Optimal Linear and Cyclic Locally Repairable Codes over Small Fields

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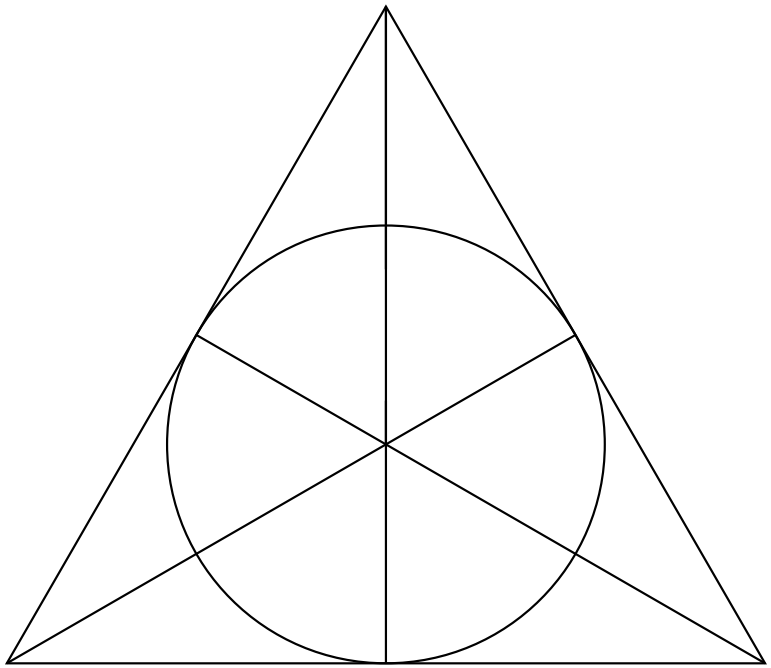
Locality and Availability

Binary $[7, 3, 4]_2$ Simplex code $\mathcal{L}$.

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$



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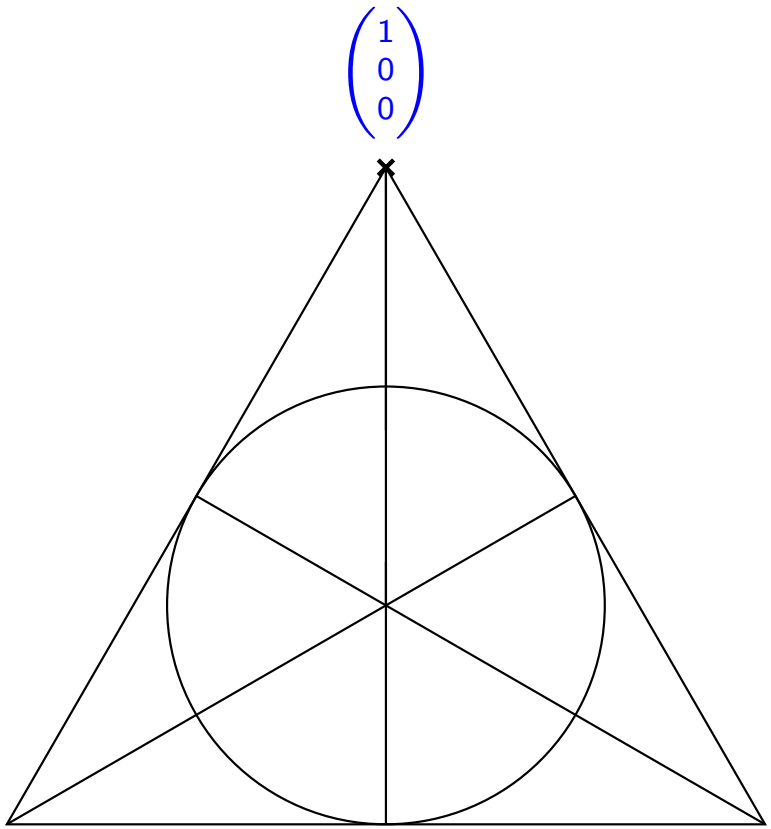

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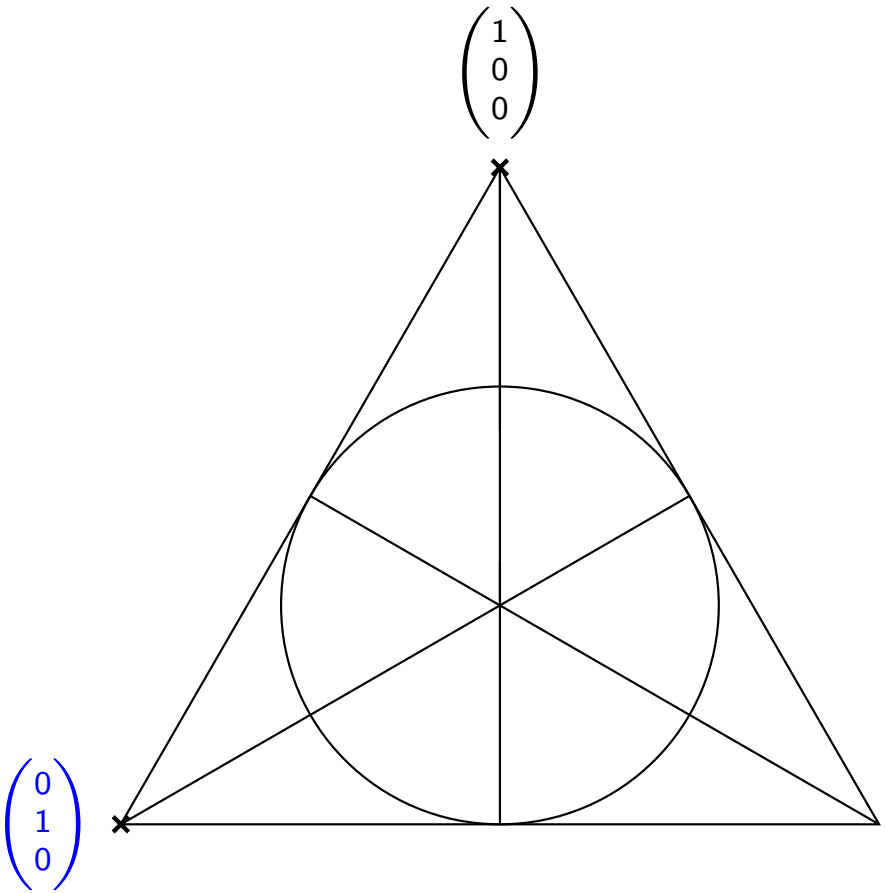

