}}+M_4TH_0\frac{e^{M_1T}-1}{M_1e^{M_1T}}+\sum_{k=1}^{m}L_k)\|x-y\|.$$

Condition (2.9) implies that A is a contraction mapping. Banach′s fixed point theorem implies that A has a unique fixed point, and so (2.10) has a unique solution. The proof is complete.

3. Main results

Function $\alpha, \beta \in PC^1(J)$ are called lower and upper solutions of equation (1.1) if

$$\begin{cases} \alpha'(t) \le f(t, \alpha(t), \alpha(\theta(t)), (K\alpha)(t), (H\alpha)(t)),\ t \ne t_k,\ t \in J, \\ \Delta\alpha(t_k) \le I_k(\alpha(t_k)),\ k = 1, 2, \ldots, m, \\ g(\alpha(0), \alpha(\eta), \alpha(T)) \le 0. \end{cases}$$

and

$$\begin{cases} \beta'(t) \ge f(t, \beta(t), \beta(\theta(t)), (K\beta)(t), (H\beta)(t)),\ t \ne t_k,\ t \in J, \\ \Delta\beta(t_k) \ge I_k(\beta(t_k)),\ k = 1, 2, \cdots, m, \\ g(\beta(0), \beta(\eta), \beta(T)) \ge 0. \end{cases}$$

Theorem 3.1. Let α, β be the lower solution and upper solution for (1.1) with $\alpha \le \beta$ on J respectively, and assume that (2.2) and (2.9) hold. Further suppose that

($\mathbf{H_1}$) There exist $M_1 > 0, M_i \ge 0, i = 2, 3, 4$, such that

$$f(t, x_1, x_2, x_3, x_4)-f(t, y_1, y_2, y_3, y_4) \ge -M_1(x_1-y_1)-M_2(x_2-y_2)-M_3(x_3-y_3)-M_4(x_4-y_4),$$

for $\alpha(t) \le y_1 \le x_1 \le \beta(t), \alpha(\theta(t)) \le y_2 \le x_2 \le \beta(\theta(t)), (K\alpha)(t) \le y_3 \le x_3 \le (K\beta)(t), (H\alpha)(t) \le y_4 \le x_4 \le (H\beta)(t),\ t \in J.$

($\mathbf{H_2}$) There exists L_k with $0 \le L_k < 1, k = 1, 2, \cdots, m$, such that

$$I_k(x) - I_k(u) \ge -L_k(x-u),\ \text{for } \alpha(t_k) \le u(t_k) \le x(t_k) \le \beta(t_k),\ k = 1, 2, \cdots, m.$$

($\mathbf{H_3}$) $g_u > 0,\ g_v \le 0,\ g_w \le 0.$

Then (1.1) possesses in $[\alpha, \beta]$ the minimal and maximal solution $u,\ v \in PC^1(J)$ respectively, where $[\alpha, \beta] = \{x \in PC^1(J) : \alpha(t) \le x(t) \le \beta(t), t \in J\}$. Moreover, there exist the monotone sequences $\{u_n\} \nearrow u$ and $\{v_n\} \searrow v$ uniformly on J, respectively.

Proof. For $\forall\psi \in [\alpha, \beta]$, we consider linear impulsive differential equation

$$\begin{cases} y'(t) = -M_1y(t) - M_2y(\theta(t)) - M_3(Ky)(t) - M_4(Hy)(t) + \eta(t), t \ne t_k,\ t \in J, \\ \Delta y(t_k) = I_k(\psi(t_k)) - L_k(y(t_k) - \psi(t_k)),\ k = 1, 2, \cdots, m, \\ g(y(0), \psi(\eta), \psi(T)) = 0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned}\eta(t) &= f(t,\psi(t),\psi(\theta(t)),(K\psi)(t),(H\psi)(t)) + M_1\psi(t) + M_2\psi(\theta(t)) \\ &\quad + M_3(K\psi)(t) + M_4(H\psi)(t).\end{aligned}$$

We will show that (3.1) has a unique solution $y \in PC^1(J)$.

Firstly, we show that there exists a unique constant $y(0) : \alpha(0) \le y(0) \le \beta(0)$ such that $g(y(0),\psi(\eta),\psi(T)) = 0$ for $\forall \psi \in [\alpha,\beta]$. From (H_3) and $g(\alpha(0),\alpha(\eta),\alpha(T)) \le 0 \le g(\beta(0),\beta(\eta),\beta(T))$, we have that

$$g(\beta(0),\psi(\eta),\psi(T)) \ge g(\beta(0),\beta(\eta),\beta(T)) \ge 0,$$

$$g(\alpha(0),\psi(\eta),\psi(T)) \le g(\alpha(0),\alpha(\eta),\alpha(T)) \le 0.$$

From continuity and monotone of g, there exists a unique constant $y(0) : \alpha(0) \le y(0) \le \beta(0)$ such that $g(y(0),\psi(\eta),\psi(T)) = 0$. By lemma 2.3, one can see that (3.1) has a unique solution $y \in PC^1(J)$. Denote $A\psi$ is a unique solution of initial value problem (3.1).