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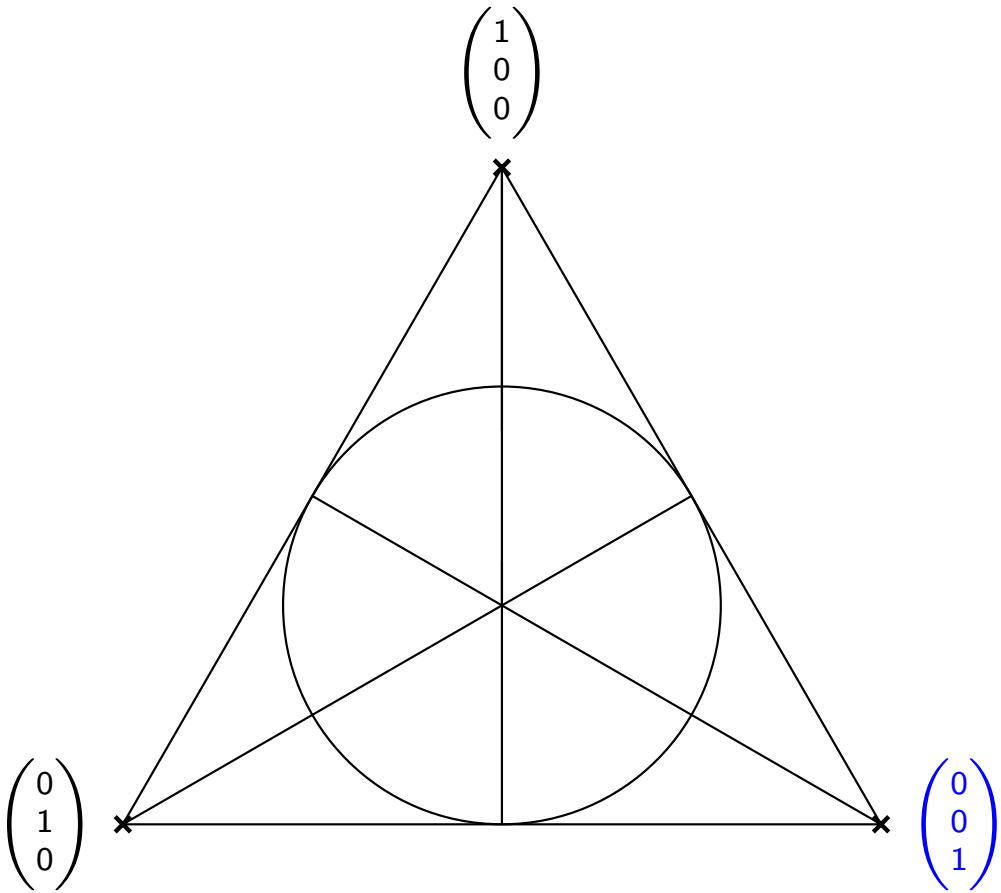

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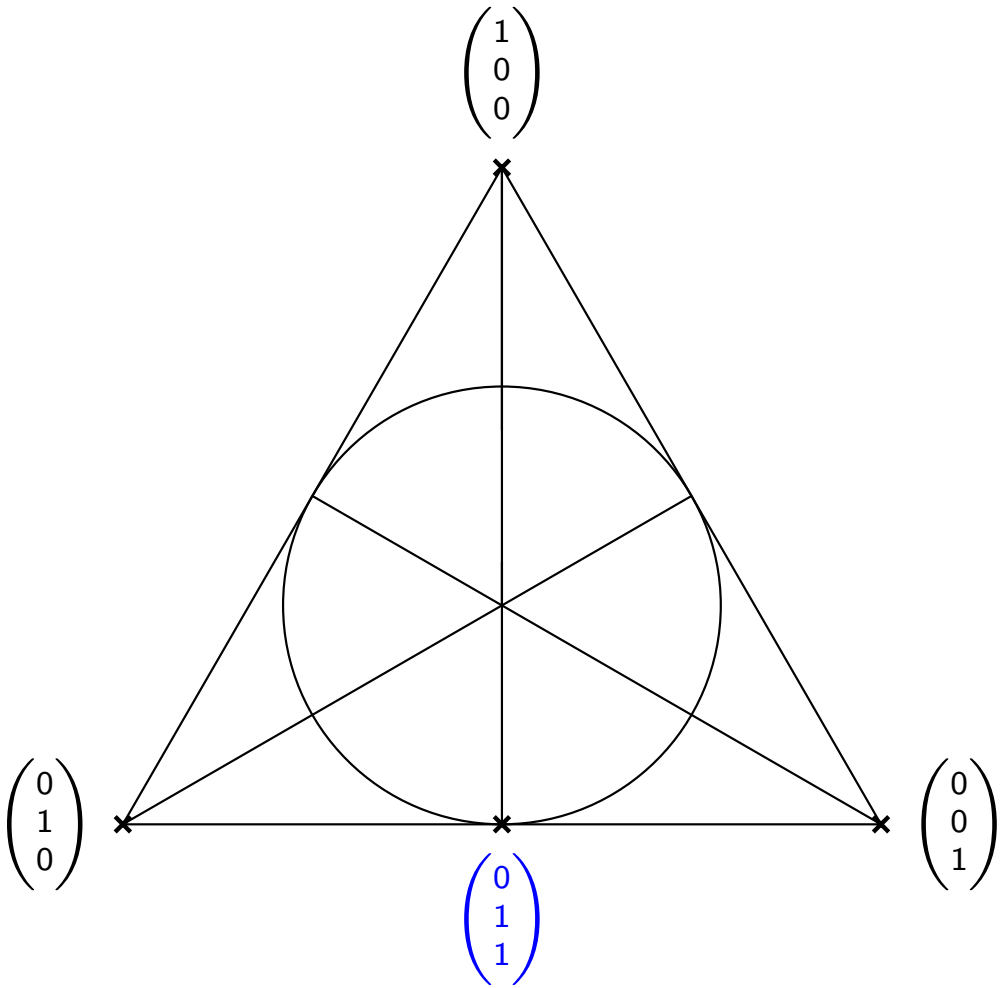