Let $u_0 = \alpha,\ v_0 = \beta$ and $u_n = Au_{n-1},\ v_n = Av_{n-1}, n = 1,2,\cdots$. We shall show that

(a) $u_0 \le Au_0, \quad Av_0 \le v_0$.

(b) A is nondecreasing in $[u_0, v_0]$.

The proof of property (a). Let $p = u_0 - u_1$, it follows that we have for $t \ne t_k$

$$\begin{aligned}p'(t) &= u_0'(t) - u_1'(t) \\ &\le f(t,u_0(t),u_0(\theta(t)),(Ku_0)(t),(Hu_0)(t)) + M_1(u_1(t)) + M_2(u_1(\theta(t))) + M_3(Ku_1)(t) \\ &\quad + M_4(Hu_1)(t) - f(t,u_0(t),u_0(\theta(t)),(Ku_0)(t),(Hu_0)(t)) - M_1u_0(t) - M_2u_0(\theta(t)) \\ &\quad - M_3(Ku_0)(t) - M_4(Hu_0)(t) \\ &\quad = -M_1p(t) - M_2p(\theta(t)) - M_3(Kp)(t) - M_4(Hp)(t),\end{aligned}$$

and for $t = t_k$

$$\begin{aligned}\Delta p(t_k) &= \Delta u_0(t_k) - \Delta u_1(t_k) \\ &\le I_k(u_0(t_k)) - I_k(u_0(t_k)) + L_k(u_1(t_k) - u_0(t_k)) \\ &= -L_kp(t_k),\end{aligned}$$

Also, from $g(u_1(0),u_0(\eta),u_0(T)) \ge g(u_0(0),u_0(\eta),u_0(T))$ and $g_u > 0$, we have $u_0(0) \le u_1(0)$, and so $p(0) = u_0(0) - u_1(0) \le 0$. By lemma 2.2, we have $p(t) \le 0$ for $t \in J$, that is, $u_0 \le Au_0$. Analogously, it is proved that $Av_0 \le v_0$.

Next, we prove the property (b). Let $\psi_1,\ \psi_2 \in [u_0,\ v_0]$ with $\psi_1 \le \psi_2$, $\omega_1 = A\psi_1, \omega_2 = A\psi_2, p = \omega_1 - \omega_2$, then from (H_1) and (H_2) , we have that for $t \ne t_k$

$$p'(t) = \omega_1'(t) - \omega_2'(t)$$

$$\begin{aligned}
&= -M_1\omega_1(t) - M_2\omega_1(\theta(t)) - M_3K(\omega_1)(t) - M_4H(\omega_1)(t) \\
&\quad +f(t,\psi_1(t),\psi_1(\theta(t)),(K\psi_1)(t),(H\psi_1)(t)) + M_1\psi_1(t) + M_2\psi_1(\theta(t)) \\
&\quad +M_3(K\psi_1)(t) + M_4(H\psi_1)(t) \\
&\quad +M_1\omega_2(t) + M_2\omega_2(\theta(t)) + M_3K(\omega_2)(t) + M_4H(\omega_2)(t) \\
&\quad -f(t,\psi_2(t),\psi_2(\theta(t)),(K\psi_2)(t),(H\psi_2)(t)) - M_1\psi_2(t) - M_2\psi_2(\theta(t)) \\
&\quad -M_3(K\psi_2)(t) - M_4(H\psi_2)(t) \\
&\leq M_1(\psi_2(t)-\psi_1(t)) + M_2(\psi_2(\theta(t))-\psi_1(\theta(t))) + M_3((K\psi_2)(t)-(K\psi_1)(t)) \\
&\quad +M_4((H\psi_2)(t)-(H\psi_1)(t)) - M_1\omega_1(t) - M_2\omega_1(\theta(t)) \\
&\quad -M_3K(\omega_1)(t) - M_4H(\omega_1)(t) + M_1\psi_1(t) + M_2\psi_1(\theta(t)) \\
&\quad +M_3(K\psi_1)(t) + M_4(H\psi_1)(t) + M_1\omega_2(t) + M_2\omega_2(\theta(t)) \\
&\quad +M_3K(\omega_2)(t) + M_4H(\omega_2)(t) - M_1\psi_2(t) - M_2\psi_2(\theta(t)) \\
&\quad -M_3(K\psi_2)(t) - M_4(H\psi_2)(t) \\
&= -M_1p(t) - M_2p(\theta(t)) - M_3(Kp)(t) - M_4(Hp)(t),
\end{aligned}$$

and for $t = t_k$

$$\begin{aligned}
\Delta p(t_k) &= \Delta\omega_1(t_k) - \Delta\omega_2(t_k) \\
&= I_k(\psi_1(t_k)) - L_k(\omega_1(t_k)-\psi_1(t_k)) - I_k(\psi_2(t_k)) + L_k(\omega_2(t_k)-\psi_2(t_k)) \\
&\leq L_k(\psi_2(t_k)-\psi_1(t_k)) - L_k(\omega_1(t_k)-\psi_1(t_k)) + L_k(\omega_2(t_k)-\psi_2(t_k)) \\
&= -L_kp(t_k),
\end{aligned}$$