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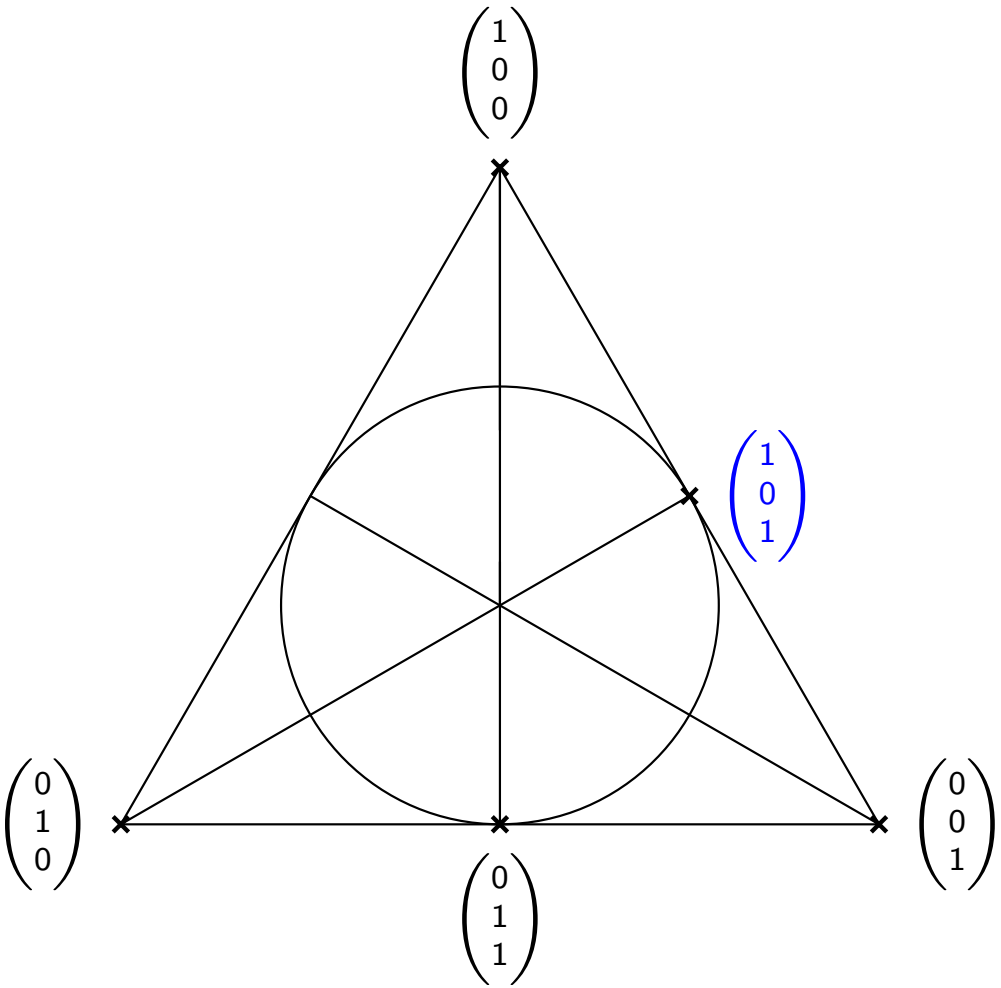

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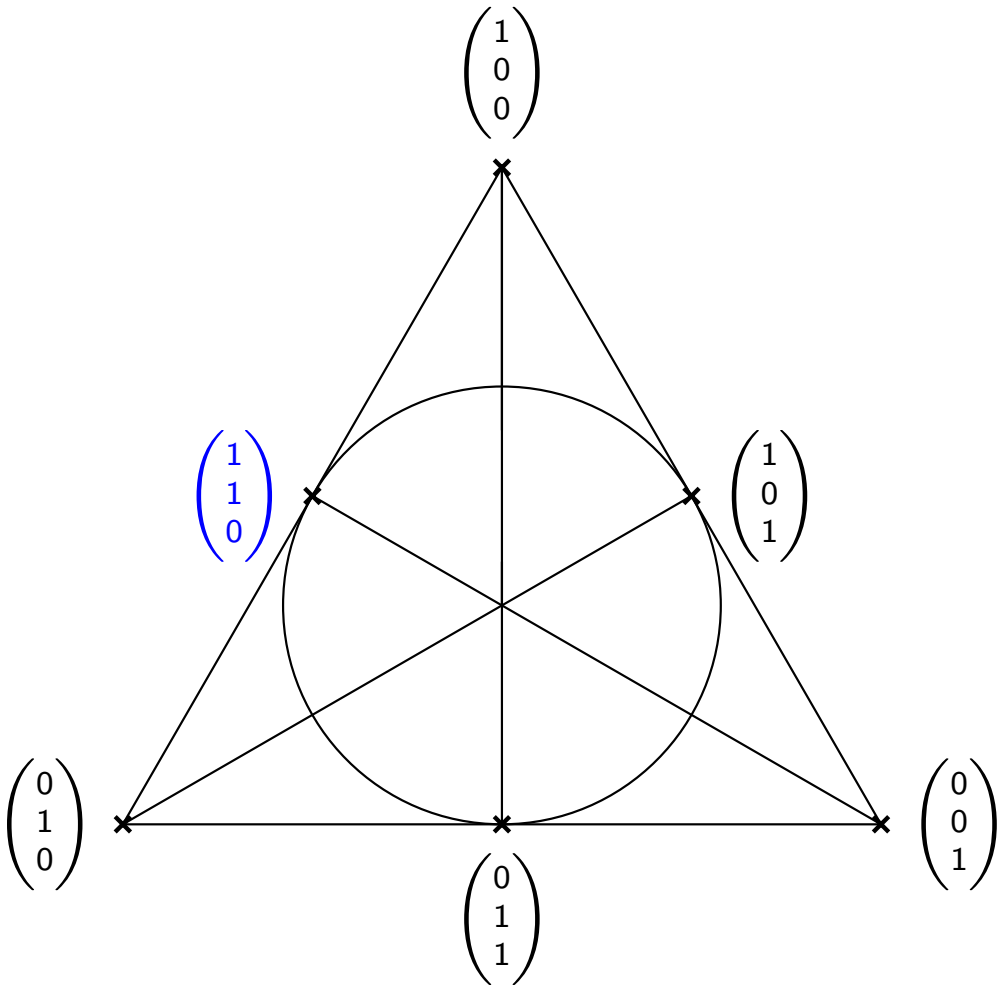

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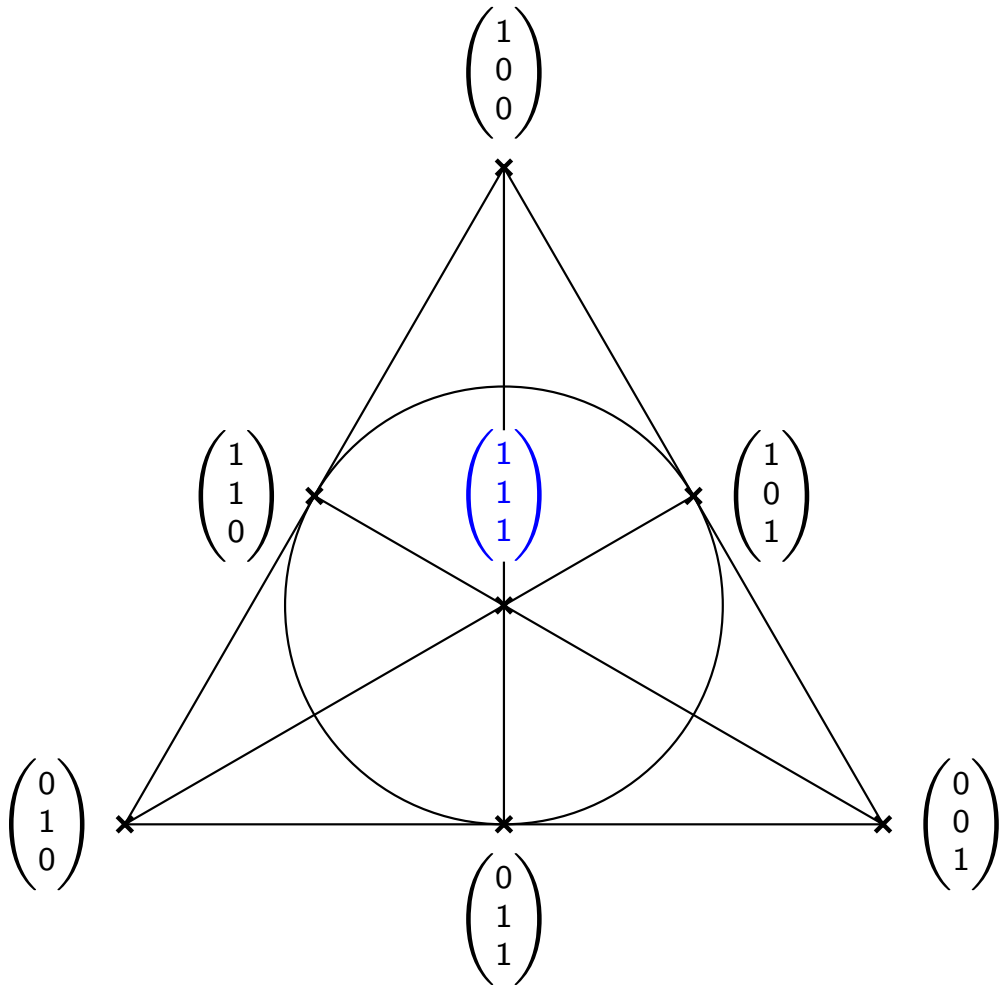

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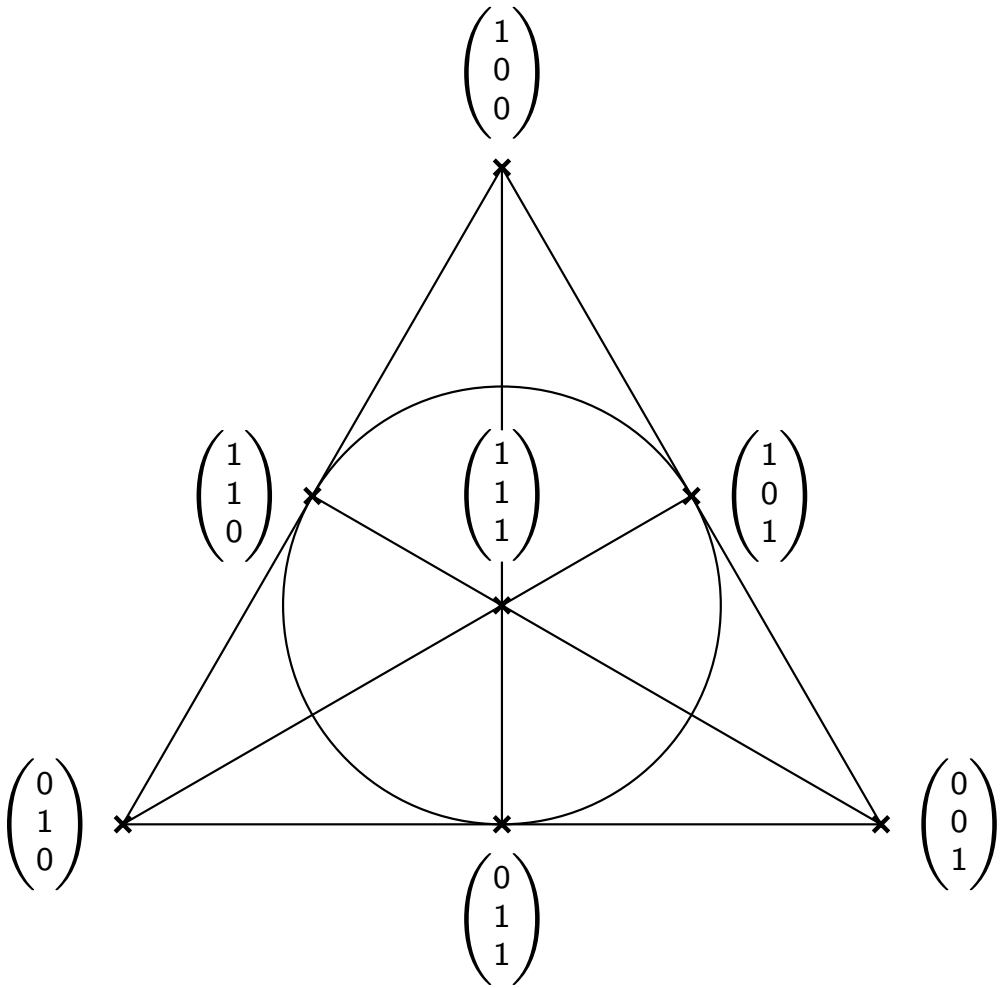

Outline

1. Locality and Availability
2. Singleton-Optimal LRC: Product Code
3. Binary Cyclic GC-Optimal LRC: Subcode of Product Code
4. Conclusion and Outlook

Outline

1. Locality and Availability
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Locality and Availability (i)

הגדרה (Locally Repairable Code (LRC))

An $[n, k, d]_q$ code $\mathcal{C}$ is (r, δ)-local

- if for all code symbols c_i, $\exists$ a punctured subcode of $\mathcal{C}$ with support containing i,
- whose length is at most $r + \delta - 1$, and
- whose minimum distance is at least δ.

A code $\mathcal{C}$ is called r-local if it is $(r, 2)$-local.

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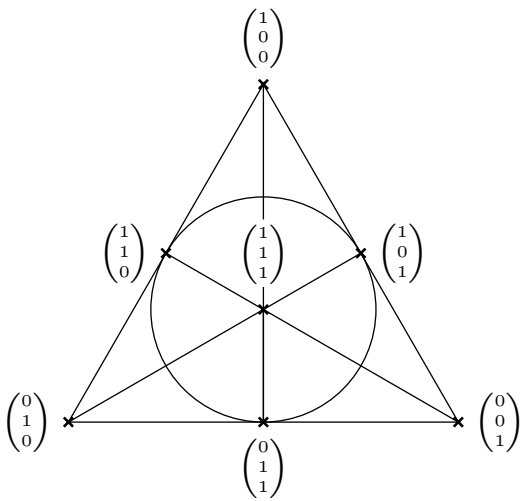

The $[7, 3, 4]_2$ Simplex code is

- 2-local $= (2, 2)$-local.

Locality and Availability (ii)

הגדרה (Availability)

An $[n, k, d]_q$ code $\mathcal{C}$ is called t-available and r-local

- if every code symbol c_i, has at least t parity-checks
- of weight $r + 1$ which intersect pairwise in (and only in) $\{i\}$.

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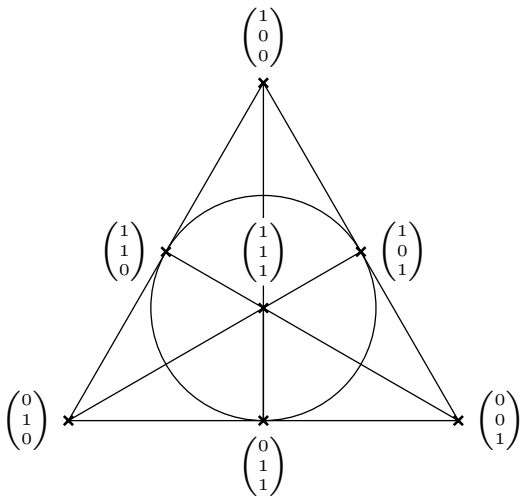

The $[7, 3, 4]_2$ Simplex code is

- 2-local and
- 3-available.