From $g(\omega_2(0),\psi_2(\eta),\psi_2(T)) = g(\omega_1(0),\psi_1(\eta),\psi_1(T))$ and (H_3), we have $\omega_2(0) \geq \omega_1(0)$, that is, $p(0) \leq 0$. It follows by lemma 2.2 that $p(t) \leq 0$ for $t \in J$, i.e. A is monotone nondecreasing in $[u_0, v_0]$.

It follows, from the properties (a) and (b), that

$$\alpha \leq u_1 \leq \cdots \leq u_n \leq v_n \cdots \leq v_1 \leq \beta, \quad \text{on } J.$$

By standard arguments, we conclude that there exist u, v such that

$$\lim_{n\to\infty} u_n = u(t), \quad \lim_{n\to\infty} v_n = v(t), \text{ uniformly on } J,$$

and

$$u_n(t) \leq u(t) \leq v(t) \leq v_n(t).$$

It is easy to show that u and v are solutions of the equation (1.1) using $u_n,\ v_n$ satisfy the relations

$$\begin{cases}
u'_{n+1}(t) = -M_1(u_{n+1}(t)) - M_2(u_{n+1}(\theta(t))) - M_3(Ku_{n+1})(t) - M_4(Hu_{n+1})(t) \\
\qquad\qquad +f(t,u_n(t),u_n(\theta(t)),(Ku_n)(t),(Hu_n)(t)) + M_1u_n(t) + M_2u_n(\theta(t)) \\
\qquad\qquad +M_3(Ku_n)(t) + M_4(Hu_n)(t), t \in J', \\
\Delta u_{n+1}(t_k) = I_k(u_n(t_k)) - L_k(u_{n+1}(t_k) - u_n(t_k)),\ k = 1,2,\cdots,m, \\
g(u_{n+1}(0), u_n(\eta), u_n(T)) = 0,
\end{cases}$$

and

$$\begin{cases} v'_{n+1}(t) = -M_1(v_{n+1}(t)) - M_2(v_{n+1}(\theta(t))) - M_3(Kv_{n+1})(t) - M_4(Hv_{n+1})(t) \\ \qquad\qquad +f(t, v_n(t), v_n(\theta(t)), (Kv_n)(t), (Hv_n)(t)) + M_1 v_n(t) + M_2 v_n(\theta(t)) \\ \qquad\qquad +M_3(Kv_n)(t) + M_4(Hv_n)(t), t \in J', \\ \Delta v_{n+1}(t_k) = I_k(v_n(t_k)) - L_k(v_{n+1}(t_k) - v_n(t_k)),\ k = 1, 2, \cdots, m, \\ g(v_{n+1}(0), v_n(\eta), v_n(T)) = 0. \end{cases}$$

Taking the limit $n \to \infty$ on both sides of above relations, we have

$$\begin{cases} u'(t) = f(t, u(t), u(\theta(t)), (Ku)(t), (Hu)(t)),\ t \neq t_k,\ t \in J, \\ \Delta u(t_k) = I_k(u(t_k)),\ k = 1, 2, \cdots, m, \\ g(u(0), u(\eta), u(T)) = 0, \end{cases} \tag{3.2}$$

and

$$\begin{cases} v'(t) = f(t, v(t), v(\theta(t)), (Kv)(t), (Hv)(t)),\ t \neq t_k,\ t \in J, \\ \Delta v(t_k) = I_k(v(t_k)),\ k = 1, 2, \cdots, m, \\ g(v(0), v(\eta), v(T)) = 0. \end{cases} \tag{3.3}$$

(3.2) and (3.3) show that u and v are solutions of equation (1.1). Finally, we prove that if $z \in [\alpha, \beta]$ is any solution of (1.1), then $u(t) \leq z(t) \leq v(t)$ on J. To this end, we assume, without loss of generality, that $u_n(t) \leq z(t) \leq v_n(t)$ for some n, since $\alpha(t) \leq z(t) \leq \beta(t)$. From property (b), we can get

$$u_{n+1}(t) \leq z(t) \leq v_{n+1}(t), \quad t \in J.$$

Since $u_0(t) = \alpha(t) \leq z(t) \leq \beta(t) = v_0(t), t \in J$, by induction, we can conclude that

$$u_n(t) \leq z(t) \leq v_n(t), \quad \text{for} \quad \text{all} \quad n,$$

Passing to the limit as $n \to \infty$, we obtain

$$u(t) \leq z(t) \leq v(t), \quad t \in J.$$

This ends the proof.

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Received: April, 2009