Bounds

(r, δ)-local, t-available $[n, k, d]_q$ LRC $\mathcal{C}$

"Singleton-optimal"
[Gopalan *et al.* 2012]

$\mathcal{C}$ satisfies:

$$d \leq n\text{-}k+1\text{-}\left(\left\lceil \frac{k}{r} \right\rceil \text{-}1\right)(\delta\text{-}1)$$

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Cyclic $[n, k, d]_2$ 2-local LRC

$\mapsto$

Additive $(n', 2^k, d')_4$ code

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Cyclic $[n, k, d]_2$ 2-local LRC
$\mapsto$
Additive $(n', 2^k, d')_4$ code

"CM-optimal"
[Cadambe–Mazumdar 2013]

$k_i \stackrel{\text{def}}{=} ri + k_{opt}^{(q)}\big(n\text{-}i(r\text{+}1), d\big)$
and
$k \leq min_{i \in \mathbb{N}} k_i$

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Product Code Constructions (i)

בנייה (Singleton-optimal Cyclic LRC [Z.–Yaakobi])

- $\mathcal{L}$: $[n_l = r + \delta - 1, k_l = r, d_l = \delta]_q$ cyclic MDS code with $D_{\mathcal{L}}$.
- $\mathcal{T}$: trivial $[n/n_l, n/n_l, 1]_q$ code.

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- $\mathcal{T}$: trivial $[n/n_l, n/n_l, 1]_q$ code.

Then, the defining set of the cyclic product code $\mathcal{C} = \mathcal{L} \otimes \mathcal{T}$ is

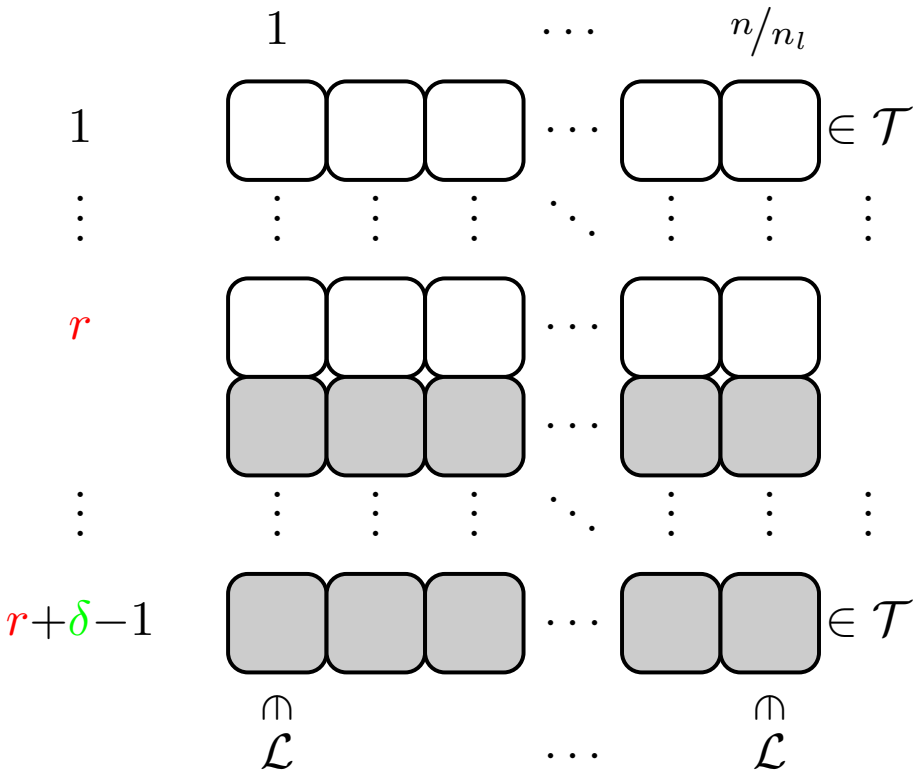
$$D_{\mathcal{C}} = D_{\mathcal{L}} \cup D_{\mathcal{L}}^{[n_l]} \cup \cdots \cup D_{\mathcal{L}}^{[n/n_l - 1]}.$$

$\mathcal{C}$ is an $[n, n/n_l r, d = \delta]_q$ cyclic (r, δ)-local code.

Proof.

Singleton-optimal. □

Product Code Construction (ii)



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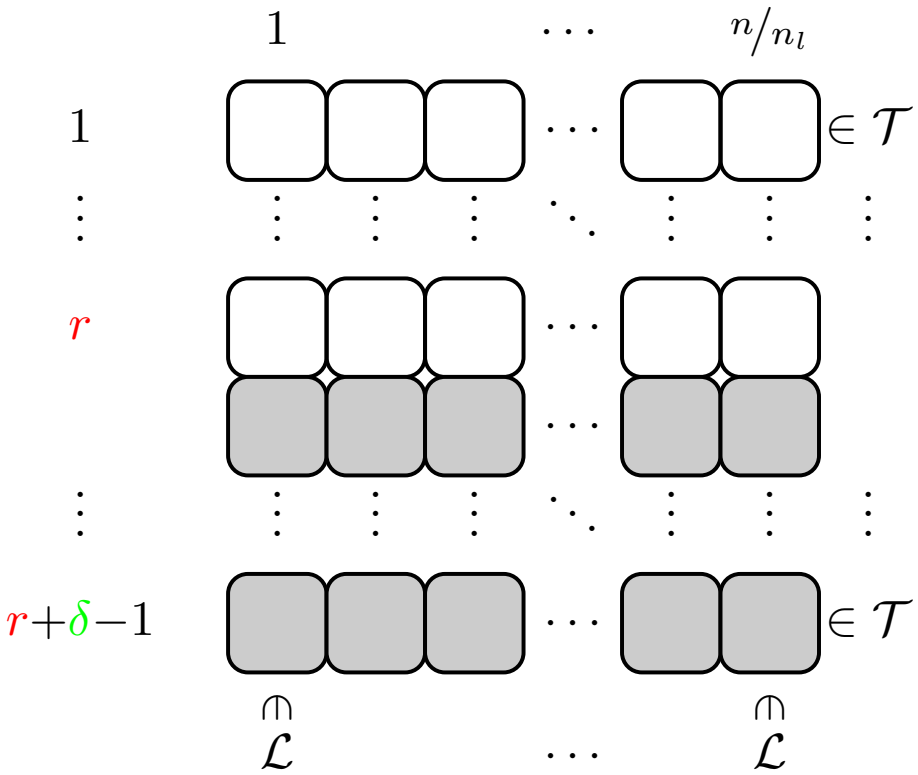

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$\mathcal{L}$: $[3, 2, 2]_2$ $(2, 2)$-local SPC with

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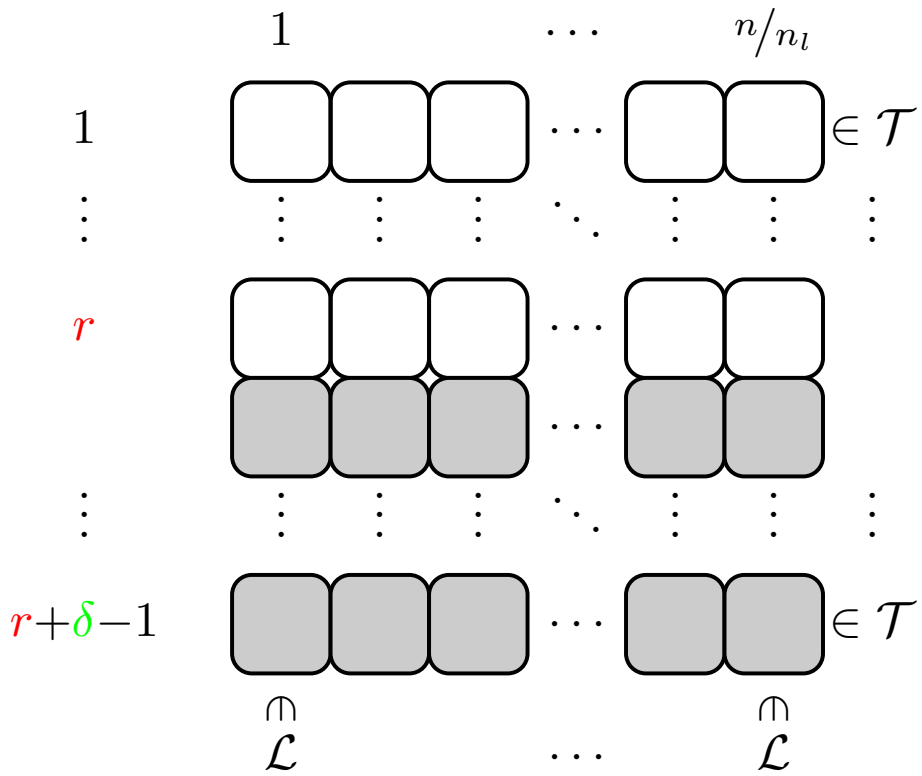

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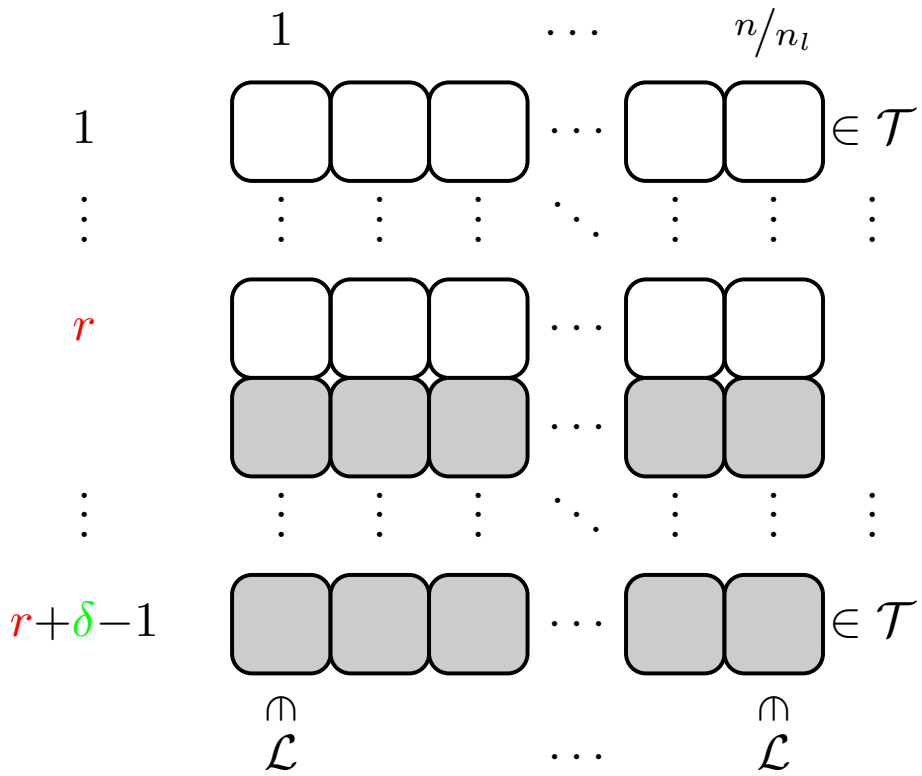

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is a $[15, 10, 2]_2$ cyclic
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GC-Optimal for Locality Two (i)

בנייה (Binary Cyclic Codes [Constr. 2, GC2014])

- $n = 2^m - 1$ and $3 \mid n$ ($\Leftrightarrow$ m even).
- $\mathcal{L}$: $[3, 2, 2]_2$ SPC with $D_{\mathcal{L}} = \{0\}$ and 2-local.

Let the defining set of $\mathcal{C}$ be

$$\begin{aligned} D_{\mathcal{C}} &= \{ \overbrace{\{0, 3, 6, 9, 12, \dots\}}^{\mathcal{L} \otimes \mathcal{T}} \cup \{1, 2, 4, \dots\} \}, \\ &= \{0, 1, 2, 3, 4, 6, \dots\}. \end{aligned}$$

$\mathcal{C}$ is an $[n = 2^m - 1, \frac{2}{3}n - m, 6]_2$ cyclic 2-local code.

Additive Code for Locality Two

הגדרה (Additive Code over $\mathbb{F}_4$)

An additive code $\mathcal{A}$ of length n over $\mathbb{F}_4$ is an additive subgroup over $\mathbb{F}_4^n$.

Let $\mathbb{F}_4 = \{0, 1, \alpha, \overline{\alpha}\}$. Then the map

$$\begin{aligned} \phi : \ \mathbb{F}_2^3 &\to \mathbb{F}_4 \\ 000 &\mapsto 0 \\ 011 &\mapsto 1 \\ 101 &\mapsto \alpha \\ 110 &\mapsto \overline{\alpha} \end{aligned}$$

projects an $[3n, k]_2$ code $\mathcal{C}$ to an $(n, 2^k)_4$ additive code $\phi(\mathcal{C})$.

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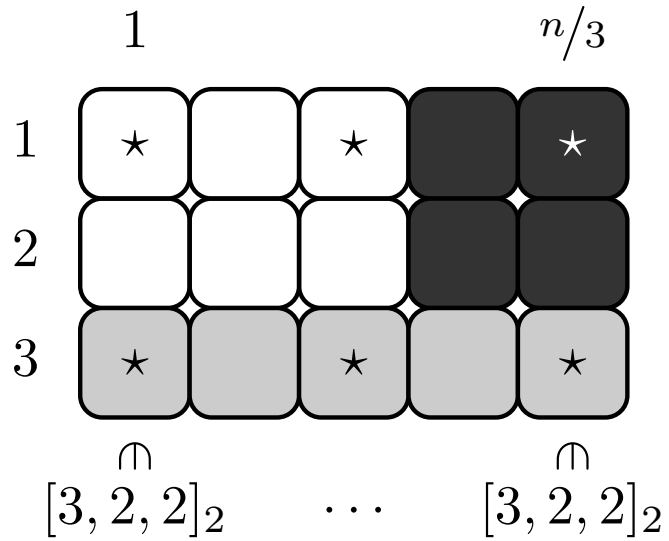
GC-optimal: The $(n', 4^{k'}, d')_4$ additive code $\mathcal{A}$ has parameters

$$n' = \frac{4^{m/2} - 1}{4 - 1}, \quad k' = n' - \frac{m}{2}, \quad d' = 3.$$

GC-Optimal for Locality Two (ii)

For $m = 4$:

$$(n' = \tfrac{4^2-1}{3} = 5, 4^{5-2}, 3)_4 \text{ additive code } \mathcal{A}$$



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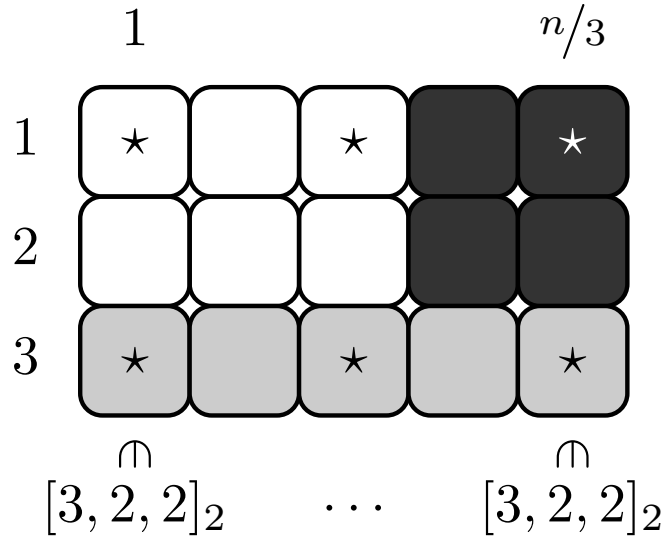

$\mathcal{C}$ is a $[15, 6, 6]_2$ cyclic 2-local code and GC- and CM-optimal.

Additive Code (ii)

משפט (Projection to Additive Code)

Let

- $\mathcal{L}$: $[n_l, k_l, d_l]_q$ locality code,
- $\mathcal{C}$: $[n, k, d]_q$ code and let $n_l | n$.

Suppose the n coordinates of a codeword in $\mathcal{C}$ can be subdivided in n/n_l groups of length n_l, where each is a codeword of $\mathcal{L}$.

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Suppose the n coordinates of a codeword in $\mathcal{C}$ can be subdivided in n/n_l groups of length n_l, where each is a codeword of $\mathcal{L}$.

- Each sub-block of n_l symbols of a codeword in $\mathcal{C}$ is projected to one symbol in $\mathbb{F}_{q^{k_l}}$.

The obtained $(n', q^k, d')_{q^{k_l}}$ additive code $\mathcal{A}$ has parameters:

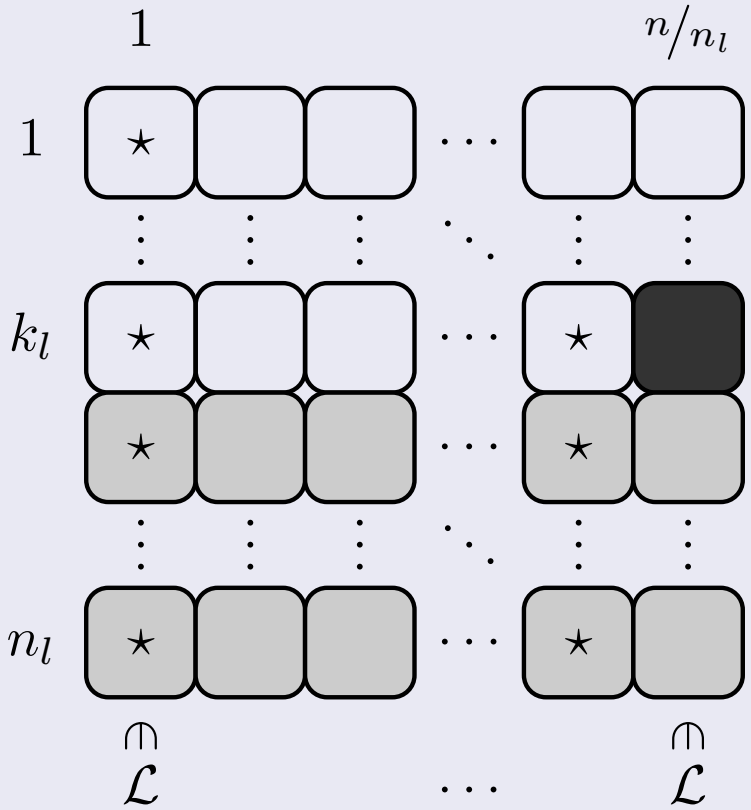
$$n' = \frac{n}{n_l} \quad \text{and} \quad d' \geq \left\lceil \frac{d}{\omega} \right\rceil,$$

where ω is the maximum weight of a codeword in $\mathcal{L}$.

Additive Code (iii)

$$(n' = \tfrac{n}{n_l}, q^k, d' \geq \left\lceil \tfrac{d}{\omega} \right\rceil)_{q^{k_l}} \text{ additive code } \mathcal{A}$$

Proof.



GC-Optimal for Locality Two and Availability Three (i)

בניה (GC-Optimal Binary 2-local and 3-available Code)

- $n = 2^m - 1$ and $7 \mid n$ ($\Leftrightarrow 3 \mid m$).
- $\mathcal{L}$: $[7, 3, 4]_2$ 2-local and 3-available Simplex code with:

$$D_{\mathcal{L}} = \{0, 3, 5, 6\}.$$

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Let the defining set be:

$$D_{\mathcal{C}} = \overbrace{\{\ldots, |-7, \square, \square, -4, \square, -2, -1, |0, \square, \square, 3, \square, 5, 6, |7, \square, \square, 10, \ldots\}}^{\mathcal{L} \otimes \mathcal{T}} \cup \{1, 2, 4, 8, \ldots, 2^{m-1}\}.$$

$\mathcal{C}$ is an $[n = 2^m - 1, \frac{3}{7}n - m, 12]_2$ cyclic 2-local and 3-available code.

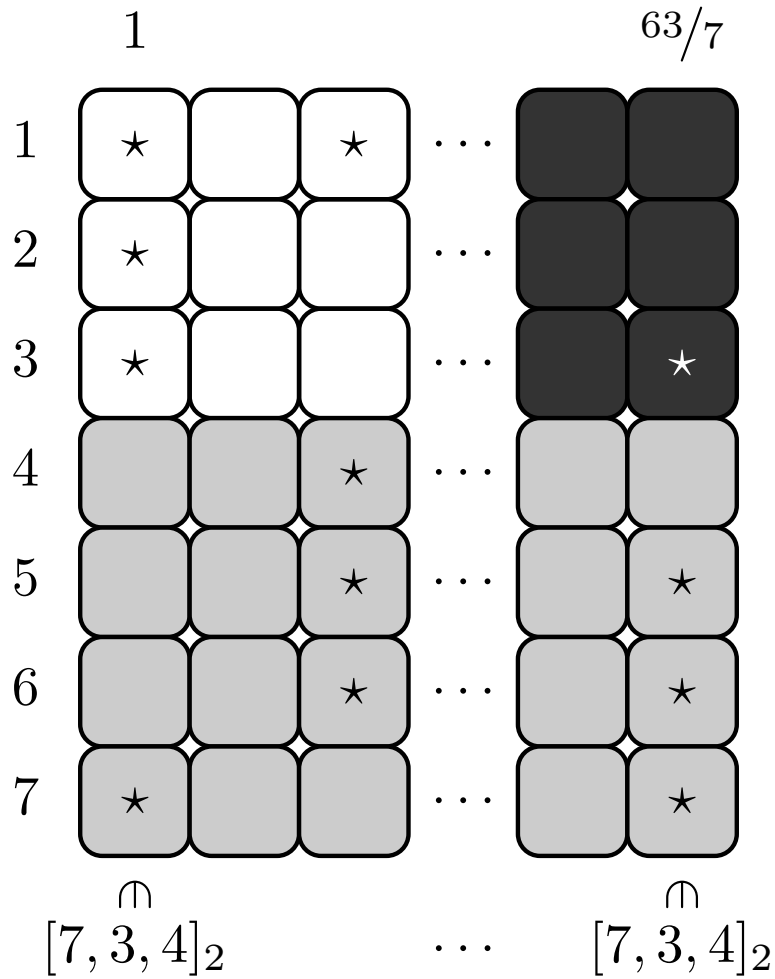
GC-optimal: The $(n', 8^{k'}, d')_8$ additive code has parameters

$$n' = \frac{8^{m/3} - 1}{8 - 1}, \quad k' = n' - \frac{m}{3}, \quad d' = 3.$$

GC-Optimal for Locality Two and Availability Three (i)

For $m = 6$:

$$(n' = \tfrac{8^2-1}{8-1} = 9, 8^{9-2}, 3)_{2^3} \text{ additive code } \mathcal{A}$$



GC-Optimal for Locality Two and Higher Availability (i)

בנייה (Binary Simplex)

- $n = 2^m - 1$ and $(2^a - 1) \mid n$ ($\Leftrightarrow a \mid m$).
- $\mathcal{L}$: $[2^a - 1, a, 2^{a-1}]_2$ Simplex 2-local and $2^{a-1}-1$-available code with

$$D_{\mathcal{L}} = \left\{0, \square, \square, 3, \square, 5, 6, \ldots, \square, 2^{a-1}+1, \ldots, 2^a - 1\right\}.$$

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Let the defining set be

$$\begin{aligned} D_{\mathcal{C}} &= \overbrace{\{D_{\mathcal{L}} \cup D_{\mathcal{L}}^{[2^a-1]} \cup D_{\mathcal{L}}^{[2(2^a-1)]} \cup \ldots\}}^{\mathcal{L}\otimes\mathcal{T}} \cup \{1, 2, 4, 8, \ldots, 2^{m-1}\} \\ &= \{\ldots, -2^{a-1}+1, \ldots, -1|, 0, 1, \ldots, 2^a - 1, |2^a, \ldots\}. \end{aligned}$$

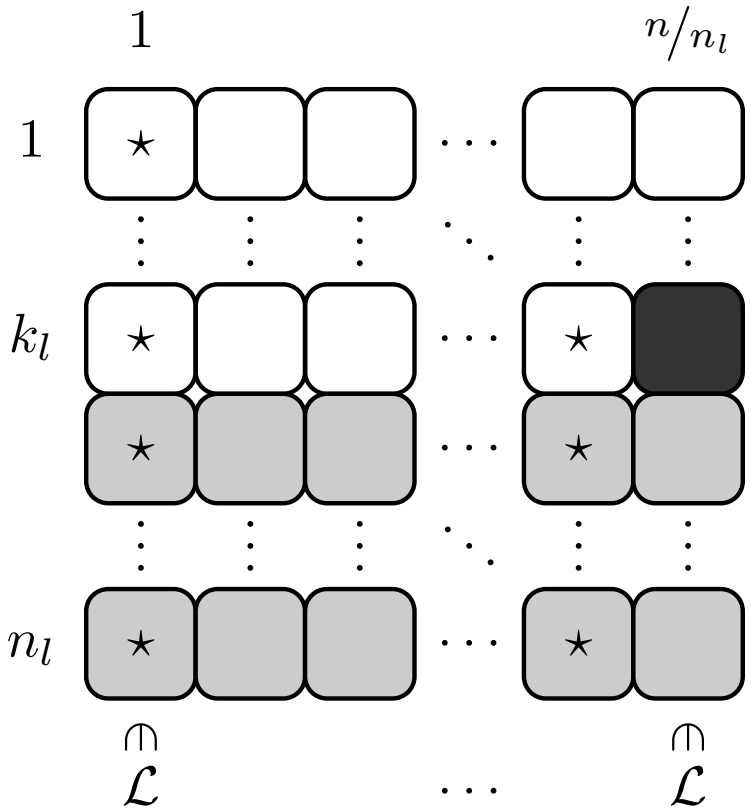
$\mathcal{C}$ is an $[n, \frac{a}{2^a-1}n\text{-}m, 2^a+2^{a-1}]_2$ cyclic 2-local, $2^{a-1}-1$-available code.

GC-Optimal for Locality Two and Higher Availability (ii)

GC-optimal: The $(n', k', d')_{2^a}$ additive code has parameters

$$n' = \frac{2^{m/a} - 1}{2^a - 1},$$
$$k' = n' - \frac{m}{2^a},$$
$$d' = \left\lceil \frac{2^a + 2^{a-1}}{2^a} \right\rceil = \left\lceil \frac{2^{a-1}(2+1)}{2^a} \right\rceil = 3.$$

From Simplex to RM Code



2-ary: Simplex code has constant-weight.

q-ary: $[q^a - 1, a, q^{a-1}(q-1)]_q$
shortened first-order (s.fo.) RM code.

Outline

1. Locality and Availability
   - Definitions
   - Existing Field-Size-Dependent Bounds
2. Singleton-Optimal LRC: Product Code
3. Binary Cyclic GC-Optimal LRC: Subcode of Product Code
   - Locality Two
   - Locality Two and Availability Three
   - Locality Two and Higher Availability
4. Conclusion and Outlook

Summary and Outlook

Framework for Optimal LRC

1. Singleton-optimal (cyclic) (r, δ)-local code.
2. GC-optimal 2-local, $2^{a-1}-1$-available binary cyclic code.
3. GC-optimal (r, δ)-local (q-ary cyclic) code [not presented].
4. GC/CM-optimal (r, δ)-local (q-ary) code [not presented].

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Outlook

- Applying other (than sphere-packing) bound for additive code $\mathcal{A}$.
- Other locality code $\mathcal{L}$.
- Efficient erasure-only global decoding for $\mathcal{C}$.

Thank you!